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Polynomial Solutions of Electric Field Equations in Anisotropic Media

Meltem Altunkaynak
Department of Mathematics, Faculty of Sciences, Dokuz Eylül University, Buca, 35160 Izmir, Turkey,
meltem.topcuoglu@deu.edu.tr, ORCID: 0000-0002-6706-2957

ABSTRACT

The time-dependent system of partial differential equations of the second order describing the electric wave propagation in electrically and magnetically anisotropic homogeneous media is considered in the paper. A method for the computation of the polynomial solutions of the initial value problem for the considered system is proposed. Symbolic computations are used and these symbolic computations are implemented in Maple. It is proved also that these polynomial solutions are approximate solutions of the considered initial value problem with smooth initial data and the inhomogeneous term. The computational experiments confirm the robustness of the suggested method for the computation of electric fields in general electrically and magnetically anisotropic media.

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**Corresponding author*

1. Introduction

At the recent time the use and development of new anisotropic materials stimulates the growing interest for modeling electric and magnetic wave propagations inside these new materials. This topic is an important interdisciplinary area of research with many cutting-edge scientific and technological applications [6],[7],[11]. The physical properties of anisotropic media essentially depend on the orientation and position. For instance, in anisotropic homogeneous media the physical properties depend on the orientation and do not depend on the position [11]. The medium can be isotropic relative to some properties and anisotropic with respect to others. For example, anisotropic crystals and dielectrics are magnetically isotropic but electrically anisotropic. Some of materials are magnetically anisotropic but electrically isotropic and some of materials are electrically and magnetically anisotropic. One of the first and a very well known approach for modeling wave propagations is a 'plane wave approach', when a wave front is considered as an unbounded plane front in the space (see, for example [3],[6],[7],[11],[12]).

Besides that the electromagnetic waves are often raised by electric currents or charges located in some points, curves and surfaces [3],[6],[7]. For example, antenna radiation above the earth's surface is an important subject in radio wave communications. The boundary-value problem of radiation from a dipole antenna above a dielectric isotropic half space was first investigated by Sommerfeld and it thus became known as the Sommerfeld dipole problem. Electromagnetic wave radiations in a dielectric free space and half space from different types of currents(pulse polarized dipole, line currents, sheet currents, and shell currents) have been studied in [3],[6],[7],[11].

The observations of electric and magnetic fields in different anisotropic media generated by electric currents give an information about the dependence of electromagnetic field behavior and the structure of media. These observations allow engineers to study properties of known anisotropic materials and design new materials with the certain response to electric and magnetic fields for given source. Most of electromagnetic scattering problems, initial value and initial boundary value problems have been solved by numerical methods, in particular, finite elements method, boundary elements methods, finite difference method, nodal method (see, for example, works [1],[9],[10],[13],[18],[20] and their references).

Nowadays computers can perform very complicated symbolic computations (in addition to numerical calculations) and this opens up new possibilities to solve initial value and initial boundary value problems. Symbolic computations can be considered as useful tools for analytical methods that can provide exact solutions of problems.

The main object studied in this paper is the following initial value problem (IVP) of radiation from the electric current in electrically and magnetically anisotropic media

$$\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}+curl_x(\mu^{-1}curl_x\mathbf{E})+\sigma\frac{\partial\mathbf{E}}{\partial t}=-\frac{\partial\mathbf{j}}{\partial t}, \quad (1)$$

$$\mathbf{E}\Big|_{t=0}=0, \quad \frac{\partial\mathbf{E}}{\partial t}\Big|_{t=0}=0. \quad (2)$$

In this paper, the IVP (1),(2) is solved using polynomial solution method. The method has been successfully applied to solve some IVPs before. In [14], [17], this method is applied to solve IVP for system of crystal optics for simulations of waves in homogeneous, anisotropic dielectrics and electrically anisotropic materials for the case when $\sigma=0$, μ is identity matrix. In [15] PS method is applied to IVP for equations of electric and magnetic fields in general electrically and magnetically anisotropic media for the case when $\sigma=0$, ε and μ are arbitrary matrices. In the paper [16], the method is applied to IVP for Maxwell's equations in bi-anisotropic materials. However, the IVP for Maxwell's equations in conducting media for the case when ε, μ are arbitrary positive definite matrices and σ is a symmetric matrix has not been studied yet.

In this paper, the IVP (1),(2) of radiation from electric current in electrically and magnetically anisotropic media is studied in the case when ε, μ and σ are arbitrary matrices. An analytical method for computing a polynomial solution of the Cauchy problem (1) with constant coefficients is studied. As an assumption the initial data and inhomogeneous term have polynomial presentations with respect to space variables. A solution of the initial value problem is found in polynomial form with undetermined coefficients depending on the time variable. For these undetermined coefficients, the recurrence relations which are used in the procedure of the coefficients recovery are found. To be able to implement this method the following studies are completed:

- Stability estimates (energy inequalities) for solutions of (1) in a finite domain of dependence (a finite domain containing characteristic cones) are described.
- Using these stability estimates, it is justified that the polynomial solutions are approximate solutions of the initial value problems with non-polynomial data.
- These theoretical results are confirmed by computational experiments which compare the exact solutions and polynomial solutions found by explained method.

This method can be applied for the isotropic, anisotropic or bi-anisotropic cases. If compared with the other methods, in the computation there are no grids and discretization which are not clear for complicated media such as anisotropic media. By the polynomial solution method, a solution of the initial value problem can be obtained easily if the initial data and inhomogeneous term have polynomial presentations with respect to space variables.

2. Maxwell system

Maxwell's equations are a set of partial differential equations that relate the electric field and magnetic field to the charge and current densities that specify the fields and give rise to electromagnetic radiation. The time dependent Maxwell equations in anisotropic homogeneous media can be written as follows [1], [11]

$$curl_x\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}, \quad (3)$$

$$curl_x\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \quad (4)$$

$$div_x(\mathbf{B})=0, \quad (5)$$

$$div_x(\mathbf{D})=\rho, \quad (6)$$

where $x=(x_1,x_2,x_3)$ be a space variable from $\mathbb{R}^3$ and t be a time variable from $\mathbb{R}$. $\mathbf{E}=(E_1,E_2,E_3)$, $\mathbf{H}=(H_1,H_2,H_3)$ are electric and magnetic fields, with components $E_i=E_i(x,t)$, $H_i=H_i(x,t)$, $i=1,2,3$; depending on x and t variables.

$\mathbf{D}=(D_1,D_2,D_3)$, $\mathbf{B}=(B_1,B_2,B_3)$ are electric and magnetic displacements with components $D_i=D_i(x,t)$, $B_i=B_i(x,t)$, $i=1,2,3$; depending on x and t variables.

$\mathbf{J}=(J_1,J_2,J_3)$ is the density of the electric current where $J_i=J_i(x,t)$, $i=1,2,3$; ρ is the density of electric charges. The conservation law of charges is given by

$$\frac{\partial\rho}{\partial t}+div_x\mathbf{J}=0.$$

In general, there are some relations that expresses $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ in terms of $\mathbf{E}$ and $\mathbf{H}$ when the media is electrically and magnetically anisotropic homogeneous, these are

$$\mathbf{D}=\varepsilon\mathbf{E},\ \mathbf{B}=\mu\mathbf{H},\ \mathbf{J}=\sigma\mathbf{E}+\mathbf{j}. \quad (7)$$

Here ε is the dielectric permittivity characterizing the electrical properties, μ is magnetic permeability characterizing the magnetical properties, σ is the conductivity and $\mathbf{j}$ is the density of the currents arising from the action of the external electromagnetic forces.

In the paper, we suppose that for $t\le 0$

$$\mathbf{H}\,|_{t\le 0}=0, \quad \mathbf{E}\,|_{t\le 0}=0, \quad \mathbf{j}\,|_{t\le 0}=0, \quad \rho\,|_{t\le 0}=0. \quad (8)$$

Using equations (3)-(6),(7) and (8), two initial value problems can be obtained. One of the IVP is for the radiation from the electric current that is

$$\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2}+curl_x(\mu^{-1}curl_x\mathbf{E})+\sigma\frac{\partial \mathbf{E}}{\partial t}=-\frac{\partial \mathbf{j}}{\partial t}, \quad (9)$$

$$\mathbf{E}\Big|_{t=0}=0, \quad \frac{\partial \mathbf{E}}{\partial t}\Big|_{t=0}=0, \quad (10)$$

and other is IVP for magnetic field defined with equations

$$\frac{\partial \mathbf{H}}{\partial t}=-\mu^{-1}curl_x\mathbf{E}, \quad (11)$$

$$\mathbf{H}(x,t)|_{t=0}=0. \quad (12)$$

In this paper, using polynomial solution method (PS-method) the initial value problem (9),(10) is solved to obtain polynomial solution of electric field $\mathbf{E}(x,t)$. Using the solution of IVP (9),(10) IVP for magnetic field $\mathbf{H}(x,t)$ is solved symbolically.

As an assumption the matrices ε, μ that characterizes the electrical and magnetical properties are taken as symmetric positive definite matrices with constant elements and the conductivity σ is taken as a symmetric positive semi definite matrix with constant elements. Also, the components $j_i(x,t)$ of density of the current $\mathbf{j}(x,t)$ are considered in the following polynomial form

$$j_i(x,t)=\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} j_i^{k,m,n}(t)x_1^k x_2^m x_3^n, \quad (13)$$

where p is a fixed number. Using existence theorems it can be shown that the solution of the problem can be written in the form

$$E_i(x,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_i^{k,m,n}(t)x_1^k x_2^m x_3^n.$$

Applying the operator $D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^\alpha \partial x_2^\alpha \partial x_3^\alpha}$ to (9),(10) for $\alpha>p$, following IVP can be obtained

$$\varepsilon\frac{\partial^2 \mathbf{E}^\alpha}{\partial t^2}+curl_x(\mu^{-1}curl_x\mathbf{E}^\alpha)+\sigma\frac{\partial \mathbf{E}^\alpha}{\partial t}=0,$$

$$\mathbf{E}^\alpha\Big|_{t=0}=0, \quad \frac{\partial \mathbf{E}^\alpha}{\partial t}\Big|_{t=0}=0,$$

where $\mathbf{E}^\alpha=D^\alpha\mathbf{E}$.

The solutions of the IVP is $\mathbf{E}^\alpha=0$ inside the conoid of dependence, since $\mathbf{j}_i^\alpha=0$ for $\alpha>p$. Thus the components of the solution of the IVP (9),(10) is in the polynomial form

$$E_i(x,t)=\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} E_i^{k,m,n}(t)x_1^k x_2^m x_3^n$$

3. Initial value problems for electric and magnetic field

In this section, let us consider the IVP (9),(10) with inhomogeneous term that has polynomial presentation with respect to space variables. Note following theorem from [4] that is

Theorem. If ε is positive definite and σ is symmetric, positive semi-definite matrix then there exists a nonsingular matrix $\mathcal{S}=(S_{ij})$; $(i,j=1,2,3)$ such that

$$\mathcal{S}^T\varepsilon\mathcal{S}=I,$$

$$\mathcal{S}^T\sigma\mathcal{S}=\mathcal{D},$$

where $\mathcal{D}=(d_1,d_2,d_3)$, $d_j\geq 0$ $(j=1,2,3)$ is the diagonal matrix with eigenvalues of $\varepsilon^{-\frac{1}{2}}\sigma\varepsilon^{-\frac{1}{2}}$ on diagonal.

Using this theorem and considering $\mathbf{E}=S\tilde{\mathbf{E}}$ in (9),(10) and multiplying with $\mathcal{S}^T$ from left hand side we obtain

$$\frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}+\mathcal{S}^T curl_x(\mathcal{M}^{-1}curl_x(\mathcal{S}\tilde{\mathbf{E}}))+\mathcal{D}\frac{\partial \tilde{\mathbf{E}}}{\partial t}=-\mathcal{S}^T\frac{\partial \mathbf{j}}{\partial t}, \quad (14)$$

$$\tilde{\mathbf{E}}(x,0)=0, \quad \frac{\partial \tilde{\mathbf{E}}(x,t)}{\partial t}\Big|_{t=0}=0, x\in R^3. \quad (15)$$

$$(x\in R^3,\ t>0)$$

Let us consider the solution of (14),(15) in the following form

$$\tilde{\mathbf{E}}(x_1,x_2,x_3,t)=\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\tilde{\mathbf{E}}^{k,m,n}(t)x_1^k x_2^m x_3^n, \quad (16)$$

where $\tilde{\mathbf{E}}^{k,m,n}(t)=\mathcal{S}^T\mathbf{E}^{k,m,n}(t)$, $\mathbf{E}^{k,m,n}(t)=(E_1^{k,m,n}(t),E_2^{k,m,n}(t),E_3^{k,m,n}(t))$. Substituting (13),(16) into (14) and (15) following IVP for ordinary differential equations are obtained

$$\frac{d^2\tilde{E}_i^{k,m,n}}{dt^2}+d_i\frac{d\tilde{E}_i^{k,m,n}}{dt}=\tilde{f}_i^{k,m,n}, \quad i=1,2,3, \quad (17)$$

$$\tilde{E}_i^{k,m,n}\Big|_{t=0}=0, \quad \frac{d\tilde{E}_i^{k,m,n}}{dt}\Big|_{t=0}=0, \quad (18)$$

here

$$\tilde{f}_i^{k,m,n}=\left(S^T\left(-\frac{\partial j}{\partial t}\right)\right)_i-S_{i1}^T B_1-S_{i2}^T B_2-S_{i3}^T B_3, \qquad i=1,2,3$$

where

$B_1=\mu_{31}^{-1}\Big((m+2)(m+1)\Big(S_{31}\tilde{E}_1^{k,m+2,n}+S_{32}\tilde{E}_2^{k,m+2,n}+S_{33}\tilde{E}_3^{k,m+2,n}\Big)\Big)-\mu_{31}^{-1}\Big((m+1)(n+1)\Big(S_{21}\tilde{E}_1^{k,m+1,n+1}+S_{22}\tilde{E}_2^{k,m+1,n+1}+S_{23}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)+\mu_{32}^{-1}\Big((m+1)(n+1)\Big(S_{11}\tilde{E}_1^{k,m+1,n+1}+S_{12}\tilde{E}_2^{k,m+1,n+1}+S_{13}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)-$

$$\mu_{32}^{-1}\Big((k+1)(m+1)\Big(S_{31}\tilde{E}_1^{k+1,m+1,n}+S_{32}\tilde{E}_2^{k+1,m+1,n}+S_{33}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)+\mu_{33}^{-1}\Big((k+1)(m+1)\Big(S_{21}\tilde{E}_1^{k+1,m+1,n}+S_{22}\tilde{E}_2^{k+1,m+1,n}+S_{23}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)-\mu_{33}^{-1}\Big((m+2)(m+1)\Big(S_{11}\tilde{E}_1^{k,m+2,n}+S_{12}\tilde{E}_2^{k,m+2,n}+S_{13}\tilde{E}_3^{k,m+2,n}\Big)\Big)+\mu_{21}^{-1}\Big((n+2)(n+1)\Big(S_{21}\tilde{E}_1^{k,m,n+2}+S_{22}\tilde{E}_2^{k,m,n+2}+S_{23}\tilde{E}_3^{k,m,n+2}\Big)\Big)-\mu_{21}^{-1}\Big((m+1)(n+1)\Big(S_{31}\tilde{E}_1^{k,m+1,n+1}+S_{32}\tilde{E}_2^{k,m+1,n+1}+S_{33}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)+\mu_{22}^{-1}\Big((k+1)(n+1)\Big(S_{31}\tilde{E}_1^{k+1,m,n+1}+S_{32}\tilde{E}_2^{k+1,m,n+1}+S_{33}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)-\mu_{22}^{-1}\Big((n+2)(n+1)\Big(S_{11}\tilde{E}_1^{k,m,n+2}+S_{12}\tilde{E}_2^{k,m,n+2}+S_{13}\tilde{E}_3^{k,m,n+2}\Big)\Big)+\mu_{23}^{-1}\Big((m+1)(n+1)\Big(S_{11}\tilde{E}_1^{k,m+1,n+1}+S_{12}\tilde{E}_2^{k,m+1,n+1}+S_{13}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)-\mu_{23}^{-1}\Big((k+1)(n+1)\Big(S_{21}\tilde{E}_1^{k+1,m,n+1}+S_{22}\tilde{E}_2^{k+1,m,n+1}+S_{23}\tilde{E}_3^{k+1,m,n+1}\Big)\Big),$$

$$B_2=\mu_{11}^{-1}\Big((m+1)(n+1)\Big(S_{31}\tilde{E}_1^{k,m+1,n+1}+S_{32}\tilde{E}_2^{k,m+1,n+1}+S_{33}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)-\mu_{11}^{-1}\Big((n+2)(n+1)\Big(S_{21}\tilde{E}_1^{k,m,n+2}+S_{22}\tilde{E}_2^{k,m,n+2}+S_{23}\tilde{E}_3^{k,m,n+2}\Big)\Big)+\mu_{12}^{-1}\Big((n+2)(n+1)\Big(S_{11}\tilde{E}_1^{k,m,n+2}+S_{12}\tilde{E}_2^{k,m,n+2}+S_{13}\tilde{E}_3^{k,m,n+2}\Big)\Big)-\mu_{12}^{-1}\Big((k+1)(n+1)\Big(S_{31}\tilde{E}_1^{k+1,m,n+1}+S_{32}\tilde{E}_2^{k+1,m,n+1}+S_{33}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)+\mu_{13}^{-1}\Big((k+1)(n+1)\Big(S_{21}\tilde{E}_1^{k+1,m,n+1}+S_{22}\tilde{E}_2^{k+1,m,n+1}+S_{23}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)-\mu_{13}^{-1}\Big((m+1)(n+1)\Big(S_{11}\tilde{E}_1^{k,m+1,n+1}+S_{12}\tilde{E}_2^{k,m+1,n+1}+S_{13}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)+\mu_{31}^{-1}\Big((k+1)(n+1)\Big(S_{21}\tilde{E}_1^{k+1,m,n+1}+S_{22}\tilde{E}_2^{k+1,m,n+1}+S_{23}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)-\mu_{31}^{-1}\Big((k+1)(m+1)\Big(S_{31}\tilde{E}_1^{k+1,m+1,n}+S_{32}\tilde{E}_2^{k+1,m+1,n}+S_{33}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)+\mu_{32}^{-1}\Big((k+2)(k+1)\Big(S_{31}\tilde{E}_1^{k+2,m,n}+S_{32}\tilde{E}_2^{k+2,m,n}+S_{33}\tilde{E}_3^{k+2,m,n}\Big)\Big)-\mu_{32}^{-1}\Big((k+1)(n+1)\Big(S_{11}\tilde{E}_1^{k+1,m,n+1}+S_{12}\tilde{E}_2^{k+1,m,n+1}+S_{13}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)+\mu_{33}^{-1}\Big((k+1)(m+1)(S_{11}\tilde{E}_1^{k+1,m+1,n}+S_{12}\tilde{E}_2^{k+1,m+1,n}+S_{13}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)-\mu_{33}^{-1}\Big((k+2)(k+1)\Big(S_{21}\tilde{E}_1^{k+2,m,n}+S_{22}\tilde{E}_2^{k+2,m,n}+S_{23}\tilde{E}_3^{k+2,m,n}\Big)\Big),$$

$$B_3=\mu_{21}^{-1}\Big((k+1)(m+1)\Big(S_{31}\tilde{E}_1^{k+1,m+1,n}+S_{32}\tilde{E}_2^{k+1,m+1,n}+S_{33}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)-\mu_{21}^{-1}\Big((k+1)(n+1)\Big(S_{21}\tilde{E}_1^{k+1,m,n+1}+S_{22}\tilde{E}_2^{k+1,m,n+1}+S_{23}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)+\mu_{22}^{-1}\Big((k+1)(n+1)\Big(S_{11}\tilde{E}_1^{k+1,m,n+1}+S_{12}\tilde{E}_2^{k+1,m,n+1}+S_{13}\tilde{E}_3^{k+1,m,n+1}\Big)\Big)-\mu_{22}^{-1}\Big((k+2)(k+1)\Big(S_{31}\tilde{E}_1^{k+2,m,n}+S_{32}\tilde{E}_2^{k+2,m,n}+S_{33}\tilde{E}_3^{k+2,m,n}\Big)\Big)+\mu_{23}^{-1}\Big((k+2)(k+1)\Big(S_{21}\tilde{E}_1^{k+2,m,n}+S_{22}\tilde{E}_2^{k+2,m,n}+S_{23}\tilde{E}_3^{k+2,m,n}\Big)\Big)-\mu_{23}^{-1}\Big((k+1)(m+1)(S_{11}\tilde{E}_1^{k+1,m+1,n}+S_{12}\tilde{E}_2^{k+1,m+1,n}+S_{13}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)+\mu_{11}^{-1}\Big((m+1)(n+1)\Big(S_{21}\tilde{E}_1^{k,m+1,n+1}+S_{22}\tilde{E}_2^{k,m+1,n+1}+S_{23}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)-\mu_{11}^{-1}\Big((m+2)(m+1)\Big(S_{31}\tilde{E}_1^{k,m+2,n}+S_{32}\tilde{E}_2^{k,m+2,n}+S_{33}\tilde{E}_3^{k,m+2,n}\Big)\Big)+\mu_{12}^{-1}\Big((k+1)(m+1)\Big(S_{31}\tilde{E}_1^{k+1,m+1,n}+S_{32}\tilde{E}_2^{k+1,m+1,n}+S_{33}\tilde{E}_3^{k+1,m+1,n}\Big)\Big)-\mu_{12}^{-1}\Big((m+1)(n+1)\Big(S_{11}\tilde{E}_1^{k,m+1,n+1}+S_{12}\tilde{E}_2^{k,m+1,n+1}+S_{13}\tilde{E}_3^{k,m+1,n+1}\Big)\Big)+\mu_{13}^{-1}\Big((m+2)(m+1)\Big(S_{11}\tilde{E}_1^{k,m+2,n}+S_{12}\tilde{E}_2^{k,m+2,n}+S_{13}\tilde{E}_3^{k,m+2,n}\Big)\Big)-\mu_{13}^{-1}\Big((k+1)(m+1)\Big(S_{21}\tilde{E}_1^{k+1,m+1,n}+S_{22}\tilde{E}_2^{k+1,m+1,n}+S_{23}\tilde{E}_3^{k+1,m+1,n}\Big)\Big).$$

The solution of the IVP (17),(18) is

$$\tilde{E}_i^{k,m,n}=\frac{1}{d_i}\int_0^t(1-e^{d_i(\tau-t)})\tilde{f}_i^{k,m,n}(\tau)d\tau, \quad i=1,2,3. \tag{19}$$

Using (16) and (19), the solution $\tilde{\mathbf{E}}(x,t)$ of (14),(15) can be obtained. Since $\mathbf{E}=\mathcal{S}\tilde{\mathbf{E}}$ then a solution of the IVP (9),(10) can be obtained in polynomial form. Using the solution of IVP (9),(10), the IVP (11),(12) can be solved by symbolic calculation.

4. Existence of solutions for the IVPs

Equations (3)-(6),(8) can be rewritten as a first order hyperbolic system in the form

$$A_0\frac{\partial \mathbf{U}}{\partial t}+\sum_{j=1}^{3}A_j\frac{\partial \mathbf{U}}{\partial x_j}+B\mathbf{U}=\mathbf{F}, \tag{20}$$

$$\mathbf{U}(x,t)\Big|_{t=0}=0, \tag{21}$$

where $\mathbf{U}=\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$, $A_0=\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix}$, $B=\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}$ and $\mathbf{F}=-\begin{pmatrix}\mathbf{j}\\ \mathbf{0}\end{pmatrix}$, and the matrix A_j is defined as

$$A_j=\begin{pmatrix}0_{3\times3} & A_j^1\\ (A_j^1)^* & 0_{3\times3}\end{pmatrix}.$$

which has the components

$$A_1^1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\qquad A_2^1=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},$$

$$A_3^1=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

Consider the symmetric positive definite matrix A_0. There exists a symmetric positive definite matrix S such that $A_0^{-1} = S^2$, that is, $A_0^{-\frac{1}{2}} = S$. Let us denote the vector

$$\mathbf{U}(x) = S\mathbf{u}(x) \tag{22}$$

Substituting (22) into (20),(21) and multiplying the resulting formula with matrix S from the left hand side we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{n} \widetilde{A}_j \frac{\partial \mathbf{u}}{\partial x_j} + \widetilde{B}\mathbf{u} = \mathbf{f},\ x \in \mathbb{R}^n,\ t > 0, \tag{23}$$

$$\mathbf{u}(x,t)\Big|_{t=0} = 0,\ x \in \mathbb{R}^n, \tag{24}$$

where

$$I_6 = SA_0S, \widetilde{A}_j = SA_jS, \widetilde{B} = SBS, \mathbf{f} = S\mathbf{F}.$$

Since S and A_j, $j = 1,2,3$ are symmetric, the matrix $\widetilde{A}_j$ will also be symmetric, which implies that (23) is a symmetric hyperbolic system.

Theorem. Let $\widetilde{A}_j$ is a symmetric matrix with constant entries and let T be a fixed positive number then for an arbitrary $f_j(x,t) \in C([0,T];\mathcal{H}^1(\mathbb{R}^3))$ we have a unique solution of (23),(24) such that

$$u_j(x,t) \in C([0,T];\mathcal{H}^1(\mathbb{R}^3)) \cap C^1([0,T];\mathcal{L}^2(\mathbb{R}^3))$$

The theorem and the proof of the theorem can be found in [8].

5. Domain of dependence and uniqueness theorem inside conoid of dependence

In this section we describe the domain of dependence for first order symmetric hyperbolic systems and prove that the solution of the system is uniquely determined inside the conoid of dependence [2], [5], [19]. And we obtain the theorem for the uniqueness of the solution of considered IVPs.

Let us consider the symmetric hyperbolic system of the form (23),(24) where each $\widetilde{A}_j(x)$ is an $m \times m$ symmetric matrix. Let

$$\mathbf{A}(\xi) = \sum_{j=1}^{n} \widetilde{A}_j \xi_j$$

and $\lambda_j(\xi)$, $j = 1,\dots,m$ be the eigenvalues of $\mathbf{A}(\xi)$. We define the constant M as

$$M = \max_{\substack{i=1,\dots,m \\ |\xi|=1}} |\lambda_j(\xi)|. \tag{25}$$

Using the constant M we define the following domains for the arbitrary point P with coordinates $(x^0,t^0) \in \mathbb{R}^n \times (0,\infty)$

$$\Gamma(P) = \left\{(x,t) : 0 \le t \le t^0, |x - x^0| \le M(t^0 - t)\right\},$$
$$S(h) = \left\{x \in \mathbb{R}^n : |x - x^0| \le M|t^0 - h|\right\},\ 0 \le h \le t^0,$$
$$R(h) = \left\{(x,t) : 0 \le t \le h, |x - x^0| = M|t^0 - t|\right\}.$$

Here $\Gamma(P)$ is the conoid of dependence with vertex P. $S(h)$ is the surface constructed by the intersection of the plane $t = h$ and the conoid $\Gamma(P)$. For $t = 0$, $S(0)$ is the base of the conoid. $R(h)$ is the lateral surface of the conoid bounded by $S(0)$ and $S(h)$.

Theorem.(see, [2]) Let $(x^0,t^0) \in \mathbb{R}^n \times (0,\infty)$, and $S(h)$, $R(h)$, $\Gamma(P)$ be as defined above, and $\mathbf{u}(x,t) \in C([0,T];\mathcal{H}^1(\mathbb{R}^n;\mathbb{R}^m)) \bigcap C^1([0,T];\mathcal{L}^2(\mathbb{R}^n;\mathbb{R}^m))$ be a solution of (23). Then the following energy inequality is valid

$$\int_{S(h)} |\mathbf{u}(x,h)|^2 dx \le e^{Kh} \int_0^h \int_{S(t)} |\mathbf{f}(x,t)|^2 dxdt,$$

if

$$\max_j \max_{k,l} \max_{x\in S(0)} \left|\frac{\partial a_{kl}^j}{\partial x_j}\right| \le L, \quad \max_{k,l} \max_{x\in S(0)} \left|b_{kl}(x)\right| \le L.$$

Proof. Let Ω be the region bounded by $S(0)$, $S(h)$, $R(h)$ and $\partial\Omega = S(0) \cup S(h) \cup R(h)$. Multiplying (23) with $\mathbf{u}$ and integrating over Ω we have

$$\int_\Omega \left\{\mathbf{u}.\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\left(\sum_{j=1}^{n} \widetilde{A}_j \frac{\partial \mathbf{u}}{\partial x_j}\right)\right\} dxdt + \int_\Omega \mathbf{u}.\widetilde{B}(x)\mathbf{u}dxdt \tag{26}$$
$$= \int_\Omega \mathbf{u}.\mathbf{f}dxdt.$$

Noting the relations

$$\mathbf{u}.\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{2}\frac{\partial |\mathbf{u}|^2}{\partial t},$$
$$\mathbf{u}.\left(\sum_{j=1}^{n} \widetilde{A}_j \frac{\partial \mathbf{u}}{\partial x_j}\right) = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\mathbf{u}.\widetilde{A}_j\mathbf{u}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(\mathbf{u}.\frac{\partial \widetilde{A}_j}{\partial x_j}\mathbf{u}\right)$$

Equation (26) can be rewritten as follows

$$\frac{1}{2}\int_\Omega \left\{\frac{\partial |\mathbf{u}|^2}{\partial t} + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\mathbf{u}.\widetilde{A}_j\mathbf{u}\right) - \sum_{j=1}^{n}\left(\mathbf{u}.\frac{\partial \widetilde{A}_j}{\partial x_j}\mathbf{u}\right)\right\} dxdt$$
$$+ \int_\Omega \mathbf{u}.\widetilde{B}(x)\mathbf{u}dxdt = \int_\Omega \mathbf{u}.\mathbf{f}dxdt$$

and applying the divergence theorem one can get

$$\frac{1}{2}\int_{\partial\Omega}\Big\{|\mathbf{u}|^2\nu_t+\sum_{j=1}^{n}\Big(\mathbf{u}.\widetilde{A}_j\mathbf{u}\Big)\,\nu_j\Big\}dS - \frac{1}{2}\int_{\Omega}\sum_{j=1}^{n}\Big(\mathbf{u}.\frac{\partial\widetilde{A}_j}{\partial x_j}\mathbf{u}\Big)dxdt+\int_{\Omega}\mathbf{u}.\widetilde{B}(x)\mathbf{u}dxdt = \int_{\Omega}\mathbf{u}.\mathbf{f}dxdt, \tag{27}$$

where $\nu=(\nu_1,\ldots,\nu_n,\nu_t)$ is the outward unit normal on $\partial\Omega$. Since $\partial\Omega=S(0)\cup S(h)\cup R(h)$ and

$$\begin{aligned}\nu &= (0,\ldots,1) \quad \text{on} \quad S(h),\\ \nu &= (0,\ldots,-1) \quad \text{on} \quad S(0),\\ \nu &= \frac{(x_1-x_1^0,\ldots,x_n-x_n^0,M^2(t^0-t))}{(t^0-t)M\sqrt{1+M^2}} \quad \text{on} \quad R(h),\end{aligned}$$

formula (27) takes the form

$$\frac{1}{2}\int_{S(h)}|\mathbf{u}|^2dx-\frac{1}{2}\int_{S(0)}|\mathbf{u}|^2dx+\frac{1}{2}\int_{R(h)}|\mathbf{u}|^2\frac{M}{\sqrt{1+M^2}}dS + \frac{1}{2}\int_{R(h)}\sum_{j=1}^{n}\Big(\mathbf{u}.\widetilde{A}_j\mathbf{u}\Big)\frac{x_j-x_j^0}{(t^0-t)M\sqrt{1+M^2}}dS - \frac{1}{2}\int_{\Omega}\Big\{\sum_{j=1}^{n}\mathbf{u}.\frac{\partial\widetilde{A}_j}{\partial x_j}\mathbf{u}-2\mathbf{u}.\widetilde{B}\mathbf{u}\Big\}dxdt = \int_{\Omega}\mathbf{u}.\mathbf{f}dxdt. \tag{28}$$

Let us denote

$$\xi_j=\frac{(x_j-x_j^0)}{(t^0-t)M}, \quad j=1,\ldots,n,$$

which satisfies $|\xi|=\sqrt{\xi_1^2+\ldots+\xi_n^2}=1$. Using this notation and

$$\mathbf{A}(\xi)=\sum_{j=1}^{n}\widetilde{A}_j\xi_j,$$

we write the following equality

$$\frac{1}{2}\int_{R(h)}\sum_{j=1}^{n}\Big(\mathbf{u}.\widetilde{A}_j\mathbf{u}\Big)\frac{x_j-x_j^0}{(t^0-t)M\sqrt{1+M^2}}dS = \frac{1}{2\sqrt{1+M^2}}\int_{R(h)}\sum_{j=1}^{n}\Big(\mathbf{u}.\widetilde{A}_j\xi_j\mathbf{u}\Big)\,dS = \frac{1}{2\sqrt{1+M^2}}\int_{R(h)}(\mathbf{u}.\mathbf{A}(\xi)\mathbf{u})\,dS. \tag{29}$$

Substituting (29) into (28) we get

$$\frac{1}{2}\int_{S(h)}|\mathbf{u}|^2dx-\frac{1}{2}\int_{S(0)}|\mathbf{u}|^2dx + \frac{1}{2\sqrt{1+M^2}}\int_{R(h)}\Big[|\mathbf{u}|^2M+\mathbf{u}.\mathbf{A}(\xi)\mathbf{u}\Big]\,dS - \frac{1}{2}\int_{\Omega}\Big\{\sum_{j=1}^{n}\mathbf{u}.\frac{\partial\widetilde{A}_j}{\partial x_j}\mathbf{u}-2\mathbf{u}.\widetilde{B}\mathbf{u}\Big\}dxdt = \int_{\Omega}\mathbf{u}.\mathbf{f}dxdt. \tag{30}$$

Consider the term $M\mathbf{I}+\mathbf{A}(\xi)$, where $\mathbf{I}$ is the identity matrix of order $m\times m$. Since $\mathbf{A}(\xi)$ is diagonalizable we can find a matrix $\mathbf{Z}$ which reduces $\mathbf{A}(\xi)$ to a diagonal matrix of its eigenvalues, denoted $\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)$. Multiplying $M\mathbf{I}+\mathbf{A}(\xi)$ with matrix $\mathbf{Z}$ from right, and with its inverse $\mathbf{Z}^{-1}$, from left we have

$$\begin{aligned}\mathbf{Z}^{-1}\left(M\mathbf{I}+\mathbf{A}(\xi)\right)\mathbf{Z} &= \mathbf{Z}^{-1}M\mathbf{I}\mathbf{Z}+\mathbf{Z}^{-1}\mathbf{A}(\xi)\mathbf{Z}\\ &= M\mathbf{I}+\mathbf{Z}^{-1}\mathbf{A}(\xi)\mathbf{Z}\\ &= \mathrm{diag}(M,M,\ldots,M)\\ &\quad +\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m).\end{aligned}$$

Noting the formula (25), we conclude that the matrix $M\mathbf{I}+\mathbf{A}(\xi)$ has positive eigenvalues, which implies that it is a positive-definite matrix, and from the definition of positive-definiteness we obtain the following inequality

$$|\mathbf{u}|^2M+\mathbf{u}.\mathbf{A}(\xi)\mathbf{u}=\mathbf{u}.(M\mathbf{I}+\mathbf{A}(\xi))\mathbf{u}\geq 0.$$

Thus (30) becomes

$$\frac{1}{2}\int_{S(h)}|\mathbf{u}|^2dx-\frac{1}{2}\int_{S(0)}|\mathbf{u}|^2dx - \frac{1}{2}\int_{\Omega}\Big\{\sum_{j=1}^{n}\mathbf{u}.\frac{\partial\widetilde{A}_j}{\partial x_j}\mathbf{u}-2\mathbf{u}.\widetilde{B}\mathbf{u}\Big\}dxdt - \int_{\Omega}\mathbf{u}.\mathbf{f}dxdt\leq 0. \tag{31}$$

Remark. Let us denote

$$\frac{\partial\widetilde{A}_j}{\partial x_j}=\Big(\frac{\partial a_{kl}^j}{\partial x_j}\Big),$$

and

$$\max_j\max_{k,l}\max_{x\in S(0)}\Big|\frac{\partial a_{kl}^j}{\partial x_j}\Big|\leq L,$$

$$\max_{k,l}\max_{x\in S(0)}\Big|b_{kl}(x)\Big|\leq L.$$

Since

$$\sum_{j=1}^{n}\mathbf{u}.\frac{\partial\widetilde{A}_j}{\partial x_j}\mathbf{u}-2\mathbf{u}.\widetilde{B}\mathbf{u}=\Big\langle\mathbf{u},\Big(\sum_{j=1}^{n}\frac{\partial\widetilde{A}_j}{\partial x_j}-2\widetilde{B}\Big)\mathbf{u}\Big\rangle,$$

then

$$\Big|\Big\langle \mathbf{u}, \Big(\sum_{j=1}^{n} \frac{\partial \widetilde{A}_j}{\partial x_j} - 2\widetilde{B}\Big)\mathbf{u}\Big\rangle\Big| = \Big|\Big\langle \mathbf{u}, \sum_{j=1}^{n} \frac{\partial \widetilde{A}_j}{\partial x_j}\mathbf{u}\Big\rangle - 2\Big\langle \mathbf{u}, \widetilde{B}\mathbf{u}\Big\rangle\Big|,$$

$$\le L\Big[\sum_{j=1}^{n}\Big\langle \mathbf{u},\mathbf{u}\Big\rangle + 2\Big\langle \mathbf{u},\mathbf{u}\Big\rangle\Big],$$

$$\le L(n+2)\Big\langle \mathbf{u},\mathbf{u}\Big\rangle,$$

$$\le L(n+2)\Big|\mathbf{u}(x,t)\Big|^2.$$

Using previous remark, the inequality (31) becomes

$$\frac{1}{2}\int_{S(h)} |\mathbf{u}|^2 dx - \frac{1}{2}\int_{S(0)} |\mathbf{u}|^2 dx - \int_{\Omega} \mathbf{u}.\mathbf{f} dxdt$$
$$\le L(n+2)\int_0^h \int_{S(h)} \Big|\mathbf{u}(x,t)\Big|^2 dxdt.$$

Denoting

$$\frac{1}{2}\int_{S(\tau)} |\mathbf{u}(x,\tau)|^2 dx = w(\tau),$$

we have

$$w(h) \le w(0) + 2(n+2)L\int_0^h w(t)dt + \int_0^h w(t)dt$$
$$+ \frac{1}{2}\int_0^h \int_{S(t)} |\mathbf{f}(x,t)|^2 dxdt.$$

Using Gronwall's Lemma we obtain

$$w(h) \le \Big(w(0) + \frac{1}{2}\int_0^h \int_{S(t)} |\mathbf{f}(x,t)|^2 dxdt\Big)e^{Kh}.$$
$$(K = 2(n+2)L + 1)$$

Then we get the energy inequality

$$\int_{S(h)} |\mathbf{u}(x,h)|^2 dx \le e^{Kh}\int_0^h \int_{S(t)} |\mathbf{f}(x,t)|^2 dxdt.$$

Thus, the stability estimate for solution of (20),(21) is

$$\int_{S(h)} |A_0^{\frac{1}{2}}\mathbf{U}(x,h)|^2 dx \le e^{Kh}\int_0^h \Big(\int_{S(t)} |A_0^{-\frac{1}{2}}\mathbf{F}(x,t)|^2 dx\Big)dt.$$

Theorem. (Uniqueness Theorem for the IVP)
Let $x \in \mathbb{R}^n$, $t > 0$ and $\widetilde{A}_j(x)$ is an $m \times m$ symmetric matrix then following Initial Value Problem has unique solution

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{n} \widetilde{A}_j \frac{\partial \mathbf{u}}{\partial x_j} + \widetilde{B}(x)\mathbf{u} = \mathbf{f},$$

$$\mathbf{u}(x,0) = \varphi(x),$$

Proof. Assume that we have two solutions u and u^* corresponding to the same data φ and the same inhomogeneous term f. Let $\hat{\mathbf{u}} = \mathbf{u} - \mathbf{u}^*$. Then $\hat{\mathbf{u}}$ satisfies

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \sum_{j=1}^{n} \widetilde{A}_j \frac{\partial \hat{\mathbf{u}}}{\partial x_j} + \widetilde{B}(x)\hat{\mathbf{u}} = 0,$$

$$\hat{\mathbf{u}}(x,0) = 0,$$

Applying energy inequality we have

$$\int_{S(h)} |\hat{\mathbf{u}}(x,h)|^2 dx \le \int_{S(0)} |\hat{\mathbf{u}}(x,0)|^2 dx = 0,$$

$$0 \le \int_{S(h)} |\hat{\mathbf{u}}(x,h)|^2 dx \le 0, \text{ for all } h \in [0,T].$$

Hence

$$|\hat{\mathbf{u}}(x,h)|^2 \equiv 0, \text{ for all } h \in [0,T], \ x \in S(h),$$

then we have

$$\mathbf{u}(x,h) - \mathbf{u}^*(x,h) \equiv 0, \text{ for all } h \in [0,T], \ x \in S(h),$$

that is,

$$\mathbf{u}(x,h) \equiv \mathbf{u}^*(x,h), \text{ for all } h \in [0,T], \ x \in S(h).$$

This proves the uniqueness theorem.

6. Implementation of the method

In this section, our aim is to implement the method to a simplified form of the problem and sketch out how the method works.

For implementation of PS method, let us consider the IVP (9),(10) with $-\frac{\partial j}{\partial t} = (f,0,0)$, $\varepsilon = I_{3\times3}$, $\mu^{-1} = \text{diag}(\mu_{11},\mu_{22},\mu_{33})$, $\sigma = \text{diag}(\sigma_{11},\sigma_{22},\sigma_{33})$. Let $f(x,t)$ has an approximation in the domain of dependence in the form

$$f(x,t) = \sum_{n=0}^{p} f^n(t)x^n, \tag{32}$$

where $x \in \mathbb{R}, t \in \mathbb{R}$ and $p \in \mathbb{N}$. The solution $\mathbf{E}(x,t) = (E_1(x,t),E_2(x,t),E_3(x,t))$ of the problem will be in the form

$$\mathbf{E}(x,t) = \sum_{n=0}^{p} \mathbf{E}(x,t)^n(t)x^n \tag{33}$$

Substituting (32) and (33) into the IVP (9),(10) we get

$$\frac{\partial^2 E_1^n}{\partial t^2} - \mu_{22}(n+2)(n+1)E_1^{n+2} + \sigma_{11}\frac{\partial E_1^n}{\partial t} = f_1^n(t)$$
$$E_1^n\Big|_{t=0} = 0, \quad \frac{\partial E_1^n}{\partial t}\Big|_{t=0} = 0, \tag{34}$$

$$\frac{\partial^2 E_2^n}{\partial t^2} + \sigma_{22}\frac{\partial E_2^n}{\partial t} = 0,\quad E_2^n\big|_{t=0} = 0,\quad \frac{\partial E_2^n}{\partial t}\Big|_{t=0} = 0, \tag{35}$$

$$\frac{\partial^2 E_3^n}{\partial t^2} + \sigma_{33}\frac{\partial E_3^n}{\partial t} = 0,\quad E_3^n\big|_{t=0} = 0,\quad \frac{\partial E_3^n}{\partial t}\Big|_{t=0} = 0. \tag{36}$$

The solutions of IVPs (35),(36) are $E_2^n(t) = 0, E_3^n(t) = 0$. Also the IVP (34) for any $n > p$ we have $E_1^n(t) = 0$. Let us start PS method computation when $n = p$. For $n = p$ we have

$$\frac{\partial^2 E_1^p}{\partial t^2} - \mu_{22}(p+2)(p+1)E_1^{p+2} + \sigma_{11}\frac{\partial E_1^p}{\partial t} = f_1^p,\quad E_1^P\big|_{t=0} = 0,\quad \frac{\partial E_1^p}{\partial t}\Big|_{t=0} = 0. \tag{37}$$

The solution of IVP (37) is of the form

$$E_1^p(t) = \frac{1}{\sigma_{11}}\int_0^t f_1^p(\tau)\left(1 - e^{\sigma_{11}(\tau - t)}\right) + \left(\mu_{22}(p+2)(p+1)E_1^{p+2}(\tau)\right)\left(1 - e^{\sigma_{11}(\tau-t)}\right)d\tau.$$

Since $E_1^{p+2} = 0$, the solution of problem (37) is

$$E_1^p(t) = \frac{1}{\sigma_{11}}\int_0^t f_1^p(\tau)\left[1 - e^{\sigma_{11}(\tau-t)}\right]d\tau.$$

Now, the computation will continue with $n = p - 1$ to calculate $E_1^{p-1}(t)$. When $n = p - 1$ we have

$$\frac{\partial^2 E_1^{p-1}}{\partial t^2} - \mu_{22}(p+1)pE_1^{p+1} + \sigma_{11}\frac{\partial E_1^{p-1}}{\partial t} = f_1^{p-1},\quad E_1^{p-1}\big|_{t=0}^{p-1},\ \frac{\partial E_1^{p-1}}{\partial t}\Big|_{t=0} = 0. \tag{38}$$

With the similar reasoning, the solution of (38) is

$$E_1^{p-1}(t) = \frac{1}{\sigma_{11}}\int_0^t f_1^{p-1}(\tau)\left[1 - e^{\sigma_{11}(\tau-t)}\right]d\tau.$$

Continuing calculation when $n = p - 2$ we have

$$\frac{\partial^2 E_1^{p-2}}{\partial t^2} - \mu_{22}p(p-1)E_1^p + \frac{\partial E_1^{p-2}}{\partial t} = f_1^{p-2}(t),\quad E_1^{p-2}\big|_{t=0} = 0,\quad \frac{\partial E_1^{p-2}}{\partial t}\Big|_{t=0} = 0. \tag{39}$$

Since $E_1^p(t)$ is calculated in previous steps, we get the solution of IVP (39) as

$$E_1^{p-2}(t) = \frac{1}{\sigma_{11}}\int_0^t f_1^{p-2}(\tau)\left(1 - e^{\sigma_{11}(\tau-t)}\right) + \left(\mu_{22}p(p-1)E_1^p(\tau)\right)\left(1 - e^{\sigma_{11}(\tau-t)}\right)d\tau.$$

Thus, the values of $E_1^i(t)$ $(0 \le i \le p)$ with decreasing i values can be calculated with the following formula

$$E_1^i(t) = \frac{1}{\sigma_{11}}\int_0^t f_1^i(\tau)\left(1 - e^{\sigma_{11}(\tau-t)}\right) + \left(\mu_{22}(i+2)(i+1)E^{i+2}(\tau)\right)\left(1 - e^{\sigma_{11}(\tau-t)}\right)d\tau.$$

The values of $E^{i+2}(\tau)$ are calculated with the previous steps. In this way all coefficients $E_1^i(t)$ of the solution $E_1(x,t)$ that is given with formula (33) can be calculated.

7. Computational examples

In this section, there are three examples. In the first example, all the components of non homogeneous term are in polynomial form. PS method is used to solve the problem and the solutions obtained by using Maple codes are checked by direct substitution to the problem. In example 2 and example 3 the components of non homogeneous term are chosen as smooth functions. Chebyshev polynomials are used for approximations of these smooth functions. Each example is chosen since we know the exact solutions of the problems with chosen matrices ε, μ and σ and inhomogeneous term. Aim is to compare the exact solution with the solution obtained by using Chebyshev polynomials and PS method.

Example 1: Let us consider a simple example for the IVP given with the equations (3)-(8). Let the matrices ε, μ and σ be identity matrices and let the inhomogeneous term $\mathbf{f}(x,t) = -\dfrac{\partial \mathbf{j}}{\partial t}$ be a vector function that has the components

$$f_1(x,t) = (x_1 + 4x_2^4 + 3x_3^2),$$
$$f_2(x,t) = x_3^2 t,$$
$$f_3(x,t) = (2x_1 + x_2)(t+2).$$

As we mention in Section 1, this problem can be written in the form of two IVPs (9),(10) and (11),(12). Our aim is to find electric and magnetic currents. Using PS method exact solution of IVP (9),(10) can be obtained.

The components of electric field $\mathbf{E}(x,t)$ are

$$E_1(x,t) = 942e^{-t} - 942 + 378e^{-t}t + 564t - 141t^2 + 48e^{-t}t^2 + 16t^3 + (4e^{-t} - 4 + 4t)x_2^4 + (3e^{-t} - 3 + 3t)x_3^2,$$
$$E_2(x,t) = 7.99e^{-t} - 7.99e^{-t}e^t + .33e^{-t}t^3e^t + 5.99e^{-t}te^t - 2e^{-t}t^2e^t + 2e^{-t}t + (.50t^2 - e^{-t} + 1 - t)x_3^2,$$
$$E_3(x,t) = (t^2 + 2t + 2e^{-t} - 2)x_1 + (.50t^2 + t + e^{-t} - 1)x_2.$$

Using the solution of the IVP (9),(10), the solution of the IVP (11),(12) can be obtained. The components of magnetic filed $\mathbf{H}(x,t)$ are

$$H_1(x,t) = -.16t^3 - .50t^2 + e^{-t} + t + .33t^3x_3 + 2x_3e^{-t} + 2x_3t - x_3t^2 - 2x_3 - 1,$$
$$H_2(x,t) = -2t + 6x_3e^{-t} + 6x_3t - 3x_3t^2 + .33t^3 + t^2 - 2e^{-t} - 6x_3 + 2,$$
$$H_3(x,t) = 8x_2^3t^2 - 16x_2^3e^{-t} - 16x_2^3t - 96x_2t^2 + 288x_2t + 384x_2e^{-t} + 96x_2e^{-t}t + 16x_2t^3 + 16x_2^3 - 384x_2.$$

By direct substitution of $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ that are obtained using PS method into the problem given in equations (3)-(8) the robustness of the method is checked.

Example 2: Let us consider the IVP (3)-(8) with the matrices ε, μ and σ that are identity matrices and let the inhomogeneous term $\mathbf{f}(x,t) = -\dfrac{\partial \mathbf{j}}{\partial t}$ be a vector function that has the components

$$f_1(x,t) = (x_1+1)^10 + (x_2+3)^3,$$
$$f_2(x,t) = x_2^3(t-1),$$
$$f_3(x,t) = (5x_2 + x_3^2)t.$$

As we mention in Section 1 and in the example 1, this problem can be written in the form of two IVPs (9),(10) and (11),(12). Similarly, our aim is to find electric and magnetic currents. Using PS method exact solution of IVP (9),(10) can be obtained.
The components of electric field $\mathbf{E}(x,t)$ are

$$E_1(x,t) = (x_1+1)^{10}\left(e^{-t}+t-1\right) + (x_2+3)^3\left(e^{-t}+t-1\right) - 6(t+3)(x_2+3)\left(e^{-t}+t-1\right) + 9(x_2+3)t^2$$
$$E_2(x,t) = \left(\frac{t^2}{2} - 2t - 2e^{-t} + 2\right)x_2^3$$
$$E_3(x,t) = (5x_2 + x_3^2)\left(\frac{t^2}{2} - e^{-t} + 1 - t\right).$$

Using the solution of the IVP (9),(10), the solution of the IVP (11),(12) can be obtained. The components of magnetic filed $\mathbf{H}(x,t)$ are

$$H_1(x,t) = -\frac{5t^3}{6} + \frac{5t^2}{2} - 5(e^{-t}+t-1),$$
$$H_2(x,t) = 0,$$
$$H_3(x,t) = -\left(3x_2^2 + 6t + 18x_2 + 3\right)\left(e^{-t}+t-1\right) + \frac{3x_2^2t^2}{2} + t^3 + 9x_2t^2 + \frac{3t^2}{2}.$$

By direct substitution of $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ that are obtained using PS method into the problem given in equations (3)-(8) the robustness of the method is checked. It is not easy to compute the solutions of such examples. As the inhomogeneous terms order increase it becomes much more complicated. This method enables us to deal with these complicated problems.

Example 3: Now let us consider problem (9),(10) and (11),(12) when ε, μ and σ be identical matrices and let the non-homogeneous term $\mathbf{f}(x,t) = -\frac{\partial \mathbf{j}}{\partial t}$ be a vector function that is not polynomial. The components of non-polynomial smooth data $\mathbf{f} = (f_1, f_2, f_3)$ are

$$f_1(x,t) = \cos(x_1)\sin(2x_2)\sin(3x_3)\delta(t),$$
$$f_2(x,t) = \sin(x_1)\cos(2x_2)\sin(3x_3)\delta(t),$$
$$f_3(x,t) = \sin(x_1)\sin(2x_2)\cos(3x_3)\delta(t).$$

The exact solution of IVPs can be easily found without using PS method. The components of the solutions are

$$\begin{aligned}
E_1(x,t) &= \frac{165}{385}\left(1-e^{-t}\right)\cos(x_1)\sin(2x_2)\sin(3x_3) \\
&\quad + \frac{8\sqrt{55}}{385}\left(e^{-\frac{t}{2}}\sin(\frac{\sqrt{55}t}{2})\right)\cos(x_1)\sin(2x_2)\sin(3x_3), \\
E_2(x,t) &= \frac{330}{385}\left(1-e^{-t}\right)\sin(x_1)\cos(2x_2)\sin(3x_3) \\
&\quad + \frac{2\sqrt{55}}{385}\left(e^{-\frac{t}{2}}\sin(\frac{\sqrt{55}t}{2})\right)\sin(x_1)\cos(2x_2)\sin(3x_3), \\
E_3(x,t) &= -\frac{495}{385}\left(e^{-t}-1\right)\sin(x_1)\sin(2x_2)\cos(3x_3) \\
&\quad - \frac{4\sqrt{55}}{385}\left(e^{-\frac{t}{2}}\sin(\frac{\sqrt{55}t}{2})\right)\sin(x_1)\sin(2x_2)\cos(3x_3),
\end{aligned} \tag{40}$$

and

$$H_1(x,t) = -\frac{\sqrt{55}}{770}\sin(x_1)\cos(2x_2)\cos(3x_3)\left(e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{55}t}{2}\right)\right)$$
$$-\frac{55}{770}\sin(x_1)\cos(2x_2)\cos(3x_3)\left(e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{55}t}{2}\right)-1\right)$$
$$H_2(x,t) = \frac{\sqrt{55}}{385}\cos(x_1)\sin(2x_2)\cos(3x_3)\left(e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{55}t}{2}\right)\right)$$
$$+\frac{55}{385}\cos(x_1)\sin(2x_2)\cos(3x_3)\left(e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{55}t}{2}\right)-1\right)$$
$$H_3(x,t) = -\frac{\sqrt{55}}{770}\cos(x_1)\cos(2x_2)\sin(3x_3)\left(e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{55}t}{2}\right)\right)$$
$$-\frac{55}{770}\cos(x_1)\cos(2x_2)\sin(3x_3)\left(e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{55}t}{2}\right)-1\right). \tag{41}$$

Using PS method we obtain polynomial form of the solution of the IVPs (9),(10) and (11),(12) and these solutions are compared by the solutions given in (40),(41) at the same fixed points and the results of the comparison of $E_1(x,t)$ are presented in the table given below. The results obtained by the PS method are in good agreement with the exact solution.

Table 1: Values of E_1 and E_1^N for $N = 24$

t	x_1	x_2	x_3	Error
7/5	1	1	1	$0.1 * 10^{-10}$
1	2	1	2	$0.3 * 10^{-10}$
2	5	2	2	$0.2 * 10^{-10}$
14/10	0	2	3	$0.1 * 10^{-9}$
2	4	4	4	$0.3 * 10^{-9}$
2	5	5	5	$0.2 * 10^{-9}$

Using the method in Section 3, a polynomial solution $\mathbf{E}^N = (E_1^N, E_2^N, E_3^N)$ of the IVP (9), (10) is calculated. The graph of the comparison for the first component of the approximate function that is $E_1^N(x,t)$ and the the first component of the explicit formula $E_1(x,t)$ are presented below.

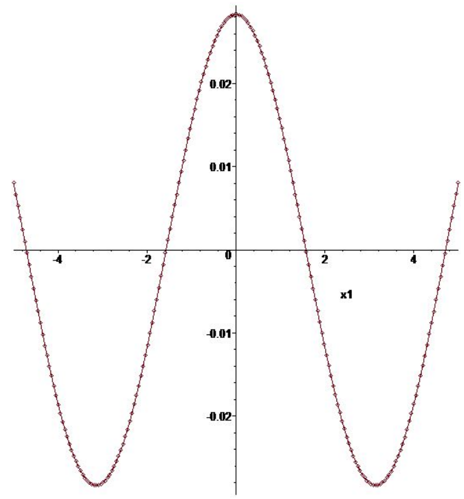

Figure 1: The graphs of the first component of the electric field $\mathbf{E}(x,t)$ computed by PS method and the explicit formula when $x_2 = 2$, $x_3 = 1$, $t = 1$.

8. Conclusion

Symbolic computations for constructing polynomial solutions for initial value problem of radiation from the electric current in electrically and magnetically anisotropic media is used for the case when ε, μ are arbitrary positive definite matrices and σ is a symmetric matrix with constant elements. Stability estimates (energy inequalities) for solutions of the system in a finite domain of dependence (a finite domain containing characteristic cones) is described. Using these stability estimates we justify that polynomial solutions are approximate solutions of the initial value problems with non-polynomial data. These theoretical results are confirmed by computational experiments which compares the exact solutions with polynomial solutions found by using polynomial solution method.

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