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Research Article

Concept Images of Prospective Mathematics Teachers for Geometric Representation of the Double Integral Concept

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Abstract – This study has focused on how prospective mathematics teachers understand the geometric representation of double integral. The basic qualitative research method was adopted as the research design in accordance with the research purpose. Six participants have been asked six questions. Later, semi-structured interviews were conducted with the participants. In this study, the data obtained from questionnaire form and interviews were analyzed with open and axial coding. As a result of this research, it was observed that the concept images of prospective mathematics teachers were grouped into two categories as “area” and “volume”. It was determined that the participants acted with an intuitive approach without having to establish a relationship between the concept definition and the concept image, the $\iint$ in the symbol of the double integral caused the participants to think of it as a two-dimensional geometric structure and their image of the concept of the single integral was very active. The findings obtained in this research show that there are problems in understanding the concept of the double integral, which is the first step of generalizing to multiple integrals, and that educators should produce solutions for this subject.

Keywords: Double integral, concept image, concept definition, prospective mathematics teachers.

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Introduction

Multivariate calculus helps us mathematically understand objects in the 3D world. Many of the functions that represent most objects in everyday life depend on more than one variable. For example, the volume of a right circular cylinder is equal to the product of the height with the area of the circle. Therefore, the application fields of multivariate calculus are wider than single-variable calculus application fields. Basic concepts of calculus find applications other than mathematics such as probability, statistics, biology, physics, economics and engineering (Stewart, 2009). However, the concepts of multivariate calculus are not well understood and there are few studies for the teaching of these concepts (Dorko & Weber, 2014).

Teacher training programs vary based on the countries' educational goals and generally consist of disciplinary subjects, educational studies and teaching practices (Comiti & Ball, 1996). For instance, the mathematics education undergraduate program is shaped around the deep field knowledge of prospective mathematics teachers, pedagogical knowledge of the field and knowledge to ensure the cognitive development of students (Loewenberg Ball et al., 2008; Shulman, 1987). While teachers' thinking structures, mental schematics and beliefs affect teaching practices in the classroom environment and students' learning (Ernest, 1988; Heid et al., 1998), teacher training programs should be able to reflect the nature of mathematics well to prospective teachers (Fennema & Franke, 1992) since students' mathematical thinking can be improved (Dossey, 1992; Loucks-Horsley et al., 2009). Therefore, mathematics teachers should have advanced mathematics thinking. Although Calculus 1 and 2 where single-variable functions are located, linear algebra and the image of pre-university mathematical concepts have been researched for over 40 years exceedingly, little is known about the image of advanced mathematical concepts (Hamza, 2012). Moreover, it is also difficult to create an appropriate concept image when advanced mathematical concepts need a high level of thinking (Tall & Vinner, 1981).

Prospective mathematics teachers take multivariate calculus as a field course and stated that their learning of multivariate calculus concepts better interprets single-variable concepts (Tchoshanov et al., 2002). In addition, it was observed that mathematics teachers who took multivariate calculus courses were more successful in understanding the basic concepts of calculus (Tchoshanov et al., 2004). Teachers who do not have a deep understanding of calculus concepts will also be incapable of educating students with the ability to pursue careers in science, technology, engineering, and mathematics (STEM) (Maltas & Prescott, 2014). Thereby societies that could not be settled with their age in the 21st century, whose economy was weak and dependent on other countries would be created. For this reason, calculus education has become a necessity to be taught conceptually in both secondary education and higher education (Sofronas & DeFranco, 2010).

One of the basic concepts of calculus, which has an increasing use in this century and has an application in real life problems, is the double integral (Steward, 2009). The double integral is the integral of a two-variable and continuous function $f\ (x, y)$ over a bounded region in the plane. More clearly; we consider a function of two variables defined on a closed rectangle

$$R = [a, b]x[c, d] = \{(x, y) \in R^2 |\ a \leq x \leq b, c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f. We are trying to find the volume of S. First, we will divide the R region into rectangles as seen in Figure 1.

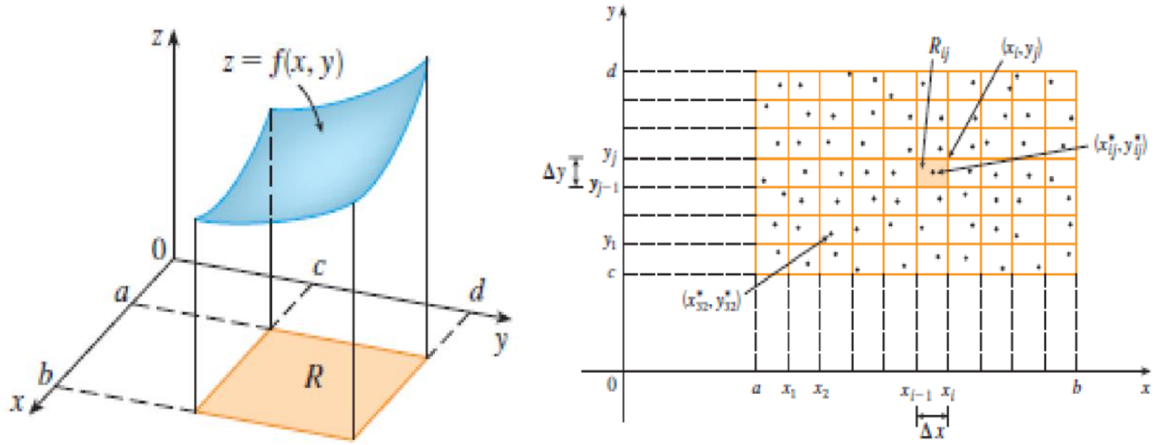

Figure 1 Dividing R into rectangles

Then the area of each rectangle ($\Delta A = \Delta x \Delta y$) and the $f(x^*_{ij}, y^*_{ij})$ value of a point inside the rectangle is multiplied. So, the volumes of the prisms formed are found. Then, if we add all the volumes, we get the approximation $V \cong \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x^*_{ij}, y^*_{ij}\right)\Delta A$ for the volume of S as seen in Figure 2.

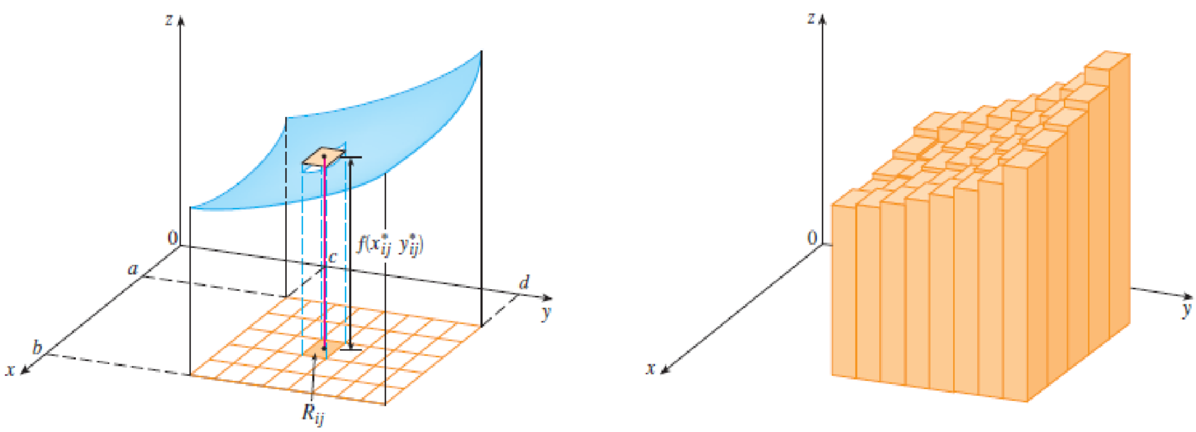

Figure 2 Volume approximations of the solid

While the width and length of each rectangle are reduced (as m and n increase) the volume sums of the prisms converge to the quantity. $V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ is defined under the graph of f and the volume of the solid S remaining above the rectangle R. This limit also occurs when f is not positive. And the double integral of f over the rectangle R is $\iint_R f(x,y)dA = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$, if this limit exists (Steward 2009). Therefore, the understanding of volume is at the heart of the double integral concept. For this reason, students are expected to see this implicit relationship in order to learn more deeply while learning the double integral concept. Otherwise, the students' understanding of the double integral concept will not be able to go beyond the operation that a calculator does. For example, students will have trouble interpreting situations where the function is negative. The curriculums expect students to see these implicit relationships by going beyond the procedural knowledge (Radzimski, 2020).

Many studies have been conducted about the single integral in the literature of mathematics education, and in some of these studies it has been discovered that since the students regard the integral calculation and area calculation the same (Rösken & Rolka 2007), they perceive the symbolic form of the integral as a special function such as $f(x) = sin\ (x)$ (Rösken & Rolka 2007; Serhan 2015), have difficulties in creating representation (Huang 2015; Serhan 2015), act with prototype examples (Jones 2018) and have difficulty in defining the relationship between image and definition (Rasslan & Tall 2002; Habineza 2013; Rösken & Rolka, 2007; Serhan 2015). There are not many studies relating to the multiple integral and it is observed that mathematics instructors have been starting to focus on multiple integral over the last five years (Martinez-Planell & Trigueros 2020). In this context; the generalization of the single-variable integral into the multiple integral (Jones & Dorko 2015), the pedagogical examination of the teaching of double and triple integral definitions by dividing them into geometric, numerical and symbolic representation layers (McGee & Martinez-Planell 2014) and how students establish the relationship between the multiple integral and the Riemann sum have been investigated (Martinez-Planell & Trigueros 2020).

There are difficulties and hardships in understanding the concept of double integral, which requires high-level cognitive competence (Dorko & Eric 2014; McGee & Martinez-Planell 2014). The concept of double integral makes it difficult for students to understand because it requires understanding function of two variables, three-dimensional geometry knowledge and visualization skills. Once students have learned the single integral, they should be able to generalize the concept of single integral both conceptually and procedurally as the first step of inductive reasoning. Problems in a mental configuration in the concept of double integral will cause problems in the concept of multiple integral as well. For this reason, the diagnostic and treatment methods of mental structures of the students for the double integral concept will also contribute to the mental structuring studies of the multiple integral concepts.

The quality of the relationship between the concept and their geometric representation also reveals the quality of understanding (Fischbein, 1993). Rösken and Rolka (2007) mentioned the importance of the analytical and geometric definitions of the concept of integral and stated that using different images instead of single and dominant images for the concept of integral will take conceptual understanding to an advanced level. Oberg (2000), on the other hand, stated that the reason for the sign problem of function that the students experience when the results of the integral processes are negative is that they cannot identify a relationship between the integral and its geometric representation. According to APOS theory, which examines how to learn advanced mathematics concepts, the relationship between multivariable functions and its geometric representation takes place only at a high level cognitive stage (McGee & Martinez 2014; Şefik & Dost, 2019). In addition, it has been determined that students have difficulty in understanding the relationship between the double integral and its geometric representation (McGee & Martinez 2014; Martinez-Planell & Trigueros 2020; Şefik & Dost 2019). Thus, their understanding (images) of the geometric representation of the concept of the double integral is of high importance. It will be beneficial to reveal how much the students' understanding of volume is in accordance with the nature of the double integral concept and in what context it is difficult to teach the concept better. In this context, this study is aimed to determine concept images of prospective mathematics teachers for the geometric representation of a double integral concept. Accordingly, the following research problem has been answered:

What are the concept images of prospective mathematics teachers regarding the geometric structure of the concept of double integral?

Theoretical framework

The whole cognitive structure formed in the individual's memory for a concept is called "concept image” (Tall & Vinner, 1981; Vinner 1983). It consists of all cognitive structures such as mental pictures, features, and processes related to the concept (Tall & Vinner, 1981; Vinner 1983). When a function word is heard, examples can be given, such as remembering the expression y=f(x) or reviving a function graph in the mind (Vinner, 1991).

The concept definition is a set of words that accurately describe a concept (Vinner, 1983). From this point of view, the definition of concept can be expressed as a structure consisting of words and/or symbols used by textbooks or relevant experts. Definitions shape our image of the concept, and once the image is formatted, the definitions are abandoned or forgotten (Vinner, 1991). The effective status of the concept images that are generated can vary depending on time, event, or even question (Tall & Vinner 1981; Vinner, 1983). This means that an image of a concept that is effective in one question may not be effective in the other question.

Image is an element that defines knowledge and gives information about whether people's thoughts belonging to the concept are appropriate for the formal structure (Smitt & Kosaslyn, 2014). The theoretical framework for "concept definition and concept image" in mathematics education covers how students think and understand a mathematical concept, in short, all the cognitive structure of a concept. In this study, the theoretical framework of the concept definition and concept image will be used to determine how prospective mathematics teachers shape the geometric representation of the double integral in their minds. Thereby, by determining the images for the geometric representation of the concept of double integral, how they conceptualize the double integral and the problems encountered will be revealed.

Method

Since the study aims to reveal the concept images of prospective mathematics teachers about the geometric representation of double integral, the basic qualitative research method was adopted as the research design. Basic qualitative research is about how people interpret their experiences and how they make sense of their own minds (Merriam, 2009). This is to reveal how prospective mathematics teachers interpret the geometric representation of the double integral and how they formed the concept image. Besides, qualitative research designs contribute to the consistency of the research phases by providing a flexible approach to the researcher in determining data collection and analysis approaches.

Participants

Students' mathematical learning changes according to the student population of learning (Kloosterman, 2002). For example, while it is sufficient for engineering students to learn procedural information by profession, conceptual understanding is not very important to them (Khiat, 2010). However, conceptual learning is very important for prospective mathematics teachers expected to reach more abstract concepts than other professions. For this reason, participants in the study consist of six prospective mathematics teachers ($P_1, P_2, P_3, P_4, P_5, P_6$) who are studying in the faculty of education in a state university in the 2017-2018 academic year and that can provide data diversity.

Calculus 4 course is offered in the spring term of the second class and focuses on the limit-derived concepts of multivariate functions and multiple integral. The teaching program is scheduled to finish the double integral concept in 12 hours. In Calculus 4 course, formal definition, algebraic operations, Fubini theorems, region transformations, area and volume calculations and improper integral subjects are taught for the double integral concept. In this context; participants were selected from the students who succeeded in lesson Calculus 4. Additionally, participants are made up of volunteers in accordance with the opinions of the relevant instructor. Examination of students' comments and exams during the course process allows the instructor to be a natural observer. Thus, it is expected that it will be possible to reach different and rich images.

Data Collection Tools

The data consists of written responses to the questionnaire form of the participants and audio recordings from semi-structured interviews with the participants about these questions. Firstly, while the concept of the double integral was offered in the Calculus 4 lesson, one of the researchers listened to the course for 3 weeks. A questionnaire form has been created taking into consideration the questions and responses students have asked in the lessons. Two expert opinions were consulted for questions prepared to ensure the content validity of the research. A pilot study has been performed with 10 students who have just learned the double integral concept. In the pilot implementation, written questionnaires have been given in their classes and during the application courses. There have been six questions (from Q1 to Q6) (Appendix 1) after the pilot study. Semi-structured interview form questions were prepared based on both the observations of Calculus 4 lectures to understand students' and lecturers' reactions to the related concept and adopted questions from previous research (e.g., Rasslan & Tall, 2002; Rösken & Rolka, 2007) according to research purposes. These questions are about concept image, interpreting symbolic and geometric representation, formal structure of double integral and calculation of Riemann Sum. After the questions of the semi-structured interview forms were formed, two expert opinions were taken to ensure the quality of both mathematics and mathematics education content of questions. Interview questions were revised and asked expert opinions until providing consensus. Some questions are revised to be provided more clearly. New questions were also added based on the research purpose. Based on expert opinions, Semi-structured interview questions are formed from easy to hard levels. Different images of students were tried to be reached by using verbal, algebraic, and graphical representations in the questions. The questionnaire form has been applied to the participants at any time in the classroom environment. They were asked to answer the questions in writing. There has been a semi-structured interview with six participants answering questionnaire form. The interviews were held at the time the participants requested. The following questions have first asked in the semi-structured interview:

I1) What do you understand from the concept of double integral?

I2) What is the formal definition of a double integral?

I3) What do you think about the geometric representation of the double integral?

After these questions, questions such as “What did you think in this question?”, “How did you calculate this question?”, “Could you explain why you thought like that” were asked in line with their answer to the questions. In addition, they have been asked additional questions based on the expressions of their participants in a semi-structured interview.

For example, when the participant said that the two-fold integral was a two-dimensional geometric structure when the integrand function was 1, the researcher asked the participant "What does the $\int_0^1 \int_0^1 dydx$ integral mean geometrically?" Vinner (1983) noted that students may have the wrong image of a concept even if they pass the exam. In addition, the conformity or relationship of the resulting concept images to the concept definition has been examined from the perspective of capacity understanding of the double integral definition described in the introduction. For example, when the participant stated in the questionnaire form that he found volume with the integral in Q2, he was asked what the integrand function meant geometrically. Thus, it has been examined whether the concept image is related to the concept definition.

The interview has been audio recorded with the permission of the participants. No time restriction has been made. The semi-structured interview lasted 45 minutes approximately.

Data Analysis

In this study, the content analysis process proposed by Strauss and Corbin (1990) is adopted. It is desired to reveal the concepts and relationships hidden in the data by subjecting them to a deep process in content analysis. It is not easy to form the appropriate concept image in advanced calculus concepts (Tall & Vinner, 1981). In order to reveal students' concept image in depth, the data were analyzed to benefit from open and axis coding techniques of grounded theory. These analysis methods allow the examination of the concept images in the most detailed way and to reveal the implicit mental pictures. Otherwise, for example, the concept images for the concept of integral will not go beyond the calculation of area and volume, and whether it is related to the concept definition will not be exactly determined. Researchers have not interfered with any participant's perspective. The data acquired with audio recording has been first converted into written form before proceeding with data analysis. The answers of the participants to the questionnaire form and the written data from the semi-structured conversation audio recordings have been assessed by transferring them to the Maxqda 2018 program. The data obtained from the questionnaire form and audio recordings were evaluated as holistic. In other words, the data obtained from the questions have not separated from each other. The concepts and connections included in the answers have tried to be analyzed by comparing them with each other.

Data analysis has been analyzed step by step and then comparatively by two authors; analysis supported by open and axial coding. Open coding is an analytical process in which the dimensions and features of the concepts are attempted to be discovered by carefully separating the data into the smallest piece (Strauss & Corbin, 1998). After the data has been separated into its smallest part, the labeling process has been performed by asking persistent questions such as "What is the major idea brought idea brought out this sentence?" (Strauss & Corbin 1998, p. 120) and continuously comparing the data. The two researchers have independently encoded a participant's interview deciphering. After that, unnecessary codes have been discarded by discussing among themselves and the codes have been finalized. Then, the transcripts of the other participants have been also examined in this perspective. Finally, the researchers have came together and discussed the differences. Member control is one of the most important strategies of qualitative studies to learn how accurately codes reflect what participants express (Rossman & Rallis, 2012). Two participants have been selected and confirmed that the coding made reflected their own thoughts, and the final version of open coding has been decided. As far as possible, analytical codes have been tried to be reached from descriptive codes (Urquhart 2012). For example, in the 6th question of the application, when participants have been asked to find the double integral result using the volumes below and above of the surface in both $-z$ and z, they marked the sum of volumes $V_1 + V_2$. P_3 has explained this as follows:

Because it is on the positive side of the region here is on the negative side. (….) If I say this place V, it will be found out. $-V$ negative. For example, we take this directly as the volume would not be negative.

This data has been first labeled as "non-negative status of volume calculation". Later, when other data have been examined since the participant considers the volume finding account and the integral account to be identical, it has been seen that the result of the integral calculation will not be negative, this data piece has been coded as "The identical state of volume and integral".

Axial coding is the process of classifying data around categories based on the characteristics and dimensions (Strauss and Corbin 1998), which we study in the most detail in open coding. After open coding, the two researchers have independently determined which categories these codes have dimensions by re-examining the written responses to the transcripts and application questions. The two researchers have finally agreed on the encodings in Table 1. Table 1 shows categories, subcategories ("concepts that pertain to a category" (Strauss and Corbin 1998, p. 101)). Open codes, and each concept will be explained in detail in the findings section.

Table 1 Concept images of prospective mathematics teachers for the geometric representation of the double integral

Category	Subcategory	Open coding
Area	As a discourse, the state of expressing the area. (Expressed as double integral discourse)	Area of non-rectangular regions (geometric shapes that cannot be calculated with a certain formula), area of the region (the area of the integral region in the double integral), area under a curve (the area under a curve within the limited interval), shadow of a curve (finding the area of the place between the curve and the plane by moving a curve to the xyz-plane), three-dimensional area (the area of that shadow by drawing a curve on the XYZ-plane, showing its shadow in two dimensions), surface area (finding the area of a surface with a double integral)
	The status of "able to calculate area" (Status in which case the area is calculated with the double integral)	If the function is 1 (the expressions of the participant who considers that integrand function is 1 as a rule), absence of function (the expressions of the participant who states that integrand function is one and there is no function), the status of area calculation by integral if the height is 1 when calculating the volume (the integrand function is geometrically high and height be).
	Connection of integral area (Relationship between integral account and area account)	Identical status of area and integral (the interpretations of the participants indicating that the result of double integral will not be negative and zero), Differentiation of area and integral (the interpretations of the participants indicating that the result of double integral may be negative and zero)
	Area calculation strategy (Interpretations made while calculating areas with double integral)	The status of non-calculation of the area if the function is not given (situations where a geometric shape is asked to express the area with a double integral; the participant states that he/she cannot solve it because no function is given)
Volume	As a discourse, the state of expressing the volume. (Expressed as double integral discourse)	The volume of the remaining space between the region and $f(x, y)$ (volume of the remaining part between the surface and the specific region while calculating the double integral), The volume of solid (volume of solid objects such as a prism, cylinder),

	the cross-section method (the cross-sectional method used when calculating the volume with the single integral), volume of irregular objects (volume of objects that cannot be directly calculated by formula such as prism, cylinder), the disk method (the disk method used when calculating the volume with the single integral)
The status of "able to calculate volume" (Status in which cases volume is calculated with the double integral)	The status of being triple integral (the concept of volume is related to triple integral concept) status of being function inside the integral (situations where the integral function of the double integral 1 does not exist), status of setting additional conditions (the operations performed when calculating the single integral volume), the status of able to calculate volume if $f(x, y) \geq 0$ is (double integral represents volume where integrand function in double integral is positive)
Connection of integral volume (Relationship between integral calculation and volume calculation)	Identical status of volume and integral (double integral operation and volume calculation are identical and the result of double integral cannot be negative and zero), Differentiation of volume and integral (the result of double integral operation may be negative and zero)
Volume calculation strategy (Interpretations made while calculating volume with double integral)	The status of unable to calculate volume if height (z) is not given (volume calculation will not be made because no integrand function is given when a solid is provided and asked to be calculated with double integral)

Findings

In this part, findings related to concept images of prospective mathematics teachers for the geometric representation of the double integral concept are provided. As a result of the analysis of the data of this research, it has been that the concept images of the prospective mathematics teachers for the geometric representation of the concept of double integral are gathered in the categories of “area” and “volume”.

Area

This category refers to the concept image of the participants who see the double integral concept as an “area" from prospective teachers.

As a discourse, the state of expressing the area

This subcategory represents the answers given by the participants who see the double integral as “area”. The following codes represent the participants' comments representing their understanding of double integral while interpreting the semi-structured interview questions (I1 and I3) and explaining their answers to the questions, Q1 and Q2, in the questionnaire form.

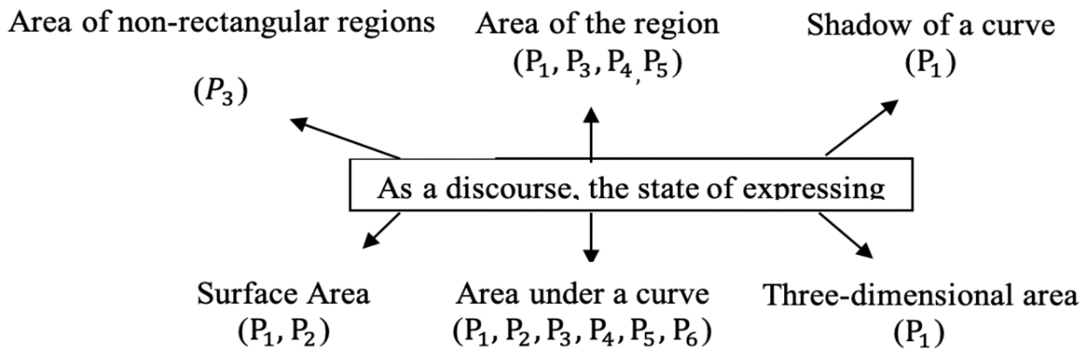

Figure 3 The classification of the expressions of the participants "what they have found" with double integral

The idea reflected by each code in Figure 3 is described below. In this way, the expressions related to the views of the participants are reflected by making direct references.

The area of non-rectangular regions refers to geometric shapes that cannot be calculated with a certain formula, such as squares, rectangles, etc., P_3 shared his thoughts on the double integral concept as follows:

"Hmm. Well. For example, there is a region, an area with no clear shape. (….) We cannot calculate them directly. So, I think the double integral has been consulted."

The area of the region represents the area of the integral region in the double integral. When the researcher asked what area we have found, P_5 said he found the area of "a closed region".

The shadow of a curve represents finding the area of the place between the curve and the plane by moving a curve to the xyz-plane. P_1 expressed how he found the area with double integral as follows and illustrated his thoughts as shown in Figure 4:

"Areaaa. I would draw a function. Like a shadow of a function, I would reduce the area down."

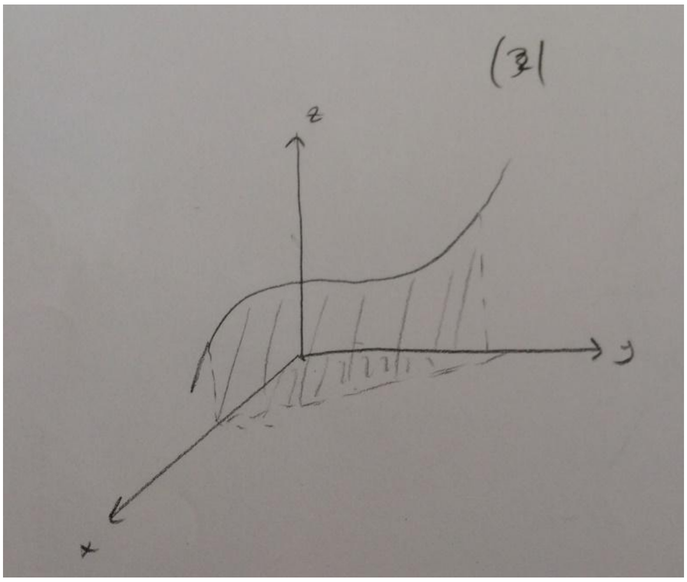

Figure 4 Interpretation of P_1 related to the double integral concept

Surface area refers to finding the area of a surface with a double integral. P_2 described the surface area as follows and illustrated its thoughts as shown in Figure 5:

“We performed it Calculus 4. This provides a region. It provides a function on it. For example, it wants the area on T region from us, by drawing them, the area or volume on *T* region of that function."

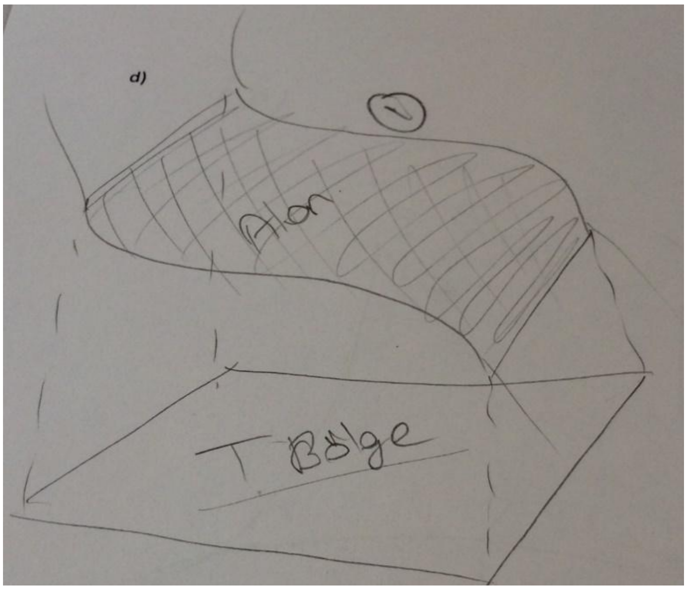

Figure 5 Interpretation of P_2 related to the double integral concept

The area under a curve represents the area under a curve within a limited interval. All participants have used this expression. P_6 has expressed the double integral as follows.

“We always found the area under this function ($y = x^2$) with the help of integral."

The three-dimensional area represents the area of that shadow by drawing a curve on the XYZ-plane, showing its shadow in two dimensions. When the researcher has asked, “What do you understand about the concept of double integral?" P_1 explained his idea as follows.

“An area on double integral. So, in the first dimension, in the single integral, I usually think of the first dimension. (….) The double integral area comes to my mind in three dimensions. “

Here, it is seen that P_1 thinks of the integral symbol in terms of dimension. The participant thinks that the single integral is one-dimensional, and the double integral is two-dimensional. For this reason, the area image is formed in the participant's mind. He illustrated the above explanations by drawing in Figure 6.

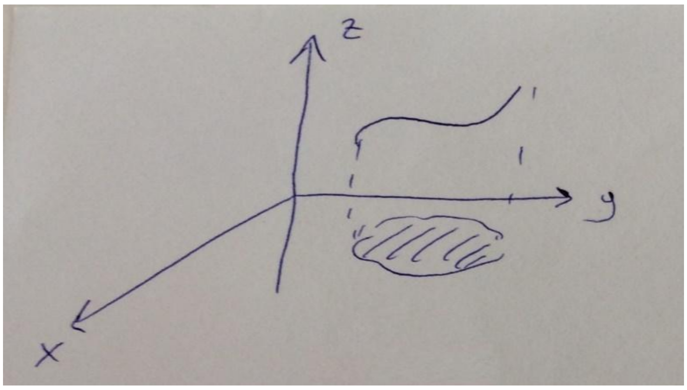

Figure 6 Interpretation of P_1 related to the double integral concept

When the above codes are examined, all codes are related to the concept of single integral rather than double integral except for the surface area code. Furthermore, the common point of these codes is that the participants consider the concept of double integral as a single variable function. Inappropriate concept images stand out in the participants. Therefore, a relationship between concept definition and concept image cannot be detected.

The status of "able to calculate area"

This subcode includes the answers given by the participants to the questions, Q2 and Q3, in the questionnaire form. In other words, it represents the concept images that explain how the participants construct the integrand function, in which cases the area is calculated with double integral and what the integrand function means geometrically. The coding for this subcategory is shown in Figure 7, followed by the meaning expressed by each code.

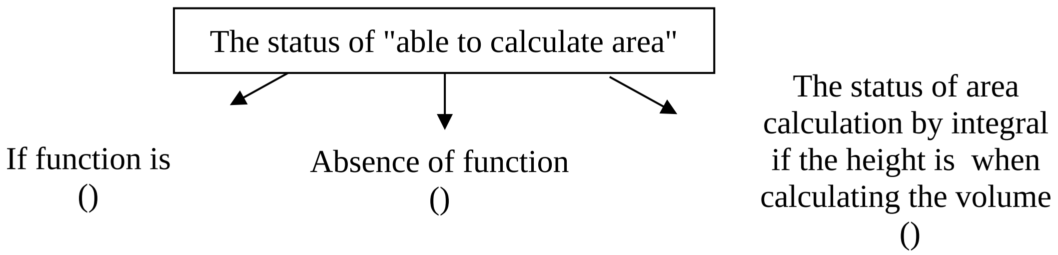

Figure 7 The classification of the interpretations of the participants on the area account with double integral

If the function is 1 represents the expressions of the participant who considers that integrand function is 1 as a rule. When the researcher asked why they wrote the integrand function 1 while calculating the area, the participants below gave the following answers.

"I'm not sure if 1 when? Hımm. If it weren't zero, we wouldn't have any area." (P_4)

The Absence of function represents the expressions of the participant who states that integrand function is one and there is no function. Some of the participants, for example:

"If there has been a function, I could interpret it as three-dimensional, but since it yields 1, it is two-dimensional." (P_6)

While P_5 explained how he solved Q3 and Q4, "stated that there should be no function. Upon this statement, the researcher asked what $\int_1^0 \int_1^0 dydx$ geometrically represented; P_5 shared his thoughts as follows.

"I call it the exact area. Because there is no function. There is no function before $dxdy$. I understand that the area is there whatever it is necessary and what it requires."

The status of area calculation by integral if the height is 1 *when calculating the volume* means that the integrand function is geometrically high and height is 1. P_3 and P_4 have been able to visualize that here and integrand function is 1 not only as a rule but also geometrically. When the researcher asked what $\int_2^3 \int_5^4 1dxdy$ geometrically meant, P_3 said:

"Here comes $z = 1$. I mean, a certain height. (….) That is the area. (….) It will be the same when we multiply it by 1. It is inferred from there. "

P_3 and P_4 were able to see the relationship between the volume perspective and the area in the concept definition of the double integral only with extra questions. They may have reached such a comment by being influenced by the questions on the questionnaire form in the semi-structured interview.

When the subcategory of “The status of “able to calculate area”” is examined, it is seen that most of the participants shared their thoughts that 1 should be written in the integral or there should be no function in order to calculate the area with double integral. Participants think that the integrand function is 1 as a rule/operation. Since the participants do not perceive the understanding of the volume in the formal structure of the double integral, they imagine the integrant function to be 1 as a rule only. Therefore, they do not realize the relationship between integrand function being 1 and height of a solid. In addition, the participants do not see “1” as a function.

Connection of integral and area

This subcategory represents the interpretations of the participants as a result of the negative result of the integral calculation when the questions, Q2 and Q3, in the questionnaire form are asked. The coding and descriptions for the subcategory are shown in Figure 8.

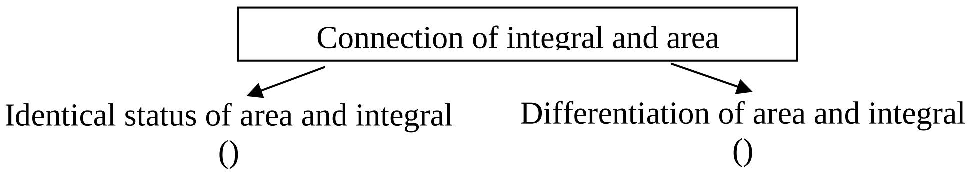

Figure 8 The classification of the interpretations of the participants on the relationship between double integral and area

The Identical status of the area and integral represents the interpretations of the participants indicating that the result of double integral will not be negative and zero. Below is a quote about P_3 interpretations of the negative result of the double integral.

“It is $-1/4$. We are supposed to never find it negative. (….) It is $-1/4$ but we are talking about area and volume.”

Differentiation of area and integral represents the interpretations of the participants indicating that the result of double integral may be negative and zero. When the result of the double integral has been negative, P_1 stated that the area could not be negative. Later, he realized that the function may be in the negative region and shared his thoughts as follows.

> “The area is not negative. I think it's wrong. (….) Maybe it can get something in a minus region, function. (….) We'll find the bottom half.”

Although the participant also said that the integral result was from the negative region in the statement above, he stated that he thought he had solved it incorrectly when the result came back negative when solving Q1 and Q2 questions. In the semi-structured interview, he stated that the answer might be due to the negative region.

When Figure 8 is examined, most of the participants see the calculation of area as identical to the double integral process. This situation shows that the relationship between concept image and definition is not established and the implicit relationships underlying the concept are not noticed.

Area calculation strategy

This subcategory represents the interpretations of the participants when calculating the area with double integral. The coding for the subcategory is shown in Figure 9.

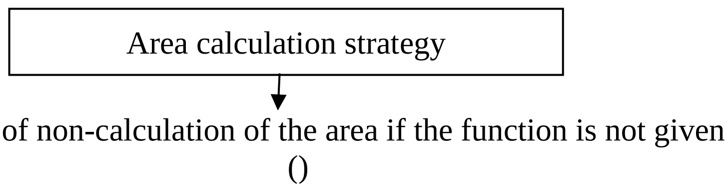

Figure 9 The classification of interpretations of the participants while calculating the area with double integral

The status of non-calculation of the area if the function is not given refers to situations in which the participant states that the participant “cannot solve because the function is not given” when a geometric shape is asked to express the area with a double integral. In the Q4, when the area of a closed rectangular region was asked to be expressed with the concept of double integral, the participants stated that they could not solve it because the integrand function was not given in the question. The expressions related to these views are given below in the form of direct references:

“He didn't define us a function. Since function has not been defined, I don't know that.” (P_1)

From the above citation, it is seen that the participants have the perception that it is an obligation to give integrand function in the questions.

Volume

This category refers to the concept image of the participants who see the double integral concept as "volume" from prospective teachers.

Status of linguistically expressed volume

This subcategory represents the answers given by the participants who see the double integral as "volume". The following codes represent the participants' comments representing their understanding of double integral while interpreting the semi-structured interview questions (I1 and I3) and explaining their answers to the questions, Q1 and Q2, in the questionnaire form.

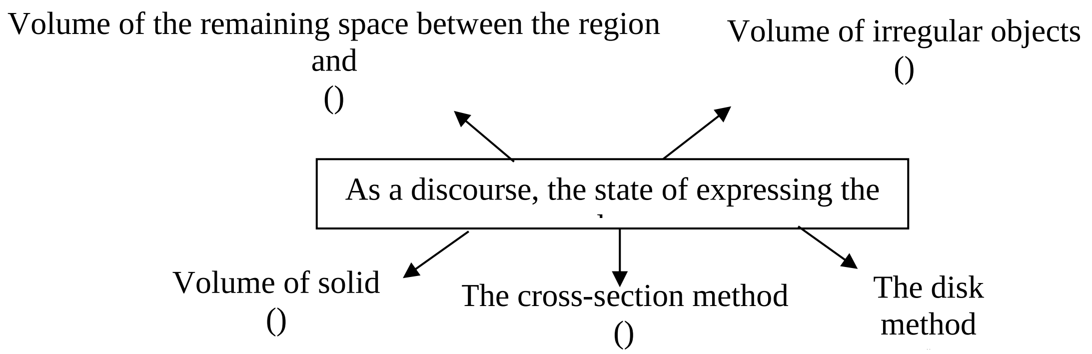

Figure 10 The classification of the expressions of the participants "what they have found" with double integral

The idea reflected by each code in Figure 10 is described below. In this way, the expressions related to the views of the participants are reflected by making direct references.

The Volume of the remaining space between the region and $f(x, y)$ indicates that there is the volume of the remaining part between the surface and the specific region while calculating the double integral. P_2 shared his thoughts on the double integral as follows:

"I might be able to find the volume between this B region and this $f(x, y)$ function."

Participants using the above attributes cannot be sure that what they are saying is true. This indicates that the participants could not make connection with the volume perspective in the concept definition. Such a concept image may have been established from the questions they solved in the lesson.

The Volume of irregular objects refers to the volume of objects that cannot be directly calculated by formulas such as a prism, cylinder. When asked by the researcher why we use the concept of double integral, P_3 reflected his opinion as follows:

“Yes. Whether it's area or volume. We use it to find the volumes and areas of objects whose picture is not exactly clear, which we cannot fully formulate."

The Volume of solid refers to the volume of solid objects such as a prism, cylinder. For example, P_5 described the concept image related to the double integral as follows:

“For example, the triangular prism can be calculated in them. It helps calculate their volumes in double integrals."

The cross-sectional method represents the cross-sectional method used when calculating the volume with a single integral. P_6 shared his thoughts on the double integral as follows:

“It is the volume of the cylinder $\pi r^2 h$. (….) If we find a double integral area, we can add something to it and find the volume. (…) you know, putting πr^2 on top of each other h time. This could lead us to a triple integral.”

The disk method represents the disk method used when calculating the volume with the single integral. When P_2 was asked what the concept of double integral reminds you, he explained his thoughts as follows.

“When we turn this curve like this, it eventually forms a conical shape there. We find its volume with integral. With double integral.”

When the above codes and quotations are examined, it is seen that the participants generally have concept images for the understanding of volume in the concept of single integral rather than the understanding of volume arising from the nature of the concept of double integral.

The status of "able to calculate volume"

This subcategory represents the participant responses indicating in which cases the double integral process indicates the volume. The coding for the subcategory is shown in Figure 11 followed by the meaning expressed by each code.

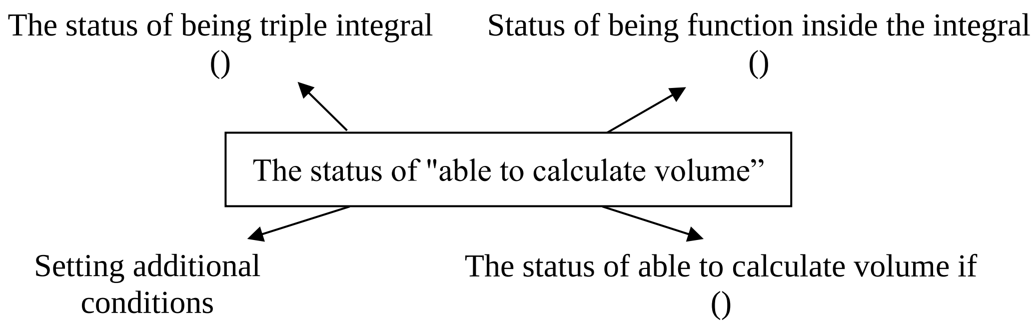

Figure 11 The classification of the interpretations of the participants on the calculation of volume in double integral

The status of being triple integral signifies that the concept of volume is related to the triple integral concept. The concept of triple integral outweighs the participants when they say "volume". Some views of the participants on volume are as follows.

"There are three different variables. I believe it's a triple integral as volume calculation. " (P_5)

"Volume to triple integral. Double integral for area. Indeed, when I hear a double word group, I think of the area that comes to mind " (P_6)

Status of being function inside the integral represents situations where the integral function of the double integral 1 does not exist. It also $f(x,y)=1$ shows that it does not see the expression as a function. For example P_2 , when there is an expression outside the integrand function 1, he said that there is volume and that he adopts it as a rule.

"Here it is the area when we take the function 1, the volume when there is a function inside, so I …Stereotyped, in my mind." (P_2)

The status of setting additional conditions describes the operations performed when calculating the single integral volume. P_6, remembered the area of each section of the solid when calculating the volume of the cylinder in the single integral and mentioned that the integral should be multiplied by "π".

The status of able to calculate volume if $f(x,y) \geq 0$ refers that double integral represents volume where integrand function in double integral is positive. P_3 said that the integrand function must be positive for the double integral to be a volume calculation. And by writing the $\int_1^2 \int_3^4 (x^2+y)dxdy$ integral as an example, he stated that this integral must be $x^2+y \geq 0$ to give volume calculation. Although it is considered here for P_3 because of the relation with the concept definition, he cannot fully interpret that the image of the function is positive. He says that this is how it is expressed during the study of textbooks and lessons. Therefore, it has been observed that the negative image of the function only causes a concept image to be created by saving the graphic as a picture drawn in that region. Because the function's mission there cannot be interpreted by P_3.

The idea of the participant's "If usually there are x and y terms in the integrand function, the volume is calculated" and the understanding of "Volume is calculated with triple integral" outweighs. When the codes here are examined, the area image is more outweighs the volume image for the double integral. Therefore, it is seen that there are problems with understanding the double integral concept. The fact that procedural knowledge is at the forefront of the participants causes a limited concept image.

Connection of integral and volume

Interpretation of the results of the double integral as positive, negative, and zero represents the answers of the participants regarding the calculation of the volumes below and above the surfaces in z and $-z$ with double integral. The coding and descriptions for this subcategory are shown in Figure 12.

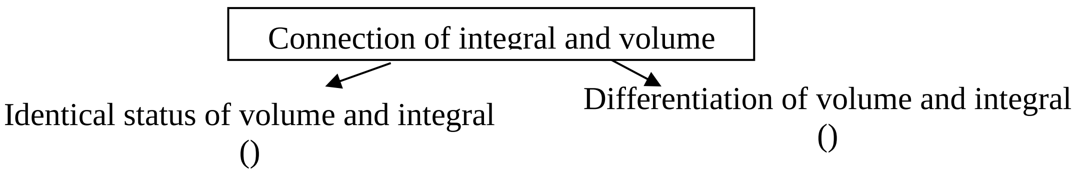

Figure 12 The classification of the interpretations of the participants relating to the relationship between double integral and volume

Identical status of volume and integral refers that double integral operation and volume calculation are identical and the result of double integral cannot be negative and zero. For example, when participants have been asked to find the double integral result using the volumes below and above the surface in both $-z$ and z in the Q6 they marked the sum of volumes $V_1 + V_2$. For example, P_3 explained marking this option as follows:

“If I say this place V, it will be found out. $-V$ negative. For example, we take this directly as the volume would not be negative"

Differentiation of volume and integral represents that the result of the double integral operation may be negative and zero. Below is a quote about P_2's interpretations of the negative result of the double integral.

Researcher: Is the integral calculation the same as the volume calculation?

P_2: It's not always the same, actually. (….) For example, if ee is at the negative and bottom area of z, we take it into absolute value.

Although P_2 above see the volume and integral calculation differently, he does not connect with the concept definition. The participant was able to comment on the absolute value of the integrand function of the volume calculation.

When Figure 12 is examined, most of the participants see the volume calculation as identical with the double integral operation. Therefore, participants think that the integral result cannot be negative or zero. Some of the participants realized that the volume could be different with double integral only later.

Volume calculation strategy

It represents the interpretations of the participants when calculating volume with double integral. The coding for this subcategory is shown in Figure 13.

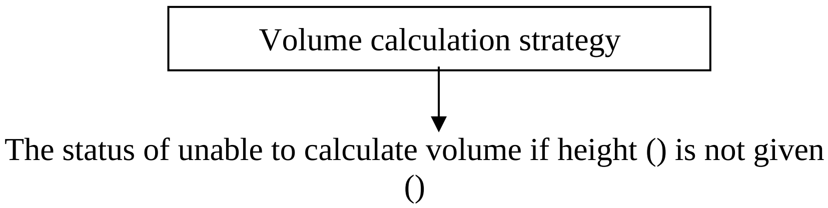

Figure 13 The classification of the interpretations of the participants when calculating volume with double integral

The status of unable to calculate volume if height (z) is not given means that volume calculation will not be made because no integrand function is given when a solid is provided and asked to be calculated with double integral. Participants explained the reasons why they could not calculate the volumes of solids given in application question 5 as follows.

"We can't find the volume. Because there's nothing. Like height." (P_1)

"I need to know what the $f(x, y)$ function is." (P_6)

From the above citations, it is seen that the participants have the perception that it is an obligation to give integrand function in the questions.

The Interplay Between Concept Definition and Concept Image for Double Integral

None of the participants could formally define the double integral (I2). In addition, there has been no finding that the participants established a relationship between the concept definition and the concept image. The interpretations of the participants in the semi-structured interview do not arise from the formal definition of double integral. When the subcategories are examined, it is seen that the interpretations of the participants result from the concept of a single integral.

Discussions

This study is aimed to reveal the concept image of prospective mathematics teachers for geometric representation of double integral.

Semi-structured interviews (I1 and I3) and answers to questions in the questionnaire form (Q1-Q4) appear to prioritize procedural skills for the concept of double integral. Due to such an approach, it is not a surprise that the participants have a limited concept image. It has been observed that the double integral is not associated with the volume understanding in accordance with the formal structure of the double integral because it takes the resulting concept images and has a volume calculation restriction. It may have been caused by the region drawings used to analyze the double integral rather than the geometry structure represented by the double integral in calculus lessons. Due to the formal structure of the double integral, the volume image is expected to be more revived in the minds of the participants, but it is seen that it is less represented. When Figure 4 and Figure 6 are examined, in fact, some of the students are aware of a concept in 3-dimensional space, but they cannot sense any further since they cannot exactly relate to the concept definition. One of the reasons why the volume image is problematic and limited in the minds of prospective mathematics teachers for a double integral can also be caused by the difficulty of visualizing 3-dimensional concepts (Seaman, 2000). In this case, symbols, rules, and prototype examples are at the forefront of memory.

Rules are at the forefront as the participants could not establish a complete relationship between concept definition and concept image in this study. For example, students have created a mental representation of the relationship of the double integral to the area as a 1 or no function in the symbolic expression rather than the height being 1 in the volume calculation. When Vinner and Drefyus (1989) examined the concept images of function in their study, it is seen that the rules become more active in memory. In the studies of Martinez-Planell and Trigueros (2020) and Jones and Dorko (2015), it is seen that between the lines, students have confused between area and volume with double integral. The reason for this confusion among students is stated to be caused by seeing the two-variable function as a two-dimensional structure by Jones and Dorko (2015).

More detailed data were obtained with the concept image-concept definition theoretical framework and grounded theory coding techniques. For example, the expression of the concepts of area and volume as discourse is an important argument in demonstrating the inadequate images of concepts formed in the mind of people and how they relate to the definition. It is observed that there are problematic and unconnected images in discourse codes. Hall (2010) stated in his study that discourse has an effect on learning calculus concepts and shows whether there is sufficient conceptualization or not. When solving double integral questions, drawing is made to determine the boundaries, but the geometric representation expressed by the integral is not drawn. In this case, it can cause students' concept image to be shaped around the area.

When the participants have been given an image of the area of a region on the XYZ-plane and a solid on the XYZ-plane and asked to calculate area (Q4) or volume with double integral (Q5), the participants had difficulty interpreting such questions. Since the integrand function is usually defined in problems, it is seen that this leads to a perception (image) such as an obligation situation in the participants. Participants do not have a connection between the function and the height, which causes problems in solving the problem. It is believed that resolving problems with the translation of different representations will lead to the creation of different and rich image concepts.

It has been identified that participants are seeing the integral calculation and finding the area and volume accounts of geometric structures identical. Therefore, the participants have placed an image where the integral result cannot be negative. Participants are unable to recognize cases arising from the negative region because they see the area and volume and the integral calculations similar. In other words, students who see the double integral as a formula that can only calculate area and volume, like a calculator, cannot notice the state of the function arising from the negative region. Additionally, this situation shows that students cannot establish a bridge between concept definition and concept image and they do not have appropriate concept image. Students limiting the calculation of the integral to area and volume only limits their conceptual understanding (Orton 1983; Sealey 2014). Even participants who state that the result of the double integral may be negative do not realize that the appearance of the function is negative. In textbooks and lectures, explanations of the result of integral as negative region by instructors overshadows the situation caused by the appearance of the function. Therefore, the Riemann sum should be given importance. The implicit relationships that underlie the concept image of Rieman's total activities can contribute to the understanding of the students. The Riemann sum will be both with the ability of students to develop a richer image of concept and to better solve their daily life problems (Sealey 2014; McGee & Martinez-Planell 2014; Martinez-Planell & Trigueros 2020).

Symbols have a significant effect on the representation of concepts in memory (Allen & Brooks, 1991). If people do not know the implicit nature of the concept, they act with the meanings in symbols (Smith & Kosslyn, 2007). We believe that the double integral has two symbols on its symbolic expression $\int\int$, which are effective in thinking as a two-dimensional geometric structure in the participants. In this study $P_1,\ P_5, P_6$ think that the concept of volume is related to the triple integral due to being three variables. Students can see two-variable function as a two-dimensional structure (Jones & Dorko, 2015). Both the complete lack of a two-variable function and the graphs drawn to determine the boundaries of the double integral symbol and the double integral in the lessons can cause the double integral to be two-dimensional. In this study, it is seen that participants load dimension meaning into the integral symbol because they cannot see the relationship of the symbolic expression of the double integral to the independent variable. Therefore, the meaning of the symbolic structure of the double integral should be given well. With outlier examples, this false concept image can be prevented. Outlier examples are effective in creating an adequate and accurate concept image of students (Tall & Vinner, 1981)

These results show that participants are acting with an intuitive approach, without having to correlate the concept definition with the concept image. Whereas definitions are an important building block in understanding the essence of mathematical concepts, in the development of concepts of students, in clarifying and communicating terms, in communicating with mathematical arguments. However, mathematicians see the concept definitions as the center of the issue in advanced mathematics, vice versa, it is seen that students act with images. (Tall & Vinner 1981; Edwards & Ward, 2004). In addition, it has shown that none of the participants can define the formal definition of the double integral and that no relationship is identified between the concept image and concept definition; the concept image and concept definition model created by Vinner (1983; 1991) have shown that participants act with an intuitive approach. Therefore, it shows that there are problems understanding relationships for the double integral concept. When the data is reviewed, the concept images for the geometrical representation of the double integral of prospective mathematics teachers can be modeled in Figure 14.

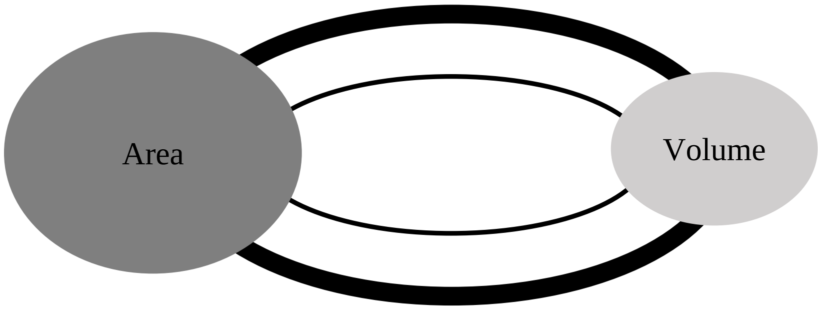

Figure 14 Views of prospective mathematics teachers for the geometric representation of the double integral

The study of Figure 14 shows that the image of the area in the minds of students is very dominant, and the concept of the integral, which is more single integral than the double integral, is seen as active.

In cognitive psychology, when the stimulus is presented quickly in sequence, the error of recognizing the stimulus is called repetition blindness (Kanwisher, 1987). The second stimulus is not seen as a separate event after the first stimulus, triggering the first stimulus and recording a single event. (Smith & Kosslyn, 2007). Repetition blindness claims that we do not create a new and separate representation of something that we have just processed as we have restricted time and we do not notice the repetition for this reason. (Smith & Kosslyn, 2007). Cognitive obstacles arise in mathematical thought processes such as reversibility, flexibility and generalization (Norman & Pichard, 1994, p.76). In the two-dimensional thinking of the double integration, the two-variable function may have some intellectual barriers, except in cases of detection problems, etc. We believe that one of the reasons why there is no appropriate concept image, especially in advanced mathematics concepts, is perception errors, such as repeat blindness. Although there are more volume questions than area questions about the double integral concept when we look at the calculus books, the area image is outweighed by the participants. This is because participants receive a single integral concept as an integral concept in a calculus course in both high school and first-class in an undergraduate degree. Therefore, it arouses the idea in which they encode the concept of integral as a single concept in general.

For this reason, it is thought that the image of the area is active in the double integral by our authors. In this regard, participants consider the multiple integral as a single integral and do not create a different image. Once students have learned the concept of a variable function, if they do not see the two-variable functions as a separate concept, they will not represent the two variances for functions separately and will act with the images of a variable function. These types of inappropriate images can be called concept blindness. The study conducted by Edwards and Ward (2004), expressed their surprise by finding that they created a new concept, images of a previously learned concept, and incorrect mental pictures. Experts who design multiple and versatile teaching should be assisted to identify perception errors of mathematical concepts and create rich and accurate image concepts. This will help strengthen the conceptual understanding of the students by designing activities that prevent perception errors.

It is not easy to create a consistent concept image in advanced calculus subjects, and it is difficult to gain deep knowledge of concepts with an inappropriate concept image (Tall and Vinner, 1981). Therefore, for a better conceptualization of the double integral concept, concrete material activities should be used. Otherwise, only the computer systems and proofs used may be weak in concept image formation. It is also effective in the formation of mental pictures in accordance with the nature of the concept of concrete materials (Fischbein, 1993). In this way, it can be explained that the area is only a feature and comes out because the height is multiplied by 1. As it seems that there is no permanent learning at the desired level in verbal expression or written proof of concept. Moreover, it will be useful to introduce the lesson with the problems that are the basis of the concept of double integral, and the features taught after that are steps towards solving these problems, and by going back to these problems again, it will be useful to reach a conceptual understanding that can solve similar problems practically. Thus, it may be possible to form appropriate concept images.

In fact, the studies to be carried out for the development of these advanced mathematical concepts have contributed to the development of mathematical thinking of mathematics teachers. It will lead to a better mathematical idea of the students they will raise indirectly. Considering the results of this study, it is recommended to carry out action studies and teaching experiments to learn the concept of better double integral.

Limitations and Recommendations

This research is limited to the concept image of six prospective mathematics teachers. Similar research may be conducted on a large group to make generalizations of the findings. Furthermore, a general model or theory would be put forward if double integral could be searched in the departments of mathematics, mathematics education, and engineering. From this point of view, the differences and similarities of concept images of students can be deepened and understood. Concept images of prospective mathematics teachers who graduate before and after 2018 may be compared to highlight the effects of the undergraduate program which is updated in 2018.

Conclusion

The concept of revealing general concept images of a concept has an important place for teaching (Vinner & Drefyus, 1989). Thus, it will help the people who plan to teach to develop their teaching methods by knowing in advance what kind of idea structure their target audience has and the cognitive obstacles they may fall into. The aim of this study is to determine how prospective mathematics teachers understand the geometric representation of double integral and to contribute to the understanding of double integral. The answers of the participants to the six questions and the data from the semi-structured interviews have been analyzed in parallel. Considering these data, concept images under two main headings “area” and “volume" have been examined.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest

No conflict of interest.

Research involving Human Participants and/or Animals

The study involves human participants. Ethics committee permission was obtained from Hacettepe University Ethics Boards and Commissions for the study's implementation (30.11.2017, No:E-35853172-433-4017).

Matematik Öğretmeni Adaylarının Çift İntegral Kavramının Geometrik Temsiline Yönelik Kavram Görselleri

Özet:

Bu çalışmada matematik öğretmen adaylarının çift katlı integralin geometrik temsilini nasıl anladıklarına odaklanılmıştır. Araştırmanın amacı doğrultusunda bu çalışmanın araştırma deseni, temel nitel araştırma yöntemi olarak benimsenmiştir. Altı katılımcıya altı soru sorulmuştur. Daha sonra katılımcılarla yarı yapılandırılmış görüşmeler yapılmıştır. Anket formu ve görüşmelerden elde edilen veriler açık ve eksensel kodlama ile analiz edilmiştir. Araştırma sonucunda matematik öğretmen adaylarının kavram imajlarının “alan” ve “hacim” olmak üzere iki kategoride toplandığı görülmüştür. Araştırmadan elde edilen veriler doğrultusunda, katılımcıların kavram tanımı ile kavram imajı arasında ilişki kurmak zorunda kalmadan sezgisel bir yaklaşımla hareket ettikleri, çift katlı integral sembolünü $\iint$ iki boyutlu geometrik bir yapı gibi düşündükleri ve tek katlı integral kavramına ilişkin imajları etkin olduğuna ulaşılmıştır. Bu araştırmada elde edilen bulgular, çoklu integrali genellemenin ilk adımı olan çift katlı integral kavramının anlaşılmasında zorluklar olduğunu ve eğitimcilerin bu konuya yönelik çözümler üretmesi gerektiğini göstermektedir.

Anahtar kelimeler: Çift katlı integral, kavram imajı, kavram tanımı, matematik öğretmen adayları.

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Appendix 1
Questions

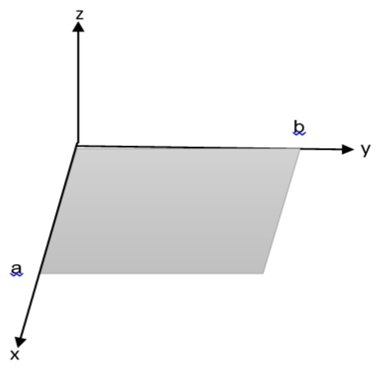
(1) Evaluate the double integral of the function $f(x,y) = xy$ over the $R = \{(x,y) \in [-1,0]x[0,1]\}$?

(2) Evalute the iterated integral.

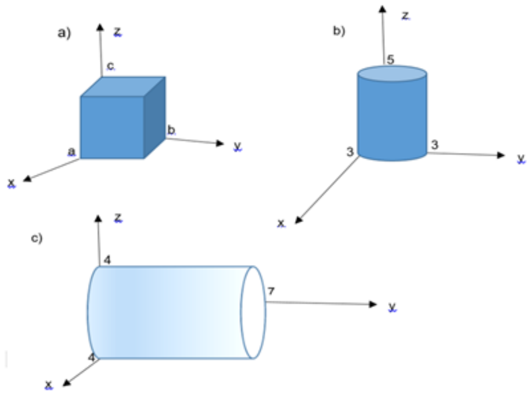
$$\int_{-1}^{0} \int_{0}^{1} xy\, dydx$$

(3) Use a double integral to find the area of the region bounded between the curves $y = x^2$ and $y = x$.

(4) Express the area of shaded rectangular region in the following figure by using the double integral.

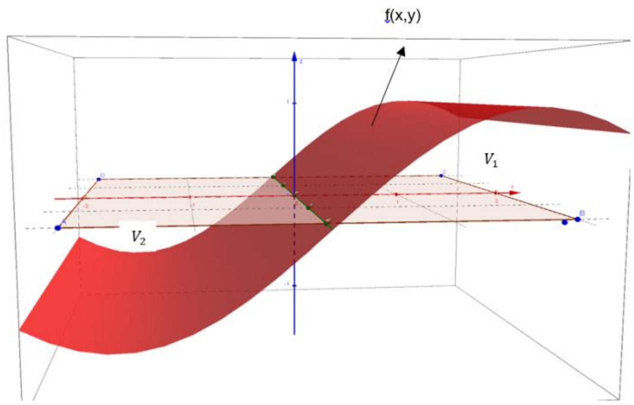


(5)



The points at which the solids listed above cut the xyz-planes are marked in the plane. Express the volume of these solids using the double integral.

(6)



The above figure shows the function $f(x, y)$ defined where $R = \{(x, y) \in [-2,2]x[-2,2]\}$. V_1 is the volume above R and below the graph of $f(x, y)$ and V_2 is the volume below R and above the graph. Accordingly, which statement provides the result of the integral given below?

$$\iint_B f(x,y)dA = ?$$

A) $V_1 + V_2$ B) $V_2 - V_1$ C) $V_1 - V_2$

D) $|V_1 - V_2|$ E) $\frac{1}{2}(V_1 + V_2)$