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RESEARCH PAPER

Three-dimensional fractional system with the stability condition and chaos control

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Abstract

A three-dimensional system is introduced in this paper and its local stability is analyzed. Our study establishes the validity and uniqueness of the linear feedback control for the proposed system and proves its existence and uniqueness. The numerical simulation algorithm described by Atanackovic and Stankovic is finally applied. The analytical results are analyzed and the dynamics of the system are explored in more detail.

Key words: Fractional-order system; stability; control; chaotic system
AMS 2020 Classification: 34A08; 34C23; 34H10

1 Introduction

Modeling and investigating actual phenomena can be accomplished with fractional dynamical systems. A dynamical system may exhibit chaos as one of its important dynamics. Dynamical chaos disappears when the fractional order falls below a threshold in a fractional-order chaotic system. There have been several articles discussing the minimum effective dimension below which the system remains chaotic, [1, 2, 3, 4, 5, 6, 7].

Chaos theory is a field of mathematics that has already attracted the attention of many researchers from different fields of science, engineering and medicine. Chaos theory describes the behavior of certain dynamical systems whose state evolves with time and are highly sensitive to initial conditions. Because of the complexity of chaotic behavior in dynamical systems, it finds applications in a variety of fields, such as science, technology and medicine [8, 9, 10, 11, 12, 13, 14]. Studying chaotic systems can be a very valuable endeavor. Sene [15] in his paper studied the applications of the fractional-order chaotic system in the sense of Caputo fractional derivative. The presence or absence of chaotic behaviors of their model was presented in terms of the Lyapunov exponents. For the model description, the circuit schematic was drawn and simulated. Naik et al. [16] in their paper studied the chaotic dynamics of a fractional-order cancer model. A detailed analysis of the equilibrium points was also considered. They also calculated the Lyapunov exponents that give the existence of chaotic behavior of the model.

Leibniz in 1695 was the first to introduce the fractional calculus followed by Liouville in 1834, Riemann in 1892 and others [17]. Fractional calculus represents the generalization of integrals and derivatives to non-integer order. After Leibniz fractional calculus has gained increasing popularity and finds applications in various fields of science, technology and medicine [18, 19, 20, 21, 22, 23]. Recently, Ozkose and Yavuz [24] in their paper studied in fractional-order case the relations between COVID-19 and diabetes diseases under the hereditary traits then validated their model by the real data from Turkey. The Adams-Bashforth-Moulton predictor-corrector method was employed for the numerical solution of their model. For the advantages of the fractional-order derivative, they considered the memory trace and hereditary traits in the model. Fractional model, similar electrode-electrolyte, electromagnetic, and wave models have been found to

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explain many systems in the physical, chemical, and biological processes.

The rest of the paper is decorated as: after the introduction in Section 1, Section 2 gives some preliminaries and discusses a chaotic three-dimensional system. Section 3 examines whether or not a proposed system solution exists and is unique. In Section 4, we introduced the stability conditions of the equilibrium points of the system. The linear feedback control system studied in Section 5 is based on the Routh-Hurwitz method. In Section 6, we present numerical simulations based on algorithmic methods and discuss the obtained results. Finally, in Section 7, we conclude the study.

2 Preliminaries

Many well known fractional derivatives including Riemann-Liouville, Grunwald-Letnikov as well as Caputo exist in the literature and are all common fractional derivative definitions. As a result, we investigate the fractional derivative of Caputo, as defined in [25]:

$$D^{\eta}g(\tau) = \frac{1}{\Gamma(m-\eta)}\int_0^t (\tau-\sigma)^{m-\eta-1}g^{(m)}(\sigma)d\sigma \tag{1}$$

$$= j^{m-\eta}\left(\frac{d^m}{d\tau^m}g(\tau)\right). \tag{2}$$

It is defined as follows: m is integer, $m-1<\eta<m$ and Γ is the Gamma function, and j^{θ} is Riemann-Liouville integral operator.

$$j^{\theta}g(\tau) = \frac{1}{\Gamma(\theta)}\int_0^t (\tau-\sigma)^{\theta-1}g^{(m)}(\sigma)d\sigma, \quad \tau>0. \tag{3}$$

Theorem 1 *[26] An autonomous linear system*

$$D^{\eta}x = Lx, \qquad x(0)=x_0,$$

where L is a $m\times m$ matrix and $0<\eta<1$ is asymptotically stable if and only if $|\arg(\mu)|>\frac{\eta\pi}{2}$ for all eigenvalues μ of L. The components of the solution $x(\tau)$ decay to zero in this case, each component of solution $x(\tau)$ decays toward or like $\tau^{-\eta}$. Also, this linear system is stable if and only if $|\arg(\mu)|\geq\frac{\eta\pi}{2}$ and those critical eigenvalues that satisfy $|\arg(\mu)|=\frac{\eta\pi}{2}$ that geometric multiplication is one. In Ref. [27], a chaotic system in three dimensions is described by:

$$\begin{cases} x_1'(\tau) = ax_1 - x_2x_3, \\ x_2'(\tau) = -bx_2 + x_1x_3, \\ x_3'(\tau) = -cx_3 + x_1^2, \end{cases} \tag{4}$$

where x_1,x_2,x_3 are state variables, $a,b,c\in\mathbb{R}^+$ are constant parameters. For $a=6, b=12, c=14$, the chaotic attractors for system (4) are displayed in Fig. 1.

Three equilibrium points exist in the system

$$O=(0,0,0), \quad Q_1=\left(\sqrt[4]{abc^2}, \sqrt[4]{a^3c^2/b}, \sqrt{ab}\right), \quad Q_2=\left(-\sqrt[4]{abc^2}, -\sqrt[4]{a^3c^2/b}, -\sqrt{ab}\right).$$

3 Solution's existence and uniqueness

Taking into account the initial value problem:

$$D^{\eta}W(\tau) = g(\tau, W(\tau)), \quad 0<\tau<\Omega, \quad W^{(k)}(0)=W_0^{(k)}, \quad k=0,1,\dots,m-1. \tag{5}$$

Theorem 2 *(Existence [28]) Let us consider*

$$E := [0, W^*]\times[W_0^{(0)}-\varepsilon, W_0^{(0)}+\varepsilon],$$

with some $W^*>0$ and some $\varepsilon>0$, and the function $g: E\to\mathbb{R}\,\chi := \min\{\chi^*, (\varepsilon\Gamma(\eta+1)/||g||_{\infty}^{\frac{1}{\eta}}\}$. Then, there exists a function $W:[0,\chi]\to\mathbb{R}$ which solves the initial value problem (5).

Theorem 3 *(Uniqueness [28]) Let us consider*

$$E := [0, \chi^*]\times[W_0^{(0)}-\varepsilon, W_0^{(0)}+\varepsilon],$$

with some $\chi^*>0$ and some $\varepsilon>0$. Therefore the function $f: E\to\mathbb{R}$ is surrounded on E and regard the second variable, meet the Lipschitz condition, i.e.,

$$|g(\tau,W)-g(\tau,V)|\leq J|W-V|,$$

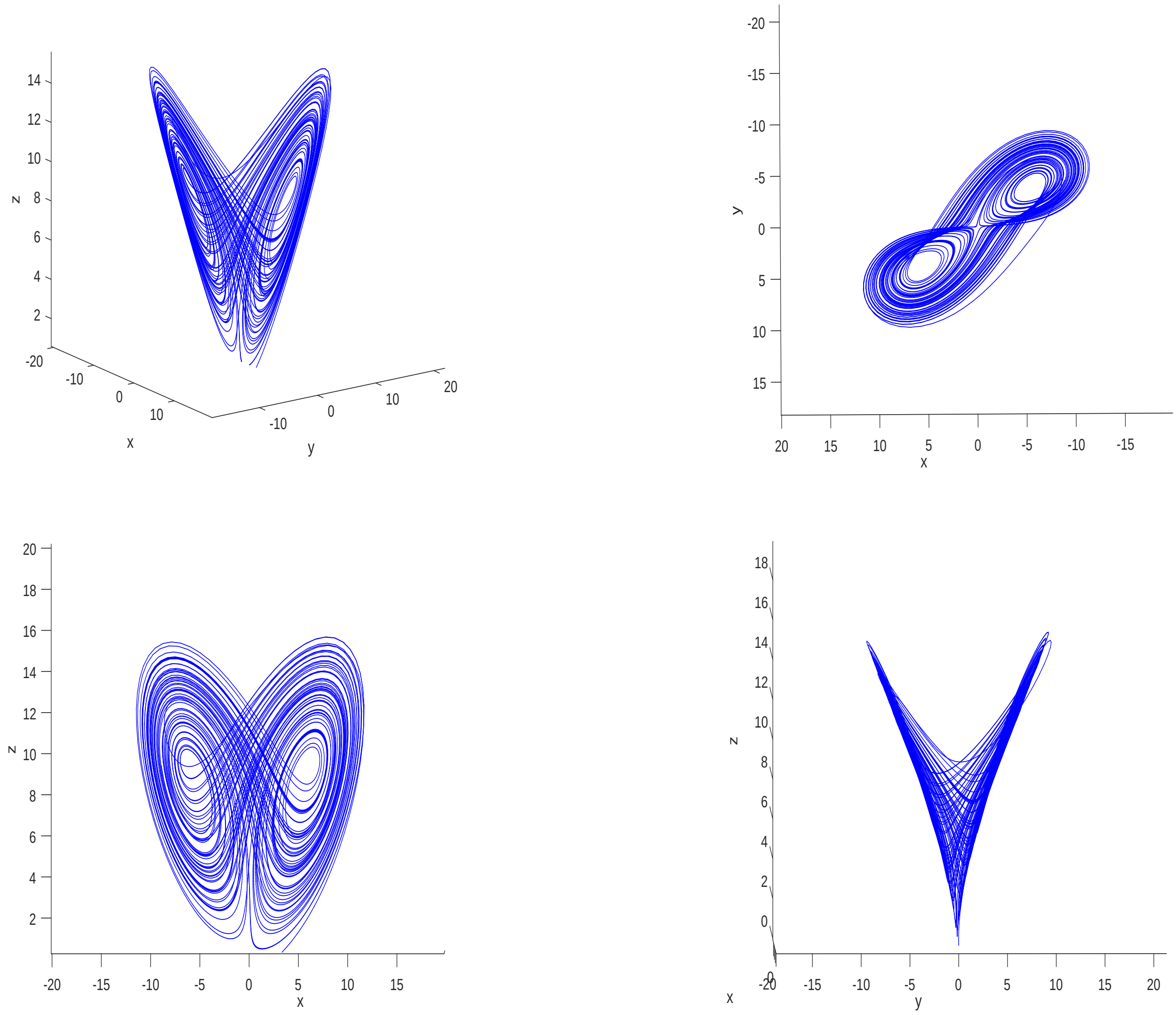

Figure 1. The chaotic attractors for system (4)

with constant $J > 0$ *independent* τ, *W and V. Then, describe* χ *as Theorem* (2), *there exists one function* $W : [0, \chi] \to \mathbb{R}$ *solving the initial value problem* (5).

Theorem 4 *In the fractional-order three-dimensional system* (4), *the initial value problem can be expressed as follows:*

$$D^{\eta}x(\tau) = Ax(\tau) + x_1(\tau)Bx(\tau) + x_2(\tau)Cx(\tau), \qquad x(0) = x_0, \tag{6}$$

where $0 < \tau < \Omega, x(\tau) = (x_1(\tau), x_2(\tau), x_3(\tau))^{\Omega} \in \mathbb{R}^3, x(0) = (x_{10}, x_{20}, x_{30})$

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If

$$\Omega := \min\{\Omega^*, (\varepsilon\Gamma(\eta + 1)/||g||_{\infty})^{\frac{1}{\eta}}\}, \quad \Omega^* > 0,$$

there exists a unique solution for (6).

Proof Consider

$$g(x(\tau)) = A\,x(\tau) + x_1(\tau)Bx(\tau) + x_2(\tau)Cx(\tau), \quad x(\tau) \in [0, \Omega^*] \times [x_0 - \varepsilon, x_0 + \varepsilon],$$

for any $\Omega^*, \varepsilon > 0$. Moreover, one has

$$\begin{aligned} |g(x) - g(y)| &= |A(x-y) + x_1Bx - y_1By + x_2Cx - y_2Cy| \\ &\leq ||A|||(x-y)| + |x_1Bx - y_1By| + |x_2Cx - y_2Cy|, \end{aligned} \tag{7}$$

where $y(\tau) \in [0, \Omega^*] \times [x_0 - \varepsilon, x_0 + \varepsilon]$, and $|\cdot|$ represents the matrix vector norm. Clearly

$$\begin{aligned} |x_1Bx - y_1By| &= |x_1Bx - y_1Bx + y_1Bx - y_1By| \\ &= |(x-x)-)Bx + y_1B(x-y)| \\ &\leq ||B||(|x| + |y_1|)(|x-y|). \end{aligned}$$

Likewise, one has the following result:

$$|x_2Cx - y_2Cy| \leq ||C||(|x| + |y_2|)(|x-y|).$$

Eq. (7) gives the result that

$$\begin{aligned} |g(x) - g(y)| &\leq [||A|| + ||B||(|x| + |y_1|) + ||C||(|x| + |y_2|)](|x-y|) \\ &\leq [||A|| + (||B|| + ||C||)(2|x_0| + 2\epsilon)](|x-y|) \\ &\leq L(||x-y||), \end{aligned}$$

where $J = ||A|| + (||B|| + ||C||)(2|x_0| + 2\varepsilon) > 0$. Hence, the fractional-order dimensional system is Lipschitz-satisfying. Then, compatible to the existence and uniqueness theorem of the fractional-order dimensional system the initial value problem of the commensurate order system (6) has a unique solution in the interval

$$\Omega := \min\{\Omega^*, (\varepsilon\Gamma(\eta+1)/||g||_\infty))^{\frac{1}{\eta}}\}.$$

■

4 Conditions for the stability of equilibrium points

The characteristic equation of system (4) is determined by

$$p(\mu) = \mu^3 + r_1\mu^2 + r_2\mu + r_3 = 0, \tag{8}$$

whose discriminant $D(p)$ is defined by

$$D(p) = R(p, p'), \tag{9}$$

and

$$D(p) = 18r_1r_2r_3 + r_1^2r_2^2 - 4r_3r_1^3 - 4r_2^3 - 27r_3^3. \tag{10}$$

If Δ_1, Δ_2 and Δ_3 are Routh-Hurwitz determinants $\Delta_1 = r_1$, $\Delta_2 = \begin{vmatrix} r_1 & 1 \\ r_3 & r_2 \end{vmatrix}$ and $\Delta_3 = r_3$. Thus we have the following stability conditions [28].
(I) If $D(p) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 = a_3$, then the equilibrium point is locally asymptotically stable for all $\eta \in (0, 1)$.
(II) The condition $r_3 > 0$, is the necessary condition for the equilibrium point to be locally asymptotically stable.

Some stability conditions for the equilibrium points Q_0, Q_1 and Q_2

The characteristic polynomial of equilibrium point Q_0 is given by:

$$p(\mu) = \mu^3 + \mu^2(c + b - a) + \mu(bc - ac) - abc. \tag{11}$$

It is clear that $r_3 = -abc < 0$, thus applying the stability condition (II) to characteristic equation (11) implies that E_0 in unstable. Similary, the equilibrium point Q_1 and Q_2 have the same characteristic polynomial, which given as:

$$p(\mu) = \mu^3 + (c + b - a)\mu^2 + (ac + bc)\mu + 4abc = 0. \tag{12}$$

Thus, applying the Routh-Hurwitz conditions and the necessary stability condition part (I), imply that equilibrium points Q_1, Q_2 are unstable.

5 Linear feedback control of the chaotic system

Here, the control of fractional three-dimensional chaotic system (4) is discussed by using the linear feedback control. The controlled fractional order chaotic system (4) is given by:

$$\begin{cases} D^{\eta}x_1(t) = ax_1 - x_2x_3 - k_1(x_1 - \bar{x}_1) \\ D^{\eta}x_2(t) = -bx_2 + x_1x_3 - k_2(x_2 - \bar{x}_2) \\ D^{\eta}x_3(t) = -cx_3 + x_1^2 - k_3(x_3 - \bar{x}_3), \end{cases} \tag{13}$$

where (k_1, k_2, k_3) are feedback control and $k_1, k_2, k_3 > 0$ and by suitable choice of feedback control according to stability conditions (I, II), we can drive the system (13) trajectories to unstable equilibrium point Q_1.

Controlling chaos for the equilibrium point Q_1

In this section, we apply stability condition of chaotic system to study chaos control. For this, we obtain the characteristic equation of the controlled system (13) evaluated at the equilibrium point by:

$$\mu^3 + \mu^2(s_1 + s_2 + k_2 + b) + \mu(s_1s_2 + 2bs_2 + k_2s_1 + k_2s_2) + (s_1s_2)k_2 + bs_1(c + k_2), \tag{14}$$

where $s_1 = (k_1 - a)$ and $s_2 = (k_3 + c)$. By applying the Routh-Hurwitz conditions (I, II) to equilibrium point (13) we find that: k_1, k_2, k_3, are all positive and defined by equation (13).
Furthermore, the inequality is enough conditions for stabilizing the controlled system (13) to the equilibrium point Q_1 and Q_2: In the system (13) we consider the fixed parameters $a = 6$, $b = 12$, $c = 4$ and using the feedback control gains $(k_1, k_2, k_3) = (2.88, 1.33, 0.52)$.
For the above mentioned value parameters and feedback control gains, it becomes clear that the trajectories of controlled system (13) with fractional order η converge to the equilibrium point Q_1.
Fig. 2 shows the trajectories of controlled fractional system for $\eta = 0.98$, which converges to the equilibrium point Q_1.
However, when $\eta = 1$ the controlled system (13) is not stable near equilibrium point Q_1 (see Fig. 3).

6 Numerical simulations and discussion

For solving system (4), we employ a numerical technique developed by Atanackovic and Stankovic [29] to solve the fractional differential equation, and we depict trajectories of system (4) using the well known Runge-Kutta method of order fourth for parameters $a = 6$, $b = 12$, $c = 4$. We have the equilibrium point $Q_1 = (135.764, 67.882, 8.485)$ and $Q_2 = (-135.764, -67.882, -8.485)$.
Fig. 1 presents numerical values for system (4) for different values. An analysis of the local stability of a three-dimensional system is presented in this paper. This study aims to verify the validity and uniqueness of linear feedback control for the proposed system and prove its existence and uniqueness. An algorithm is finally applied to the numerical simulation.

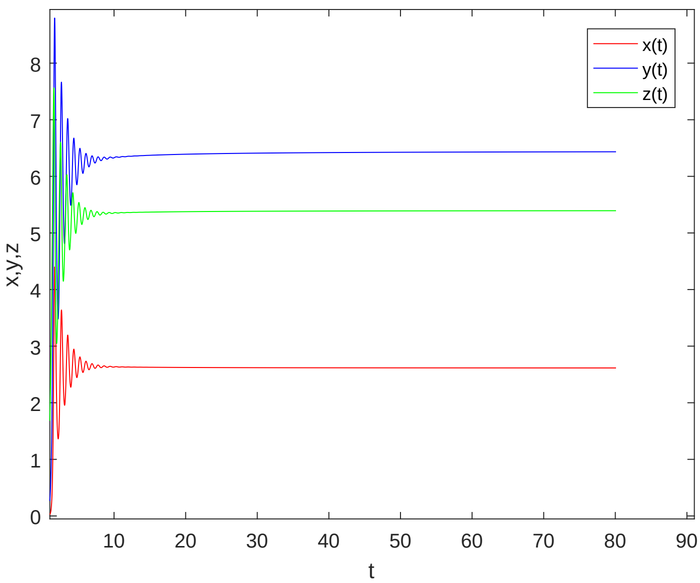

Figure 2. The trajectories of the controlled system (13) by $\eta = 0.98$

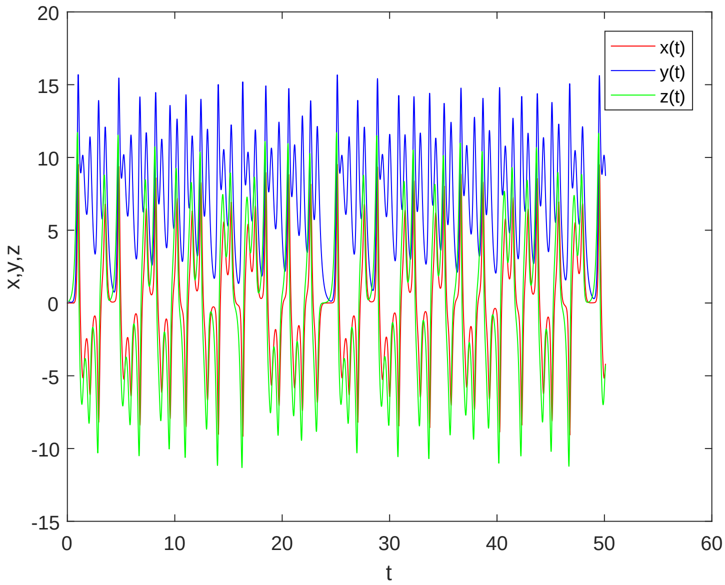

Figure 3. The trajectories of the controlled system (13) by $\eta = 1$

7 Conclusions

In the present paper, we examined a three-dimensional fractional order chaotic system. The conditions that ensure the existence and uniqueness of its solution were identified. We employed Routh-Hurwitz method to determine the stability conditions. We also employed the feedback control of the chaotic system with fractional order. Through the numerical simulations, the performance and authenticity of the proposed method were presented. The trajectories of the model (4) through the well known Runge-Kutta method of order fourth were depicted. It is concluded from the obtained results that the fractional power of the derivative has a significant effect on the dynamic process. Also, it is observed that the smaller fractional power of the derivative is the chaotic behavior of the system.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interests.

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Author's contributions

M.G.: Conceptualization, Methodology, Software, Writing-Original draft. R.K.G.: Conceptualization, Methodology, Supervision, Investigation. Z.E.: Software, Methodology, Data Curation. All authors discussed the results and contributed to the final manuscript.

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