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RESEARCH PAPER

Analysis of fractional-order vaccinated Hepatitis-B epidemic model with Mittag-Leffler kernels

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Abstract

The current paper investigates a newly developed model for Hepatitis-B infection in sense of the Atangana-Baleanu Caputo (ABC) fractional-order derivative. The proposed technique classifies the population into five distinct categories, such as susceptible, acute infections, chronic infections, vaccinated, and immunized. We obtain the Ulam-Hyers type stability and a qualitative study of the corresponding solution by applying a well-known principle of fixed point theory. Furthermore, we establish the deterministic stability of the proposed model. For the approximation of the ABC fractional derivative, we use a newly proposed numerical method. The obtained results are numerically verified by MATLAB 2020a.

Key words: Fractional calculus; fractional-order model; hepatitis-B disease; ABC derivative; fixed-point theorem; numerical simulation

AMS 2020 Classification: 34A08; 34D20; 34K60; 92C50; 92D30

1 Introduction

Many pandemics and endemics around the world are first explored using a mathematical model based on data from various hospitals. These models explain the human disease origin, current development, and forecast. Mathematical formulations can then be used by researchers and scholars to discover the treatment or cure. The treatment may come in the form of precaution or vaccination for affected individuals of a certain population. Thus, vaccinations against several diseases such as pertussis, measles, polio, Hepatitis-B and influenza have been given and have led to healthy recovery, as mentioned in [1, 2, 3, 4, 5, 6]. Numerous mathematical models for various diseases have been developed recently, including stochastic, deterministic, and difference equation systems with several vaccination parameters for diagnosed infections, as seen in [5]. Among the most serious diseases is Hepatitis-B, which is transmitted by infected individuals. More than one million people have died worldwide due to this outbreak. [7, 8] shows that around 200 million individuals were infected by the pandemic and that three and a half billion people were in a chronic condition. A combination of long-term planning and frequent vaccinations can reduce the spread of the disease in the population in the case of a major epidemic [8]. Strong immunization and dose for the infected persons will rapidly decrease the cases of HBV. To investigate the qualitative analysis Hepatitis-B disease, we consider [9] model as below:

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$$
\begin{aligned}
\acute{\Omega}(\sigma) &= \Pi\Lambda - \frac{\gamma\Omega(\sigma)\mathcal{J}^{a}(\sigma)}{N} + \lambda\mathcal{H}(\sigma) - (\delta+\mu)\Omega(\sigma),\\
\acute{\mathcal{J}}^{a}(\sigma) &= \frac{\gamma\Omega(\sigma)\mathcal{J}^{a}(\sigma)}{N} - (\mu+\alpha+\kappa)\mathcal{J}^{a}(\sigma),\\
\acute{\mathcal{J}}^{c}(\sigma) &= \kappa\mathcal{J}^{a}(\sigma) - (\mu+\rho+\eta)\mathcal{J}^{c}(\sigma),\\
\acute{\mathfrak{R}}(\sigma) &= \alpha\mathcal{J}^{a}(\sigma) + \eta\mathcal{J}^{c}(\sigma) - \mu\mathfrak{R}(\sigma),\\
\acute{\mathcal{H}}(\sigma) &= \Lambda(1-\Pi) + \delta\Omega(\sigma) - (\lambda+\mu)\mathcal{H}(\sigma),
\end{aligned}
\tag{1}
$$

where the parameters and variables in the above equation are listed below:

- vaccinated cases, susceptible population, acute infection cases, chronic carriers cases and immunized cases have been represented by $\mathcal{H}(\sigma)$, $\Omega(\sigma)$, $\mathcal{J}^{a}(\sigma)$, $\mathcal{J}^{c}(\sigma)$ and $\mathfrak{R}(\sigma)$ respectively.
- Λ: Recruitment/Birth rate.
- λ: Waning vaccine-induced immunity.
- μ: Natural death rate.
- ρ: The HBV death rate.
- γ: Contact rate among infected and non-infected individuals.
- α: The rate of recovery for infected individuals.
- η: The rate of recovery of individuals infected chronically.
- κ: The rate at which acute cases are transformed into chronic cases.
- Π: Ratio of new-borns who have not received proper immunization.
- δ: Hepatitis immunization rate.

Riemann-Liouville, Euler, and Fourier made significant contributions to the development of ordinary calculus in the 18th century. At the time, many authors made significant contributions to the field of fractional calculus (FC), see [10, 11, 12]. This is due to the fact that ordinary calculus lacks the applications of modern calculus in many mathematical modeling domains, such as the process of memory and hereditary data. FC, as a general form of integer order calculus, has significantly larger freedom in their derivative than is found in integer-order derivative due to its local behavior. Several applications of FC are discussed in [13, 14, 15, 16, 17, 18, 19, 20, 21]. Researchers and scientists have become more interested in analyzing non-integer order (FO) of differential and integral calculus because of its applications. Non-local and non-singular concepts were introduced in FC articles, replacing singular and local kernels with these new concepts. The most useful feature of this newly developed kernel is its memory property combined with the system's hereditary. Atangana, Baleanu, and Caputo (ABC) [13] proposed a novel FO operator, in 2016, depending on the general non-local and non-singular kernel of Mittag-Leffler (ML) mapping. As seen in [22, 23, 24, 25, 26], the ABC order fractional order operator has been used in many mathematical schemes describing different physical problems. More specifically, this generalized ML function is a precise tool to deal with real word problems.
Scientists and researchers across a wide range of fields are working to stop or slow down the spread of these diseases, as evidenced by these references [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44]. Many authors have investigated various models for a number of diseases; for a detailed study see [45, 46, 47, 48, 49, 50]. Epidemic models by the researcher are widely used nowadays to investigate the dynamic spreading of disease and to follow the efficient method for its controlling. Ordinary differential equation with integer-order derivative could be generalized to the fractional-order larger degree freedom derivative of fractional order. Also FO differential equations with $0 < \wp \leq 1$ have been studied in [26]. Generally, biological models and FDEs are theoretically related to memory-based systems [23, 25, 50]. Furthermore, the history factor is a major source of disease transmission. In the coming days, the development of totality and historical effects on existing levels will be a source of transmission. Heterogeneity and historical effect show the spread of the previous infections. Thus, the mentioned properties can be examined via fractional derivatives, as well as their effect on disease transmission [39, 50].
The purpose of this work is to investigate the dynamic behavior of the fractional HBV epidemic model's solutions (2). We established the stability and equilibria analysis for the given system based on the diseases free equilibrium point. We additionally addressed some basic principles and gave theoretical solutions. Furthermore, we simulate the unknown quantities to verify and explain the fractional order mathematical model. We note from the existing research that limited study has been done on non-integer order epidemic models using ABC derivatives. There has not been enough research done on the ML kernel-based arbitrary order HBV vaccinated model. Thus, the primary motivation for this study is to develop a HBV FO-vaccinated system. Under the ABC derivative with $\wp \in (0,1]$ we reexamine the HBV model (1) in the following fractional form

$$
\begin{cases}
{}^{ABC}D^{\wp}\,\Omega(\sigma) = \Pi\Lambda - \dfrac{\wp\Omega(\sigma)\mathcal{J}^{a}(\sigma)}{N} + \lambda\mathcal{H}(\sigma) - (\delta+\mu)\,\Omega(\sigma),\\
{}^{ABC}D^{\wp}\,\mathcal{J}^{a}(\sigma) = \dfrac{\wp\Omega(\sigma)\mathcal{J}^{a}(\sigma)}{N} - (\mu+\alpha+\kappa)\,\mathcal{J}^{a}(\sigma),\\
{}^{ABC}D^{\wp}\,\mathcal{J}^{c}(\sigma) = \kappa\mathcal{J}^{a}(\sigma) - (\mu+\rho+\eta)\,\mathcal{J}^{c}(\sigma),\\
{}^{ABC}D^{\wp}\,\mathfrak{R}(\sigma) = \alpha\mathcal{J}^{a}(\sigma) + \eta\mathcal{J}^{c}(\sigma) - \mu\mathfrak{R}(\sigma),\\
{}^{ABC}D^{\wp}\,\mathcal{H}(\sigma) = \Lambda\,(1-\Pi) + \delta\Omega(\sigma) - (\lambda+\mu)\,\mathcal{H}(\sigma).
\end{cases}
\tag{2}
$$

with the initial conditions

$$
S(0) \geq 0,\quad \mathcal{J}^{a}(0) \geq 0,\quad \mathcal{J}^{c}(0) \geq 0,\quad \mathfrak{R}(0) \geq 0,\quad \mathcal{H}(0) \geq 0. \tag{3}
$$

Our remaining paper has the following arrangement as follows. In Section Fundamental we recall some basic results of fractional calculus.

In this way, we manage the remainder of the contents of the current examination as follows. During Section 2, essential definitions are provided. The existence theory of model and stability is given in Section 3. Also, numerical methods to solve the considered problems are given in Section 4. Indeed, numerical results of the proposed models under practising different values of fractional orders are supplied in section Section 5. Finally, the conclusion of the current investigation can be observed in Section 6.

2 Preliminaries

In this section, we give some important definitions which we will use in the rest of the paper [23, 25, 26, 50].

Definition 1 *Let $\xi(\sigma)$ be a function satisfying $\xi(\sigma) \in \mathbb{H}^1[0, T]$ the $\mathbb{ABC}$ fractional derivative of order $0 \leq \ell \leq 1$ and is defined by*

$$^{ABC}D_t^{\wp}(\xi(\sigma)) = \frac{\mathbb{M}(\wp)}{1-\wp}\int_0^t \mathbf{E}_{\wp}\left[\frac{-\wp}{1-\wp}\left(t-z\right)^{\wp}\right]\frac{d}{dz}\xi(z)dz, \tag{4}$$

where $\mathbb{M}(\wp) = \frac{\wp}{2-\wp}$ is the normalization constant, $\mathbb{M}(0) = \mathbb{M}(1) = 1$ and $\mathbf{E}_{\wp}$ is the Mittag-Leffler operator given by

$$\mathbf{E}_{\wp}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\wp k+1)}.$$

Theorem 1 *[51] The Atangana-Baleanu fractional differential equation*

$$^{ABC}D_t^{\wp}\xi(\sigma) = f(\sigma)$$

has a unique solution in the form

$$\xi(\sigma) = \frac{1-\wp}{\mathbb{M}(\wp)}f(\sigma) + \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp)}\int_0^{\sigma} f(\vartheta)(\sigma-s)^{\wp-1}d\vartheta. \tag{5}$$

Definition 2 *[52] Let*

$$\begin{cases} ^{ABC}D_0^{\wp}\xi(\sigma) = f(t, \xi(\sigma)), \\ \xi(0) = \xi_0 \end{cases}$$

is a non-linear fractional ordinary differential equation. The new formula for the numerical scheme of the ABC fractional derivative can be written as

$$\begin{aligned} \xi_{m+1} = \xi_0 + \frac{1-\wp}{\mathbb{M}(\wp)}f(\xi(t_m), t_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\Bigg\{\frac{h^{\wp}f(\xi_n, t_n)}{\Gamma(\wp+2)}[(m+1-n)^{\wp}(m-n+2+\wp) \\ -(m-n)^{\wp}(m-n+2+2\wp)] - \frac{h^{\wp}f(\xi_s, t_s)}{\Gamma(\wp+2)}[(m+1-n)^{\wp+1} - (m-n)^{\wp}(m-n+1+\wp)] + E_m^{\wp}\Bigg\}, \end{aligned} \tag{6}$$

where $E_m^{\wp}$ is given by

$$E_m^{\wp} = \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp)}\sum_{k=0}^{m}\int_{t_n}^{t_n-1}\frac{(y-t_n)(y-\sigma_{n-1})}{2}\frac{\partial^2}{\partial y^2}[f(\xi(y), y)]_{y=\lambda_y}(\sigma_{m+1}-y)^{\wp-1}dy. \tag{7}$$

Theorem 2 *Let B be a convex subset of Z and suppose that the two operators Υ_1, Υ_2 with*
(1). $\Upsilon_1 u + \Upsilon_2 u \in \mathbb{B}$ for each $u \in B$.
(2). Υ_1 is "contraction".
(3). A continuous and compact set is Υ_2.
satisfying the operator equation $\Upsilon_1 u + \Upsilon_2 u = u$, has one or more solution(s).

3 Existence theory of model (2)

In this part, we established the existence and uniqueness of the solution for the proposed system (2). We find the solution and stability of the proposed model under ABC derivative with FO using Banach fixed point principles. We rearrange the proposed model in the following way

$$\begin{cases} \aleph_1(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) = \Pi\Lambda - \frac{\wp\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} + \lambda\mathcal{H}(\sigma) - (\delta+\mu)\Omega(\sigma), \\ \aleph_2(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) = \frac{\wp\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} - (\mu+\alpha+\kappa)\mathcal{J}^a(\sigma), \\ \aleph_3(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) = \kappa\mathcal{J}^a(\sigma) - (\mu+\rho+\eta)\mathcal{J}^c(\sigma), \\ \aleph_4(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) = \alpha\mathcal{J}^a(\sigma) + \eta\mathcal{J}^c(\sigma) - \mu\mathfrak{R}(\sigma), \\ \aleph_5(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) = \Lambda(1-\Pi) + \delta\Omega(\sigma) - (\lambda+\mu)\mathcal{H}(\sigma). \end{cases} \tag{8}$$

For $0 < \wp \leq 1$, using (8) we write the proposed model in the following form

$$\begin{aligned} {}^{ABC}D^{\wp}_{+0}\mathcal{H}(\sigma) &= \aleph(\sigma, v(\sigma)), \\ v(0) &= v_0. \end{aligned} \tag{9}$$

By Theorem 1, the system (9) becomes

$$v(\sigma) = v_0(\sigma) + \Big[\aleph((\sigma, v(\sigma)) - \aleph_0(\sigma)\Big]\frac{1-\wp}{M(\wp)} + \frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta, \tag{10}$$

where

$$v(\sigma) = \begin{cases} \Omega(\sigma) \\ \mathcal{J}^a(\sigma) \\ \mathcal{J}^c(\sigma) \\ \mathfrak{R}(\sigma) \\ \mathcal{H}(\sigma) \end{cases}, v_0(\sigma) = \begin{cases} \Omega_0 \\ \mathcal{J}^a_0 \\ \mathcal{J}^c_0 \\ \mathfrak{R}_0 \\ \mathcal{H}_0 \end{cases}, \aleph(\sigma, v(\sigma)) = \begin{cases} \aleph_1(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) \\ \aleph_2(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) \\ \aleph_3(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) \\ \aleph_4(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) \\ \aleph_5(\sigma, \Omega, \mathcal{J}^a, \mathcal{J}^c, \mathcal{H}, \mathfrak{R}) \end{cases}, \aleph_0(\sigma) \begin{cases} \aleph_1(0, \Omega_0, \mathcal{J}^a_0, \mathcal{J}^c_0, \mathcal{H}_0, \mathfrak{R}_0) \\ \aleph_2(0, \Omega_0, \mathcal{J}^a_0, \mathcal{J}^c_0, \mathcal{H}_0, \mathfrak{R}_0) \\ \aleph_3(0, \Omega_0, \mathcal{J}^a_0, \mathcal{J}^c_0, \mathcal{H}_0, \mathfrak{R}_0) \\ \aleph_4(0, \Omega_0, \mathcal{J}^a_0, \mathcal{J}^c_0, \mathcal{H}_0, \mathfrak{R}_0) \\ \aleph_5(0, \Omega_0, \mathcal{J}^a_0, \mathcal{J}^c_0, \mathcal{H}_0, \mathfrak{R}_0) \end{cases}. \tag{11}$$

Using (10) and (11), define two operators Υ_1 and Υ_2, using (10)

$$\begin{aligned} \Upsilon_1 u &= v_0(\sigma) + \Big[\aleph(\sigma, v(\sigma)) - \aleph_0(\sigma)\Big]\frac{1-\wp}{M(\wp)}, \\ \Upsilon_2 u &= \frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta. \end{aligned} \tag{12}$$

Next, we have to determine the qualitative analysis for the proposed model by using fixed point principle.
(L_1) there exist some constants ϵ_1 and ϵ_2,

$$|\aleph(\sigma, v(\sigma))| \leq \epsilon_1 |v(\sigma)| + \epsilon_2.$$

(L_2) there exists a positive constant K_p, for each $u, u_1 \in \mathbb{X}$,

$$|\aleph(\sigma, v(\sigma)) - \aleph(\sigma, v_1(\sigma))| \leq K_p\|u - u_1\|.$$

Theorem 3 *The system* (10) *has at least one solution, if* (L_1) *and* (L_2) *hold, then the proposed system* (2) *also has a unique solution if*

$$\frac{(1-\wp)K_p}{M(\wp)} < 1.$$

Proof First we have to show that Υ_1 is contraction by using Banach contraction principle. Let $u_1 \in \mathbf{B}, : \mathbf{B} = \{u \in \mathfrak{Z} : ||u|| \leq r, r > 0\}$ be a closed convex set. From the operator Υ_1 defined in (12), we have

$$\begin{aligned} \|\Upsilon_1 u - \Upsilon_1 u_1\| &= \frac{(1-\wp)}{M(\wp)}\max_{\sigma\in[0,T]}\Big|\aleph(\sigma, v(\sigma)) - \aleph(\sigma, v_1(\sigma))\Big|, \\ &\leq \frac{(1-\wp)p}{M(\wp)}\|u - u_1\|. \end{aligned} \tag{13}$$

Hence, the operator Υ_1 is closed and therefore contraction.
Next, we have to show that the operator Υ_2 is compact, continuous and bounded. Also, obviously the operator Υ_2 is defined on all domains, so $\aleph$ is continuous. Let $u \in \mathbb{B}$, we have

$$\begin{aligned} |\Upsilon_2(u)| &= \max_{\sigma\in[0,T]}\frac{\wp}{M(\wp)\Gamma(\wp)}\left\|\int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta\right\|, \\ &\leq \frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^{\sigma}(\sigma - s)^{\wp-1}|\aleph(\vartheta, v(\vartheta))|d\vartheta, \\ &\leq \frac{T^{\wp}}{M(\wp)\Gamma(\wp)}[\epsilon_1 r + \epsilon_2]. \end{aligned} \tag{14}$$

So by (14) the operator Υ_2 is bounded. For equi-continuous, $\sigma_1 > \sigma_2 \in [0, T]$, such that

$$\begin{aligned} |\Upsilon_2 v(\sigma_1) - \Upsilon_2 v(\sigma_2)| &= \frac{\wp}{M(\wp)\wp(\wp)}\left|\int_0^{\sigma_1}(\sigma_1 - s)^{\wp-1}\aleph(y), v(y)d\vartheta - \int_0^{\sigma_2}(\sigma_2 - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta\right|, \\ &\leq \frac{[\epsilon_1 r + \epsilon_2]}{M(\wp)\Gamma(\wp)}[\sigma_1^{\wp} - \sigma_2^{\wp}]. \end{aligned} \tag{15}$$

As $\sigma_1 \to \sigma_2$, R.H.S of (15) tends to 0. Also, by continuous operator Υ_2 we have

$$|\Upsilon_2 v(\sigma_1) - \Upsilon_2 v(\sigma_2)| \to 0, \ \ as \ \ \sigma_1 \to \sigma_2.$$

Hence we proved that Υ_2 is continuous and bounded. So Υ_2 is also uniformly continuous. Using "Arzela$'$-Ascoli theorem", we have that Υ_2 is relatively compact and therefore completely continuous. By (3) and (10) it is easy to obtain that the system has at least one solution.

Uniqueness of the solution

Theorem 4 *Assume* (L_2) *and the integral form* (10) *has a unique solution. Then the system* (2) *has also a unique solution if*

$$\left[\frac{(1-\wp)K_p}{M(\wp)} + \frac{T^{\wp}K_p}{M(\wp)\Gamma(\wp)}\right] < 1.$$

Proof Let the operator $\mathcal{T} : \mathfrak{Z} \to \mathfrak{Z}$ be defined by

$$\mathcal{T}v(\sigma) = v_0(\sigma) + \Big[\aleph(\sigma, v(\sigma)) - \aleph_0(\sigma)\Big]\frac{1-\wp}{M(\wp)} + \frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta, \ \ \sigma \in [0, T]. \tag{16}$$

and $u, u_1 \in \mathfrak{Z}$, then

$$\begin{aligned} ||\mathcal{T}u - \mathcal{T}u_1|| \leq & \frac{(1-\wp)}{M(\wp)}\max_{\sigma\in[0,T]}\Big|\aleph(\sigma, v(\sigma)) - \aleph(\sigma, v_1(\sigma))\Big|, \\ & + \frac{\wp}{M(\wp)\Gamma(\wp)}\max_{\sigma\in[0,T]}\Big|\int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta - \int_0^{\sigma}(\sigma - s)^{\wp-1}\aleph(\vartheta, v_1(\vartheta))d\vartheta\Big|, \\ & \leq \left[\frac{(1-\wp)K_p}{M(\wp)} + \frac{\wp T^{\wp}K_p}{M(\wp)\Gamma(\wp)}\right]||u - u_1||, \\ & \leq \Theta||u - u_1||, \end{aligned} \tag{17}$$

where

$$\Theta = \left[\frac{(1-\wp)K_p}{M(\wp)} + \frac{T^{\wp}K_p}{M(\wp)\Gamma(\wp)}\right]. \tag{18}$$

By (17), the operator $\mathcal{T}$ is contraction. Therefore, the equation (10) has a unique solution. Consequently, the proposed system (2) has also a unique solution.

Ulam-Hyers stability

Next, we obtain the stability of the proposed system, consider small change $\varphi \in C[0, T]$ satisfying $0 = \wp(0)$, we have
(1) $|\varphi(\sigma)| \leq \xi$, *for* $\xi > 0$
(2) ${}^{ABC}D^{\wp}_{+0}(v(\sigma)) = \aleph(\sigma, v(\sigma)) + \varphi(\sigma)$, for all $\sigma \in [0, T]$.

Lemma 1 *The solution to the changed problem can be expressed by*

$$\begin{cases} {}^{ABC}_{0}D^{\wp}_{+0}v(\sigma) = \aleph(\sigma, v(\sigma)) + \varphi(\sigma), \\ v(0) = v_0, \end{cases} \tag{19}$$

satisfying

$$\left|v(\sigma) - \left(v_0(\sigma) + \Big[\aleph(\sigma, v(\sigma)) - \aleph_0(\sigma)\Big]\frac{1-\wp}{M(\wp)} + \frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^{\sigma}(\sigma - \vartheta)^{\wp-1}\aleph(\vartheta, v(\vartheta))d\vartheta\right)\right| \leq \wp_{T,\wp}\xi, \tag{20}$$

where

$$\wp_{T,\wp} = \frac{\Gamma(\wp)(1-\wp) + T^{\wp}}{M(\wp)\Gamma(\wp)}.$$

Proof The proof is obvious, therefore the details are omitted.

Theorem 5 *Consider* (L_2) *together with equation* (20), *the solution of equation* (10) *is UH stable and hence, the analytical solution for the proposed system is UH stable for* $\Theta < 1$.

Proof Let $u_1 \in \mathfrak{Z}$ be a unique solution and u $\in \mathfrak{Z}$ be any solution of equation (10), we have

$$\begin{aligned} |\nu(\sigma)-\nu_1(\sigma)| &= \left|\nu(\sigma)-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu_1(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu_1(\vartheta))d\vartheta\right)\right|, \\ &\le \left|\nu(\sigma)-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu(\vartheta))d\vartheta\right)\right| \\ &+\left|\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu(\vartheta))d\vartheta\right)\right. \\ &\left.-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu_1(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu_1(\vartheta))d\vartheta\right)\right|, \\ &\le \xi\wp_{T,\wp}+\frac{(1-\wp)K_p}{M(\wp)}||u-u_1||+\frac{\wp T^\wp K_p}{M(\wp)\Gamma(\wp)}||u-u_1|| \\ &\le \xi\wp_{T,\wp}+\Theta||u-u_1||. \end{aligned} \tag{21}$$

From (21), we can write

$$||\nu-\nu_1|| \le \frac{\xi\wp_{T,\wp}}{1-\Theta}. \tag{22}$$

From (22), we obtained that the solution of (10) is Ulam-Hyers stable and hence by considering $\aleph_\nu(\xi)=\wp_{T,\wp}\xi$, $\aleph_\nu(0)=0$ the solution is generalized Ulam-Hyers Stable. This proves that the solution of the considered model is Ulam-Hyers stable and also generalized Ulam-Hyers stable.

Now we postulate the assumptions given below
(1) $|\varphi(\sigma)| \le \nabla(\sigma)\xi$, *for* $\xi > 0$
(2) ${}^{ABC}D^{\wp}_{+0}(\nu(\sigma)) = \aleph(\sigma,\nu(\sigma))+\varphi(\sigma)$, for all $\sigma \in [0,T]$.

Lemma 2 *The next equation will satisfy* (19)

$$\begin{aligned} &\left|\nu(\sigma)-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu(\vartheta))d\vartheta\right)\right| \\ &\le \nabla(\sigma)\xi\wp_{T,\wp}. \end{aligned} \tag{23}$$

Proof The proof is obvious, therefore the details are omitted.

Theorem 6 *By Lemma* (2)*, the solution to the considered system is Ulam-Hyers-Rassias (UHR) stable and hence, the generalized UHR stable.*

Proof Let $u_1 \in \mathfrak{Z}$ be a unique solution and $u \in \mathfrak{Z}$ be a solution of (10), we have

$$\begin{aligned} |\nu(\sigma)-\nu_1(\sigma)| &= \left|\nu(\sigma)-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu_1(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu_1(\vartheta))d\vartheta\right)\right|, \\ &\le \left|\nu(\sigma)-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu(\vartheta))d\vartheta\right)\right| \\ &+\left|\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu(\vartheta))d\vartheta\right)\right. \\ &\left.-\left(\nu_0(\sigma)+\left[\aleph(\sigma,\nu_1(\sigma))-\aleph_0(\sigma)\right]\frac{1-\wp}{M(\wp)}+\frac{\wp}{M(\wp)\Gamma(\wp)}\int_0^\sigma(\sigma-\vartheta)^{\wp-1}\aleph(\vartheta,\nu_1(\vartheta))d\vartheta\right)\right|, \\ &\le \nabla(\sigma)\xi\wp_{T,\wp}+\frac{(1-\wp)K_p}{M(\wp)}||u-u_1||+\frac{\wp T^\wp K_p}{M(\wp)\Gamma(\wp)}||u-u_1||, \\ &\le \nabla(\sigma)\xi\wp_{T,\wp}+\Theta\|u-u_1\|. \end{aligned} \tag{24}$$

From (24), we get

$$\|\nu-\nu_1\| \le \frac{\nabla(\sigma)\xi\wp_{T,\wp}}{1-\Theta}. \tag{25}$$

Therefore, the solution of (10) is stable.

4 Numerical solution of the proposed model

Numerous numerical techniques have been suggested for the approximation of the fractional derivative of Atangana-Baleanu in the sense of Caputo. In [53, 54] Atangana and Owolabi proposed a new form of the Adams-Bashforth approach based on the Mittag-Leffler kernel for the ABC fractional derivative approximation. The purpose of this section of the article is to demonstrate how to apply the numerical technique described in [52], which has recently been proven for accuracy and reliability [55, 56] to solve any fractional differential equation. The numerical technique for approximating ABC is defined in (2). To learn more about this numerical approach in detail, we suggest our readers to [52].
To determine the approximate solution to the given model, we use theorem (1) for each of $\Omega(\sigma)$, $\mathcal{J}^a(\sigma)$, $\mathcal{J}^c(\sigma)$, $R^\nu(\sigma)$, $V^\nu(\sigma)$ of the system (2) and obtain the following result:

$$
\begin{aligned}
\Omega(\sigma) &= S(0) + \frac{1-\wp}{\mathbb{M}(\wp)}\left[\Pi\Lambda - \frac{\gamma\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} + \lambda V(\sigma) - (\delta+\mu)\,\Omega(\sigma)\right] \\
&+ \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp}\int_0^\sigma(\sigma-y)^{\wp-1}\left[\Pi\Lambda - \frac{\gamma\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} + \lambda V(\sigma) - (\delta+\mu)\,\Omega(\sigma)\right]dy, \\
\mathcal{J}^a(\sigma) &= \mathcal{J}^a(0) + \frac{1-\wp}{\mathbb{M}(\wp)}\left[\frac{\gamma\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} - (\mu+\alpha+\kappa)\,\mathcal{J}^a(\sigma)\right] + \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp}\int_0^\sigma(\sigma-y)^{\wp-1}\left[\frac{\wp\Omega(\sigma)\mathcal{J}^a(\sigma)}{N}\right. \\
&\left. - (\mu+\alpha+\kappa)\,\mathcal{J}^a(\sigma)\right]dy, \\
\mathcal{J}^c(\sigma) &= \mathcal{J}^c(0) + \frac{1-\wp}{\mathbb{M}(\wp)}\left[\kappa\mathcal{J}^a(\sigma) - (\mu+\rho+\eta)\,\mathcal{J}^c(\sigma)\right] + \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp}\int_0^\sigma(\sigma-y)^{\wp-1}\left[\kappa\mathcal{J}^a(\sigma) - (\mu+\rho+\eta)\,\mathcal{J}^c(\sigma)\right]dy, \\
\mathfrak{R}(\sigma) &= \mathfrak{R}(0) + \frac{1-\wp}{\mathbb{M}(\wp)}\left[\alpha\mathcal{J}^a(\sigma) + \eta\mathcal{J}^c(\sigma) - \mu\mathfrak{R}(\sigma)\right] + \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp}\int_0^\sigma(\sigma-y)^{\wp-1}\left[\alpha\mathcal{J}^a(\sigma) + \eta\mathcal{J}^c(\sigma) - \mu\mathfrak{R}(\sigma)\right]dy, \\
\mathcal{H}(\sigma) &= \mathcal{H}(0) + \frac{1-\wp}{\mathbb{M}(\wp)}\left[\Lambda\,(1-\Pi) + \delta\Omega(\sigma) - (\lambda+\mu)\,\mathcal{H}(\sigma)\right] + \frac{\wp}{\mathbb{M}(\wp)\Gamma(\wp}\int_0^\sigma(\sigma-y)^{\wp-1}\left[\Lambda\,(1-\Pi)\right. \\
&\left. + \delta\Omega(\sigma) - (\lambda+\mu)\,\mathcal{H}(\sigma)\right]dy.
\end{aligned}
\tag{26}
$$

At $\sigma = \sigma_{m+1}$ and by applying (6) on (26), we get the result as

$$
\begin{aligned}
\Omega_{m+1} &= \Omega_0 + \frac{1-\wp}{\mathbb{M}(\wp)}\nabla_1(S(\sigma_m),\sigma_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\left\{\frac{h^\wp\nabla_1(\Omega_n,\sigma_n)}{\Gamma(\wp+2)}\left[(m+1-n)^\wp(m-n+2+\wp)\right.\right. \\
&\left. - (m-n)^\wp(m-n+2+2\wp)\right] - \left.\frac{h^\wp\nabla_1(\Omega_{n-1},\sigma_{n-1}}{\Gamma(\wp+2)}\left[(m+1-n)^{\wp+1} - (m-n)^\wp(m-n+1+\wp)\right]\right\} + E^\wp_{1,m}, \\
\mathcal{J}^a_{m+1} &= \mathcal{J}^a_0 + \frac{1-\wp}{\mathbb{M}(\wp)}\nabla_2(\mathcal{J}^a(\sigma_m),\sigma_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\left\{\frac{h^\wp\nabla_2(\mathcal{J}^a_n,\sigma_n)}{\Gamma(\wp+2)}\left[(m+1-n)^\wp(m-n+2+\wp)\right.\right. \\
&\left. - (m-n)^\wp(m-n+2+2\wp)\right] - \left.\frac{h^\wp\nabla_2(\mathcal{J}^a_{n-1},\sigma_{n-1}}{\Gamma(\wp+2)}\left[(m+1-n)^{\wp+1} - (m-n)^\wp(m-n+1+\wp)\right]\right\} + E^\wp_{2,m}, \\
\mathcal{J}^c_{m+1} &= \mathcal{J}^c_0 + \frac{1-\wp}{\mathbb{M}(\wp)}\nabla_3(\mathcal{J}^c(\sigma_m),\sigma_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\left\{\frac{h^\wp\nabla_3(\mathcal{J}^c_n,\sigma_n)}{\Gamma(\wp+2)}\left[(m+1-n)^\wp(m-n+2+\wp)\right.\right. \\
&\left. - (m-n)^\wp(m-n+2+2\wp)\right] - \left.\frac{h^\wp\nabla_3(\mathcal{J}^c_{n-1},\sigma_{n-1}}{\Gamma(\wp+2)}\left[(m+1-n)^{\wp+1} - (m-n)^\wp(m-n+1+\wp)\right]\right\} + E^\wp_{3,m}, \\
\mathfrak{R}_{m+1} &= \mathfrak{R}_0 + \frac{1-\wp}{\mathbb{M}(\wp)}\nabla_4(\mathfrak{R}(\sigma_m),\sigma_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\left\{\frac{h^\wp\nabla_4(\mathfrak{R}_n,\sigma_n)}{\Gamma(\wp+2)}\left[(m+1-n)^\wp(m-n+2+\wp)\right.\right. \\
&\left. - (m-n)^\wp(m-n+2+2\wp)\right] - \left.\frac{h^\wp\nabla_4(\mathfrak{R}_{n-1},\sigma_{n-1}}{\Gamma(\wp+2)}\left[(m+1-n)^{\wp+1} - (m-n)^\wp(m-n+1+\wp)\right]\right\} + E^\wp_{4,m}, \\
\mathcal{H}_{m+1} &= \mathcal{H}_0 + \frac{1-\wp}{\mathbb{M}(\wp)}\nabla_5(\mathcal{H}(\sigma_m),\sigma_m) + \frac{\wp}{\mathbb{M}(\wp)}\sum_{n=0}^{m}\left\{\frac{h^\wp\nabla_5(\mathcal{H}_n,\sigma_n)}{\Gamma(\wp+2)}\left[(m+1-n)^\wp(m-n+2+\wp)\right.\right. \\
&\left. - (m-n)^\wp(m-n+2+2\wp)\right] - \left.\frac{h^\wp\nabla_5(\mathcal{H}_{n-1},\sigma_{n-1}}{\Gamma(\wp+2)}\left[(m+1-n)^{\wp+1} - (m-n)^\wp(m-n+1+\wp)\right]\right\} + E^\wp_{5,m},
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
\nabla_1 &= \Pi\Lambda - \frac{\gamma\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} + \lambda\mathcal{H}(\sigma) - (\delta+\mu)\,\Omega(\sigma), \\
\nabla_2 &= \frac{\gamma\Omega(\sigma)\mathcal{J}^a(\sigma)}{N} - (\mu+\alpha+\kappa)\,\mathcal{J}^a(\sigma), \\
\nabla_3 &= \kappa\mathcal{J}^a(\sigma) - (\mu+\rho+\eta)\,\mathcal{J}^c(\sigma), \\
\nabla_4 &= \alpha\mathcal{J}^a(\sigma) + \eta\mathcal{J}^c(\sigma) - \mu\mathfrak{R}(\sigma), \\
\nabla_5 &= \Lambda\,(1-\Pi) + \delta\Omega(\sigma) - (\lambda+\mu)\,\mathcal{H}(\sigma),
\end{aligned}
$$

and $E^\wp_{1,m}$, $E^\wp_{2,m}$, $E^\wp_{3,m}$, $E^\wp_{4,m}$, $E^\wp_{5,m}$ are of the form of $E^\wp_m$ given in (7).

5 Numerical simulations

In order to make our obtained results more understandable, we continue with some approximations, such as simulations of the system (2). By the values of all of the parameters in a biologically possible manner given in table 1 and to perform simulations to study the qualitative analysis of deterministic and fractional stability, it is necessary to assign values to all parameters of model (2).

Parameters	Value	Source
Λ	0.8	estimated
μ	0.03	estimated
ρ	0.005	estimated
γ	0.05	estimated
α	0.03	estimated
η	0.07	estimated
κ	0.009	estimated
λ	0.07	estimated
δ	0.07	estimated
Π	0.04	estimated
$S(0)$	100	estimated
$\mathcal{J}^a(0)$	10	estimated
$\mathcal{J}^c(0)$	70	estimated
$\mathfrak{R}(0)$	60	estimated
$\mathcal{H}(0)$	50	estimated

Table 1. Parameters values

Our simulation has been performed by applying a newly introduced numerical scheme for the approximation of ABC fractional derivative to model (1). Further, the parameters from Table 1 can be used. Time ranges from $[0-20]$ and the initial population for the compartment susceptible class $\Omega(\sigma)$, acutely infected class $\mathcal{J}^a(\sigma)$, chronically carrier class $\mathcal{J}^c(\sigma)$, recovery class $\mathfrak{R}(\sigma)$, and immunized class $\mathcal{H}(\sigma)$ have been chosen from Table 1 as well. Figures 1,2 and 3 illustrate the simulation of several compartments in the proposed model as a result of applying the previously given data. Figures 2(a)–2(e) represent the comparison of model (2) and (1), and Figures 1(a)–1(e) represent the comparison of the (2) and (1), when $\wp = 1.0$. Secondly, we apply iterative approaches developed from (27) to simulate the model (2) under the ABC non-integer order operator. All of the compartments $(\Omega(\sigma), \mathcal{J}^a(\sigma), \mathcal{J}^c(\sigma), \mathfrak{R}(\sigma), \mathcal{H}(\sigma))$ of the said fractional order system were plotted for the table 1 parameters for various arbitrary order values as $\wp = 1.0, 0.95, 0.90, 0.85$. Non-integer order operators of the ABC type have been used. According to these figures 2(a)–2(e), we can see the dynamic behavior of several compartments of the system (2). At first, the decay in the susceptible class is pretty rapid, but subsequently becomes stable with time. Similarly, infection cases decayed at various fractional orders of $\wp$. The recovery achieves their maximum at this point. The graphical findings show that the proposed model is dependent on the fractional order $\wp$ and provides more flexible data about the behavior of the model that cannot be achieved with the classic integer-order model.

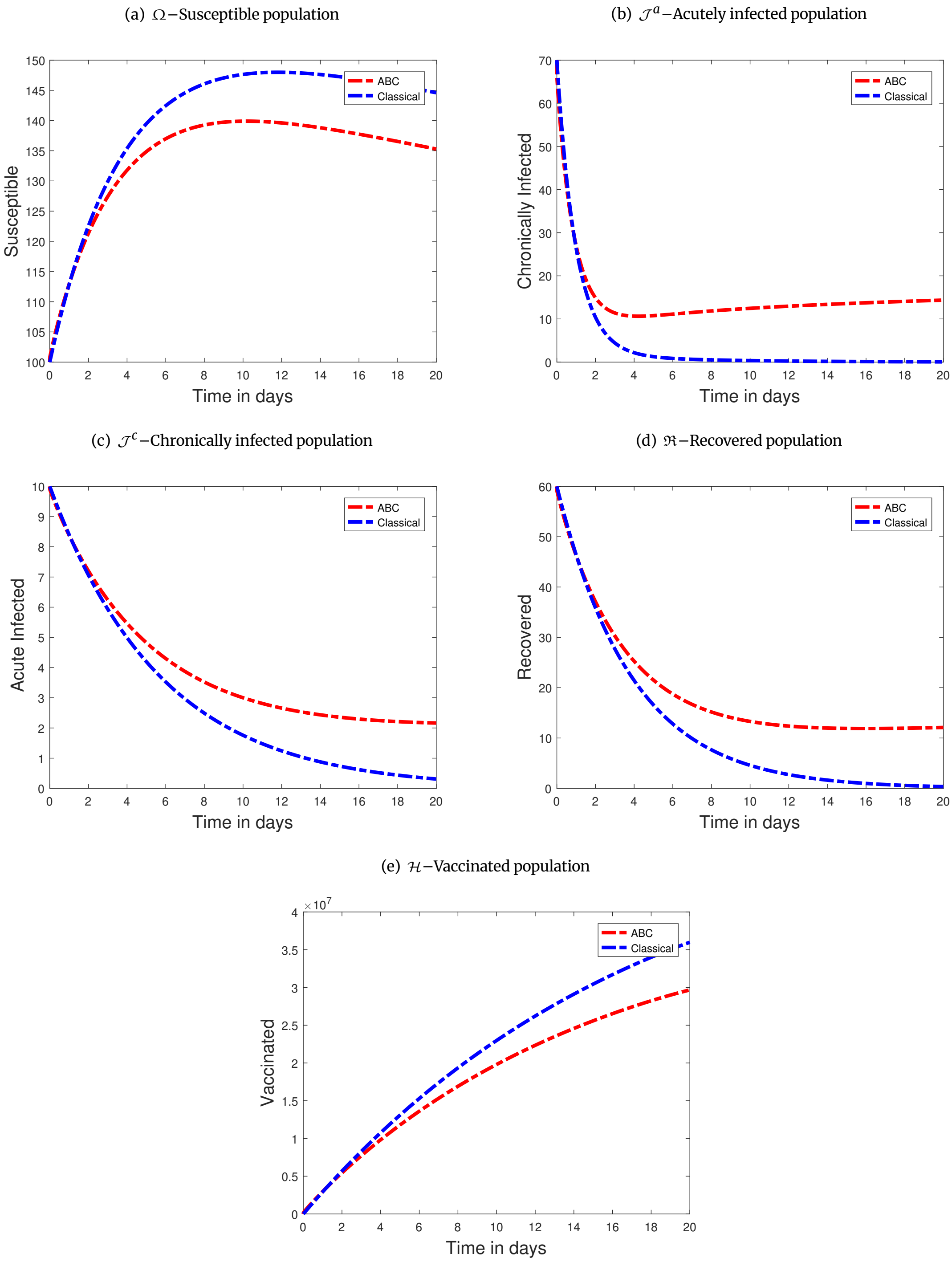

Figure 1. Simulations of susceptible, acute infections, chronic carriers, recovered and vaccinated individuals of model (1), when $\wp = 0.90$.

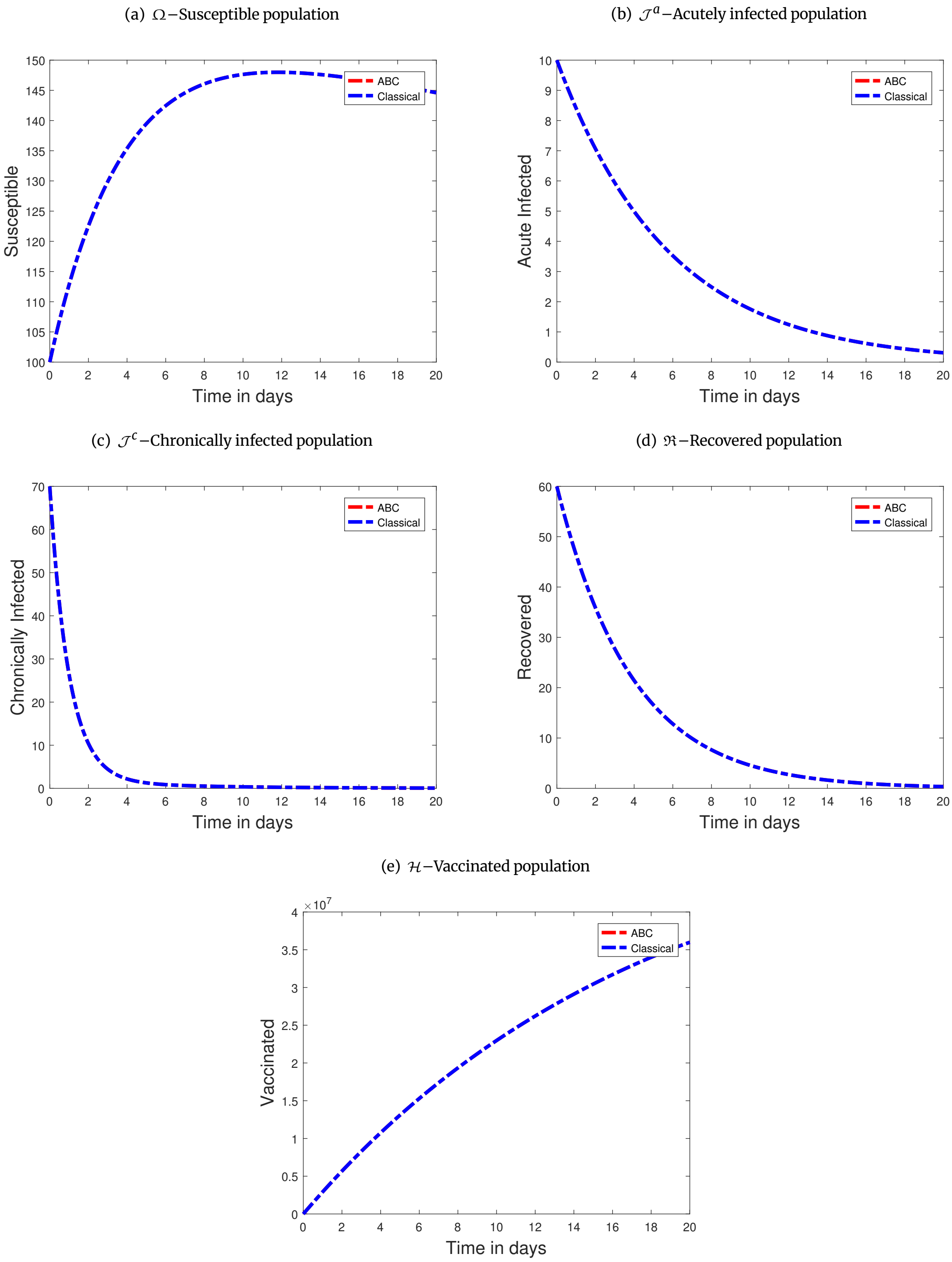

Figure 2. Simulations of susceptible, acute infections, chronic carriers, recovered and vaccinated individuals of model (2), when $\wp = 1$.

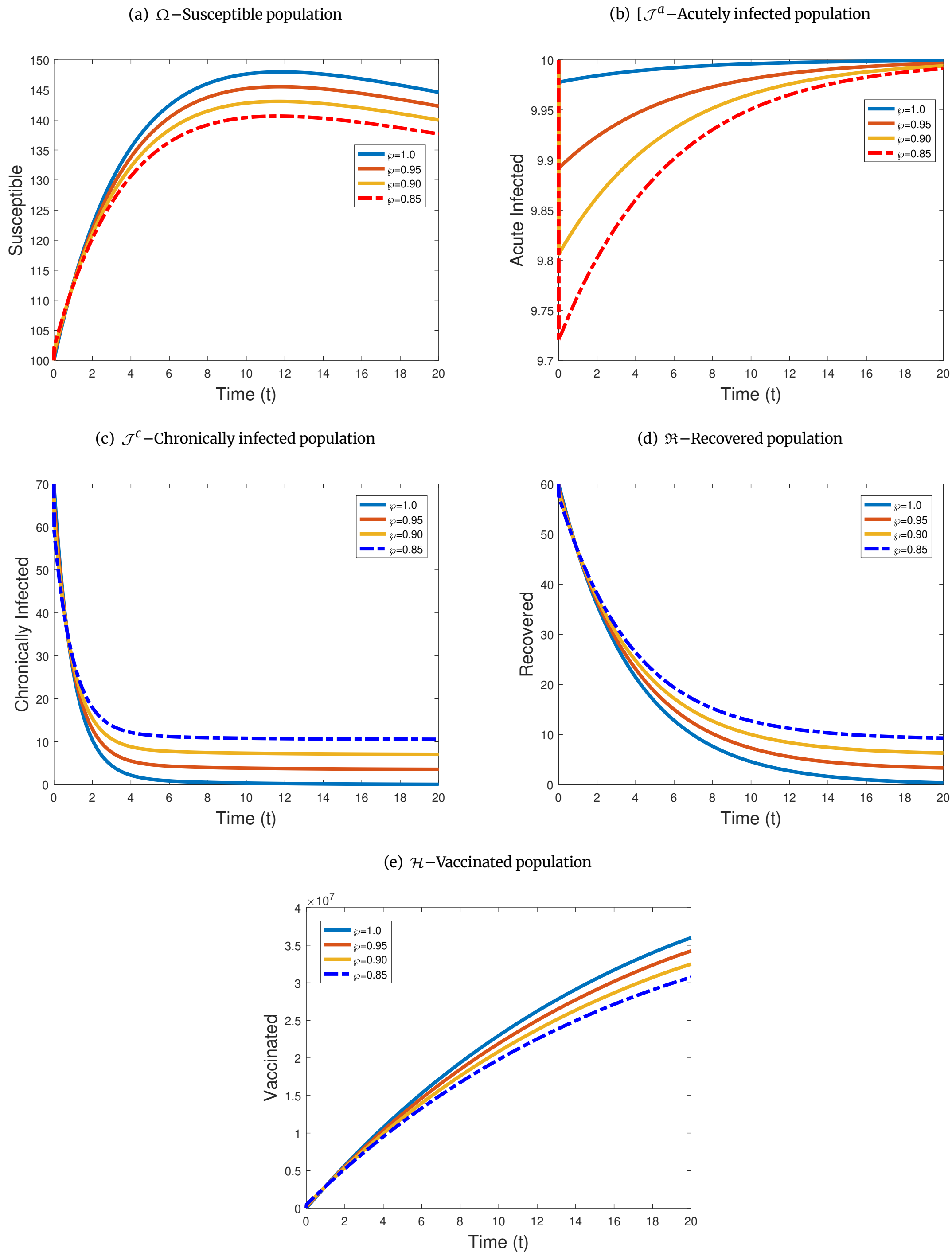

Figure 3. Simulations of susceptible, acute infections, chronic carriers, recovered and vaccinated individuals of model (2) when $\wp$ = (1.0, 0.95, 0.90, 0.85).

6 Conclusions

In the end we concluded that the proposed fractional system describing the vaccinated Hepatitis-B behaviors has been studied qualitatively using the contraction theorems of closed norm space. This investigation is taken under non-singular kernel related to the ABC fractional order derivative. By disease free equilibrium points, the proposed system also has been studied for existence, uniqueness, and stability. With the use of a newly defined numerical technique, a numerical simulation has been made for the approximation of ABC fractional derivative at different fractional orders. Such type of techniques can also can also be achieved by other different fractional operators like He's derivative. Taking the initial populations bigger than zero for $t > 0$ we have simulated the compartment of the proposed model. Moreover, we can declare that the result properly satisfying the initial data when the proposed system's right hand side approaches to zero under certain conditions. We can also control the epidemic by taking the optimal control strategy and with suitable variable or a parameter to minimised the infected class like unscreened blood, the reuse of dental and surgical instruments, etc. As seen in the graphs, the fractional derivative gives a more accurate and flexible data for investigating the complexity of the dynamics of the HBV model, see Figures 1–3.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interests.

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Author's contributions

A.D.: Conceptualization, Methodology, Software, Data Curation, Investigation, Writing-Original draft. M.Z.A.: Conceptualization, Methodology, Supervision, Investigation. All authors discussed the results and contributed to the final manuscript.

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