Mathematical Modelling and Numerical Simulation with Applications, 2022, 2(2), 73–87

https://www.mmnsa.org
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.2022.007

RESEARCH PAPER

On a new approach to distributions with variable transmuting parameter: The concept and examples with emerging problems

Jordan Hristov [1,*,‡]

[1]Department of Chemical Engineering, University of Chemical Technology and Metallurgy, Sofia, Bulgaria

[*]Corresponding Author
[‡]jordan.hristov@mail.bg or jyh@uctm.edu (Jordan Hristov)

Abstract

A new concept in the transmutation of distribution applying variable transmuting function has been conceived. Test examples with power function by quadratic and cubic transmutations have been demonstrated by the applications of the error-function and standard logistic function variable transmuting functions. The efficiency and properties of the new approach by numerical examples addressing the rate constants of the transmuting functions and the shape parameter of the test power function have been demonstrated. An additional example with a quadratic transmutation of the exponential distribution through the error function as a variable transmuting parameter has been developed.

Key words: Transmutation; variable transmuting parameter; transmuted distributions; power function
AMS 2020 Classification: 62E17; 62P35; 62P99

1 Introduction

Distributions are widely implemented to fit experimental data dominantly in statistical applications. Applications of certain statistical tools are strongly dependent on the used probabilistic models of considered data. The increase in the variety of statistical data that should be fitted (modelled) revealed that many classical distributions are unsatisfactory in statistical data fitting. Hence, there are appeals to create more generalized distributions allowing modeling of more complicated phenomena more flexibly. Motivated by the need to add more parameters to distribution functions thus making them more flexible in data analyzes [1]. In this context, there are several attempts to consider compound distributions [2, 3, 4], exponentiated distributions [5], beta class of distributions [6, 7], generalized exponential distribution [8], weighted distributions [9]. The weighted distributions, for instance, take into consideration the verification method for adjustment probability distributions by introducing weights [10] which to some extent is close to the transmutation method considered in this work.

Here we address a class of weighted distributions developed by the so-called transmutation method [11] which results in a specific class of mixture distributions. In the approach conceived by Shaw and Buckley [11] the generalization of the distributions is achieved by the application of a transmutation map. Precisely, the transmutation map is a functional composition of the cumulative distribution function (cdf) of a certain distribution with the inverse cumulative distribution (quantile) function of another [1]. In statistical publications, there are numerous examples of transmutations of classical distributions such as Weibull distribution [12, 13, 14, 15, 16, 17], power distribution [18], minimax distribution [19], linear exponential distribution [20, 21, 22], Frechet distribution [23], Gumbel distribution [20, 23], Gamma distributions [20], etc.

This article introduces the idea to replace the transmutation parameter with a function (called here also activation function) dependent on the probability variable and varying only in range , as in the case when the transmutation parameter is a discrete value and resulting in a

➤ **Received:** 29.04.2022 ➤ **Revised:** 15.06.2022 ➤ **Accepted:** 17.06.2022 ➤ **Published:** 19.06.2022

specific class of mixture distributions. For a better understanding of the main idea of the transmutations, the technique is explained in the next section (Background) from a general point of view and with two simple examples further used in this work.

2 Background

The increasing number applications fitting real-world data, from life science, economics or advanced technologies, are raising problems for more flexibility which some of statistical distributions cannot provide adequate answers. Particularly, to capture the skewness and kurtosis (see the definitions in Appendix (Section 10) associate with such applications it was introduced a transmutation mapping [11]. This map is *a functional composition of a cumulative distribution function (cdf) of a particular distribution with the inverse cumulative distribution (quantile) of another distribution* [1]. This approach increases the distribution flexibility to fit experimental data. The common approach is to use *discrete values of the transmuting parameters* (commonly denoted by the symbol λ) [1, 11, 16, 21, 22, 24, 25, 26, 27, 28, 29]. This work addresses transmutations of statistical distributions by variable transmuting parameters, precisely, transmuting parameters dependent on random variables. For the sake of clarity, and creating the exposition gradually understandable, as well as to present the new approach we will start with some basic definitions explained next.

Theory of distribution transmutations

If there are absolutely *continuous cumulative distribution functions* (*cdf*s) $F_1(x)$ and $F_2(x)$ with the corresponding *pdf*s $f_1(x)$ and $f_2(x)$, on a common sample space, then the general rank of transmutation, following Shaw and Buckley [11], is formulated as

$$G_{R_{12}}(u) = F_2\left[F_1^{-1}(u)\right], \quad G_{R_{21}}(u) = F_1\left[F_2^{-1}(u)\right], \tag{1}$$

where both functions $G_{R_{12}}(u)$ and $G_{R_{21}}(u)$ map the compact $[0,1]$ into itself as well as they are mutually inverse with $G_{R_{ij}}(0) = 0$ and $G_{R_{ij}}(1) = 1$ when $i = 1, 2$.
In accordance with the definition of [11] a random variable x has a transmuted distribution of family of rank k if the cumulative distribution function (*cdf*) is defined in a general form as [11, 16, 24, 21]

$$F(x) = G(x) + \left[1 - G(x)\right]\sum_{i=1}^{k}\lambda_i\left[G(x)\right]^i, \tag{2}$$

with $\lambda_i \in [-1, 1]$ for $i = 1, 2, 3, \ldots, k$ and $-k \leq \sum_{i=1}^{k}\lambda_i < 1$.
The general transmuted family reduces to the base function (base cumulative distribution) (*cdf*) $G(x)$ for $\lambda_i = 0$. Two simple transmutations, undoubtedly explaining the idea of this mapping, and used in this work, are briefly presented next.

Transmutations of quadratic and cubic ranks: Examples

Before demonstrating simple examples we have to stress the attention on two important issues in applications of the transmutation approach, namely

- When the task is to demonstrate how the transmutation of certain rank transforms the base function (distribution) then the choice of the *transmuting parameter* λ (can be termed also as *activating parameter*) is to some extent arbitrary, with discrete values, from $\lambda \in [-1, 1]$, as it will be done in the following examples. This can be considered as a *forward problem*.That is, in the forward problem a given value of λ activates (results in) a particular shifted distribution.

- When a certain transmuted version of base function (distribution) has to be applied in fitting procedure to a given set of statistical data, then the determination of λ is a task related to an inverse problem. That is, in such a case the problem is to find particular value of λ so that the transmuted distribution to fit the statical data. This can be considered as a *backward problem*. This step is beyond the scope of this work, because the primary task addressed here is to demonstrate how the new approach in generation of transmuted distributions works.

It is worth noting that, in both cases, the tasks are performed with discrete values of the transmuting parameter λ as it follows from the basic formulation of Shaw and Buckley [11].
Here, for the seek of the clarity of the method applied and its development conceived in this work, we give demonstrative examples of the two most popular transmutations [11] with particular sets of values of λ chosen arbitrarily from the range $\lambda \in [-1, 1]$ as it has been done in all works cited above.

Quadratic transmutation

For $k = 1$, applying (2), we have a *cdf*

$$F_1(x) = (1 + \lambda)G(x) - \lambda G^2(x), \quad |\lambda| < 1, \tag{3}$$

and a corresponding *pdf* $f_1(x) = \frac{d}{dx}F_1(x)$

$$f_1(x) = (1 + \lambda)g(x) - 2\lambda g(x)G(x), \quad g(x) = \frac{dG(x)}{dx}. \tag{4}$$

It is obvious that for $\lambda = 0$ the transmuted function $F_1(x)$ reduces to $G(x)$. To illustrate this, we present examples of quadratic transmutations of the power function (distribution) (Eqs. (5) and (6)), with a shape parameter α and cumulative density function (*cdf*) [18]

$$G(x) = 1 - (1 - x)^{\alpha}, \quad 0 < x < 1, \quad \alpha > 0, \tag{5}$$

and corresponding probability density function (*pdf*)

$$g(x) = \alpha(1 - x)^{\alpha-1}, \quad 0 < x < 1, \quad \alpha > 0. \tag{6}$$

Applying (3) and (4) to (5), we get

$$F_1 = (1 + \lambda)\left[1 - (1 - x)^{\alpha}\right] - \lambda\left[1 - (1 - x)^{\alpha}\right]^2, \tag{7}$$

$$f_1 = (1 + \lambda)\left[\alpha(1 - \alpha)^{\alpha-1}\right] - 2\alpha\left[1 - (1 - x)^{\alpha}\right]\left[\alpha(1 - x)^{\alpha-1}\right]. \tag{8}$$

This function is a special case of beta distribution describing random data confined in the open interval $(0, 1)$ [1, 18]. The numerical tests shown in figure 1 demonstrate the variations of the cumulative density function for various values of the transmuting parameter λ as well as the effect of the shape parameter for $\alpha < 1$ and $\alpha > 1$. The changes in both the skewness and kurtosis are obvious.

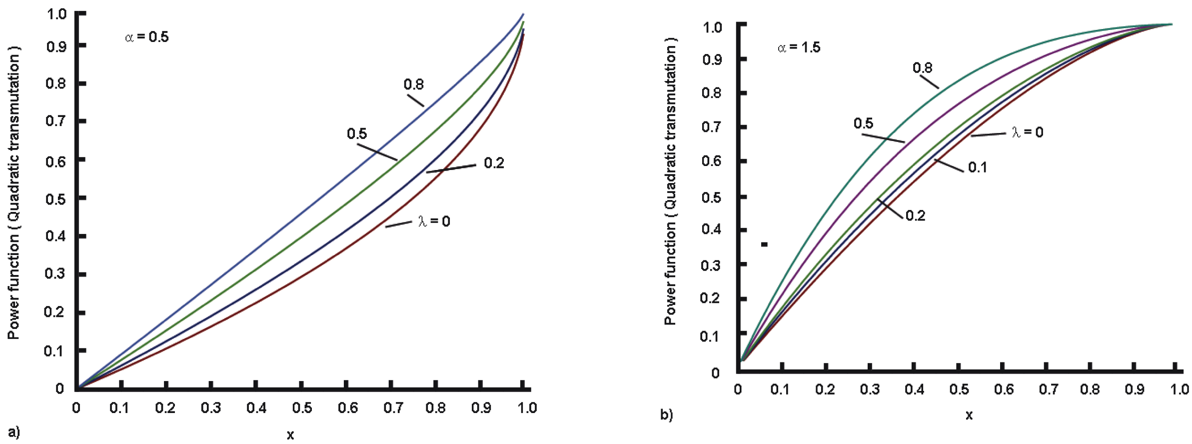

Figure 1. Two cases of quadratic transmuted power function (distribution): a) with $\alpha = 0.5$; b) with $\alpha = 1.5$

Cubic transmutation

For $k = 2$, applying (2), we have a *cdf*

$$F_2(x) = G(x) + \lambda_1 G(x)\left[1 - G(x)\right] + \lambda_2 G^2(x)\left[1 - G(x)\right]. \tag{9}$$

This can be presented also as

$$F_2(x) = (1 + \lambda_1) G(x) + (\lambda_2 - \lambda_1) G^2(x) - \lambda_2 G^3(x), \tag{10}$$

and a corresponding *pdf*

$$f_2(x) = (1 + \lambda_1) g(x) + 2(\lambda_2 - \lambda_1) g(x) G(x) - 3\lambda_2 g(x) G^2(x), \tag{11}$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [-1, 1]$, and $-2 < \lambda_1 + \lambda_2 < 1$. For this specific case, following Granzotto et al. [16], we have $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$.

If we try to minimize the number of transmuting parameters, it is possible to suggest that $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$ where $|\lambda| < 1$. Then, from (10) we get a simpler form of $F_2(x)$, namely

$$\bar{F}_2(x) = (1 + \lambda) G(x) - 2\lambda G^2(x) + \lambda G^3(x), \tag{12}$$

$$\bar{f}_2(x) = (1 + \lambda) g(x) - 4\lambda g(x) G(x) + 3\lambda g(x) G^2(x). \tag{13}$$

In the case with the power distribution (5), we have

$$\bar{F}_2 = (1+\lambda)\left[1-(1-x)^{\alpha}\right] - 2\lambda\left[1-(1-x)^{\alpha}\right]^2 + \lambda\left[1-(1-x)^{\alpha}\right]^3, \tag{14}$$

$$\bar{f}_2 = (1-\lambda)\left[\alpha(1-x)^{\alpha-1}\right] - 4\lambda\left[1-(1-x)^{\alpha}\right]\left[\alpha(1-x)^{\alpha-1}\right] + 3\lambda\left[1-(1-x)^{\alpha}\right]^2\left[\alpha(1-x)^{\alpha-1}\right]. \tag{15}$$

Examples of cubic transmutations of the power function (distribution), the same as used in the tests of the quadratic transmutation, are shown in Figure 2. The effect of the transmutation on the distributions is more obvious since there is a strong effect of the shape parameter α.

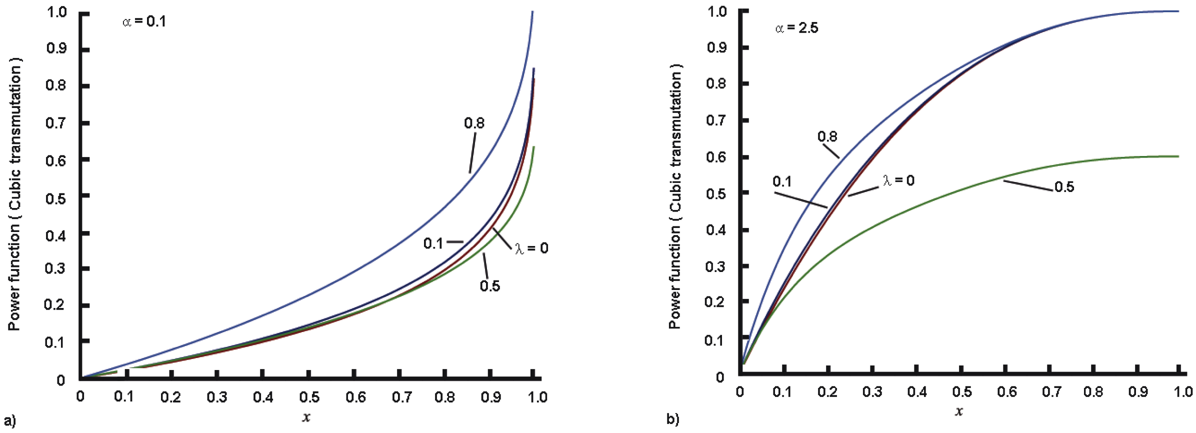

Figure 2. Two cases of cubic transmuted power function: Left- with $\alpha = 0.1$; Right- with $\alpha = 2.5$

There exist many examples of transmuted distributions [1] (see the detailed analysis in this article and references quoted therein) but here we address a different approach rather than selecting values of λ (the case with $k = 1$) or pairs (λ_1, λ_2) when $k = 2$, as it was done in most of the articles cited above. It is worth remarking that for the sake of clarity in the explanation of the idea developed in this article the numerical examples with power function are provided with $\alpha = 0.5$ and $\alpha = 1.5$, but there are no restrictions to demonstrate similar behaviours with different values of the shape parameter α; the effect of the shape parameter on the transmuted distribution in the light of the new concept is beyond the scope of this work.

3 Aim

This article conceives a new approach in transmutation of distributions by applying variable transmuting parameters (activation functions) instead of its particular (discrete) counterparts used in the original concept. To some extent, this leads to new distributions obeying all desired properties of transmuted basic functions. The power function is used as a test distribution with two transmuting (activating) functions: the Gaussian Error-function and the Standard Logistic Function. In fact, this is an experimental work, in sense of experimental mathematics, on the *forward transmuting problem*, when a new idea about transmutation of functions (distributions) is directly demonstrated.

4 Further paper organization

In the sequel the concept of a variable transmuting parameter (Section 5) with two functions as examples: Error Function $erf(x)$ (Section 5) and the Standard Logistic Function $LogF(x)$ (Section 5) is presented. Further, two examples with the application of these continuous transmuting functions (Section 6) to the power function distribution demonstrate all features and problems emerging in application of this new approach.

5 Variable transmuting parameter: The concept

Now, we have to stress the attention on the fact that *there is no rule for choice of the value of the transmuting parameter* λ, *when the forward problem is at issue*, despite the restriction $\lambda \in [-1, 1]$ as the references [11, 16, 21, 24] (and see the references therein) widely used to obtain new distributions in statistics. Only, by selections of pairs λ_1,λ_2 it is possible to get a variety of transmuted functions [1, 11, 16, 20, 21, 22, 24, 25, 26, 27, 28, 29]. One attempt to resolve the problem is the simplification of the cubic transmutation (12) avoiding the use of λ_1 and λ_2. In what follows we conceive the idea that λ can be dependent on the argument of the transmuted function, but at the same time to satisfy the condition $\lambda \in [-1, 1]$, that is

$$\lambda(x) = \Lambda(x), \quad \Lambda(x) \in [-1, 1], \quad x \in (-\infty, \infty). \tag{16}$$

In this context, both the quadratic (17) and cubic (18) transmuted profiles are

$$F_1^{\lambda}(x) = G(x) + \wedge(x)\left[G(x) - G^2(x)\right], \tag{17}$$

$$F_2^{\lambda}(x) = G(x) + \wedge(x)\left[G(x) - 2G^2(x) + G^3(x)\right]. \tag{18}$$

Therefore, we have a superposition of the basic function $G(x)$ and a term (functional relationship) deforming the entire transmuted profile (distribution). In such a case the *pdfs* of these transmuted *cdfs* are

$$f_1^{\lambda}(x) = f(x) + L(x)\left[G(x) - G^2(x)\right] + \wedge(x)\,g(x)\left[1 - 2G(x)\right], \quad L(x) = \frac{d\wedge(x)}{dx}, \tag{19}$$

$$f_2^{\lambda}(x) = g(x) + L(x)\left[G(x) - 2G^2(x) + G^3(x)\right] + \wedge(x)\,g(x)\left[1 - 2G(x) + 3G^2(x)\right]. \tag{20}$$

It is important to stress the attention on the requirement coming from the new formulation the function $\wedge(x)$ to be *smooth and differentiable with respect to* x.

The functional relationship of $\wedge(x)$

We realize that there exists a variety of such functions defined by (16), but skipping a discussion on this problem which is beyond the scope of the present study, we suggest the following functional relationships:

Error function

In this case, we suggest

$$\wedge(x) = erf\,[p \cdot x] = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2 \cdot z^2}dz, \quad p > 0, \quad x(-\infty, \infty). \tag{21}$$

This is an *ad hoc* selection of $\wedge(x)$ where the derivative with respect to x is

$$L(x) = \frac{d\wedge(x)}{dx} = \frac{2p}{\sqrt{\pi}}e^{-p^2x^2}. \tag{22}$$

In addition, the integral of $\wedge(x)$ is

$$\int erf\,(px)dx = \frac{(px)\,erf(px)}{p} + \frac{1}{p}\frac{e^{-p^2x^2}}{\sqrt{\pi}} + C. \tag{23}$$

The parameter p controls the rate of growth of $\wedge(x)$ as it is shown in Fig. 3 (left panel). It is obvious that for $p = 1$, we get the basic $erf(x)$, growing rapidly to 1.

Logistic function

Here we select only the standard logistic function along the axis $x \gg 0$, namely

$$V(x) = \frac{1}{1 + e^{-px}}, \quad x \gg 0, \quad p > 0, \quad 0.5 \le V(x) \le 1, \tag{24}$$

with the following basic properties

$$V(x) = \frac{1}{1 + e^{-px}} = \frac{e^{px}}{1 + e^{px}}, \quad \frac{dV(x)}{dx} = \frac{ke^{-px}}{(1 + e^{-px})^2}, \quad \int V(x)dx = \frac{1}{p}\ln\left(1 + e^{px}\right) + C. \tag{25}$$

As in the case of the error function (21) the parameter p controls the rate of growth (see figure 3 (right panel)) There are no restriction using other versions of the logistic function but here the standard version was chosen for its simplicity allowing to demonstrate the main idea of the variable transmuting parameter (function).

6 Distributions with variable transmuting parameter: Demonstrative examples

Example 1: Power distribution with Error-function as a variable activation function

Here, we consider again the power function (5), which allows comparing the new approach in the transmutation with the classical approach (with discrete λ) demonstrated in Section 2.

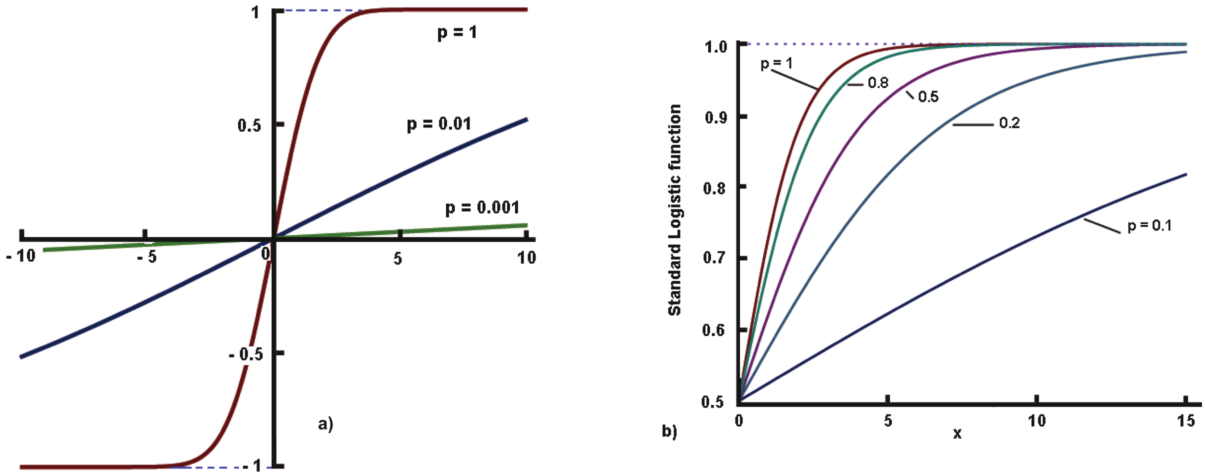

Figure 3. Two examples of variable functions $\wedge(x)$. Left: Error function (ERF); Right: Standard Logistic function (SLF). The dotted lines show the lower (-1) and the upper (1) limits of variations of the functions (the same in all figures in the sequel).

Quadratic transmutation

Plots of the quadratic transmuted *cfd*s power function are shown in Figure 4. The behaviours of the transmuted functions reveal that in general the character of the effect of the transmuting parameter on the skewness and kurtosis resembles the effects of the discrete values of λ.

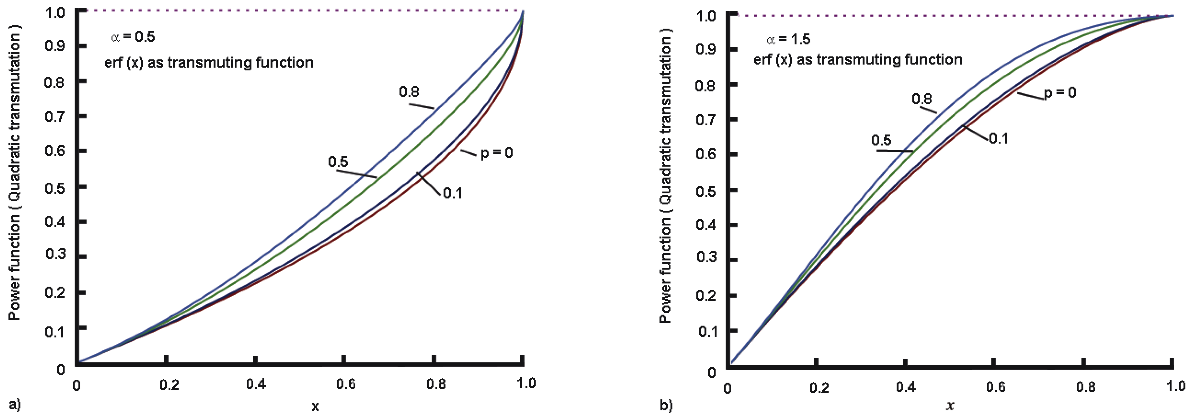

Figure 4. Two examples of variable functions $\wedge(x)$ to the quadratic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$. See the plots in Fig. 1

The corresponding *pdf*s are shown in Figures 5a-5b.

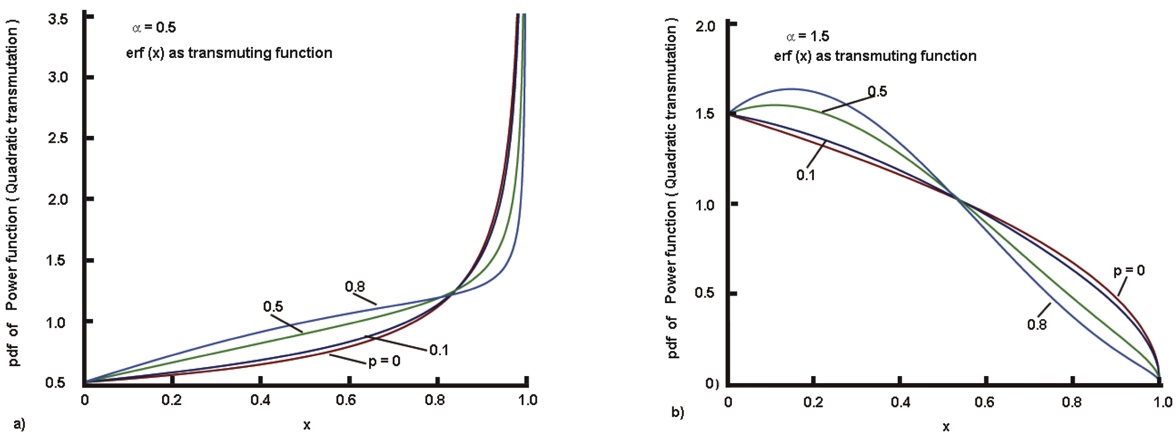

Figure 5. Two examples of *pdf*s with variable functions $\wedge(x) = erf(x)$ to the quadratic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$.

Cubic transmutation

The cubic transmuted profiles ($k = 2$) in Figure 6 have almost the same behaviour as the quadratic counterparts ($k = 1$) but now they are located too close and the effect on the skewness and kurtosis is not so distinguished as in the case with $k = 1$. Only in the central zone, we can see some differences (see the inserts).

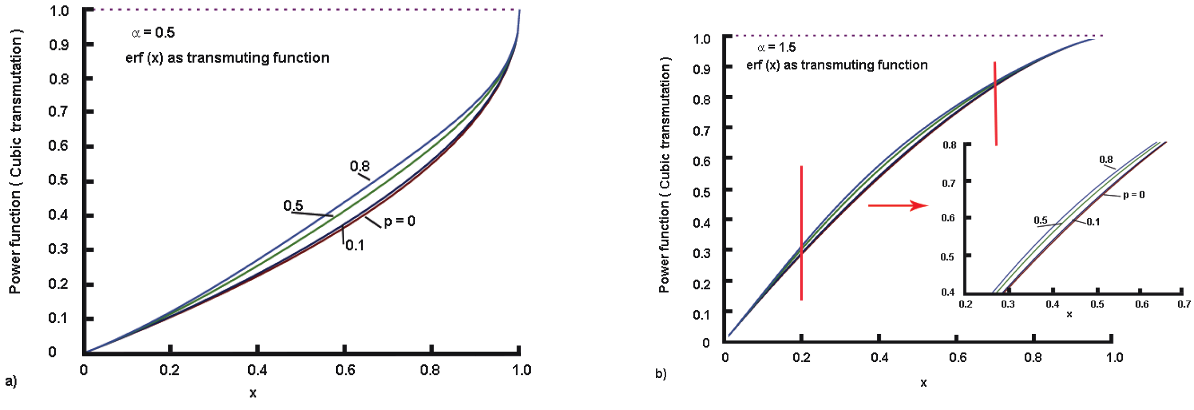

Figure 6. Two examples of variable functions $\Lambda(x)$ to the cubic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$

The corresponding *pdf*s are shown in Figures 7a-7b.

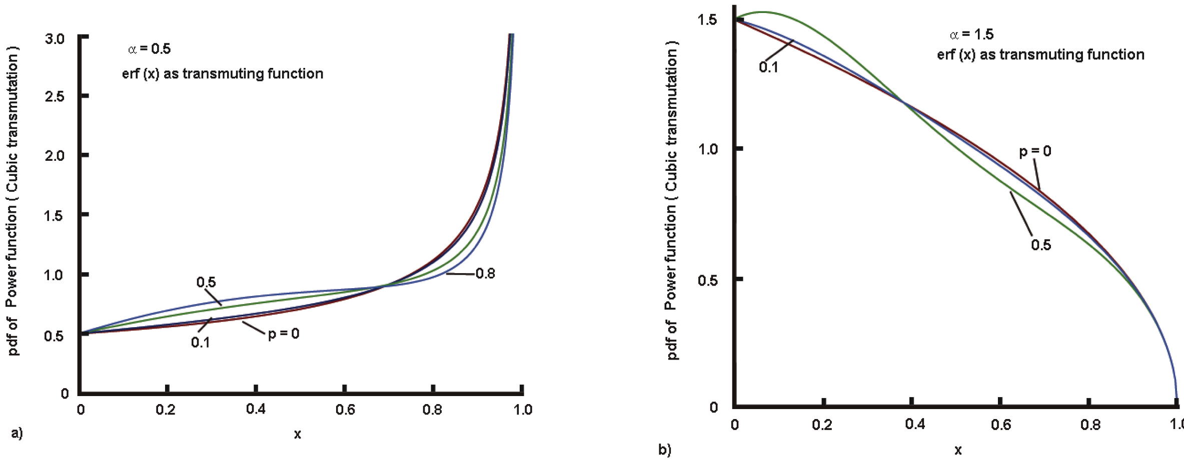

Figure 7. Two examples of *pdf*s with variable functions $\Lambda(x)$ to the cubic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$.

Reliability analysis: Survival and Hazard functions of transmuted distributions

The survival function $S_k(x)$ of $F_k(x)$ is the probability of an item not falling prior to a given x and is defined as

$$S_k(x) = 1 - F_k(x), \tag{26}$$

and the Hazard function $H_k(x)$ is given by

$$H(x) = \frac{f_k(x)}{S_k(x)}. \tag{27}$$

These functions, related to quadratic transmutation at issue, are shown in Fig. 8.

Example 2: Power function with standard logistic function as a variable transmuting parameter

Quadratic transmutation

The quadratic transmuted profiles of the power function, with different shape parameters, are shown in Figure 9. It is obvious that in both cases, with respect to the values of the shape parameter α) there are sufficient shifts in the distributions with respect to the basic version when $\lambda = 0$. The shifts are towards the case with $\lambda \to 1$. The corresponding **pdf**s, shown in Figure 10 reveal more distinguishable plots but

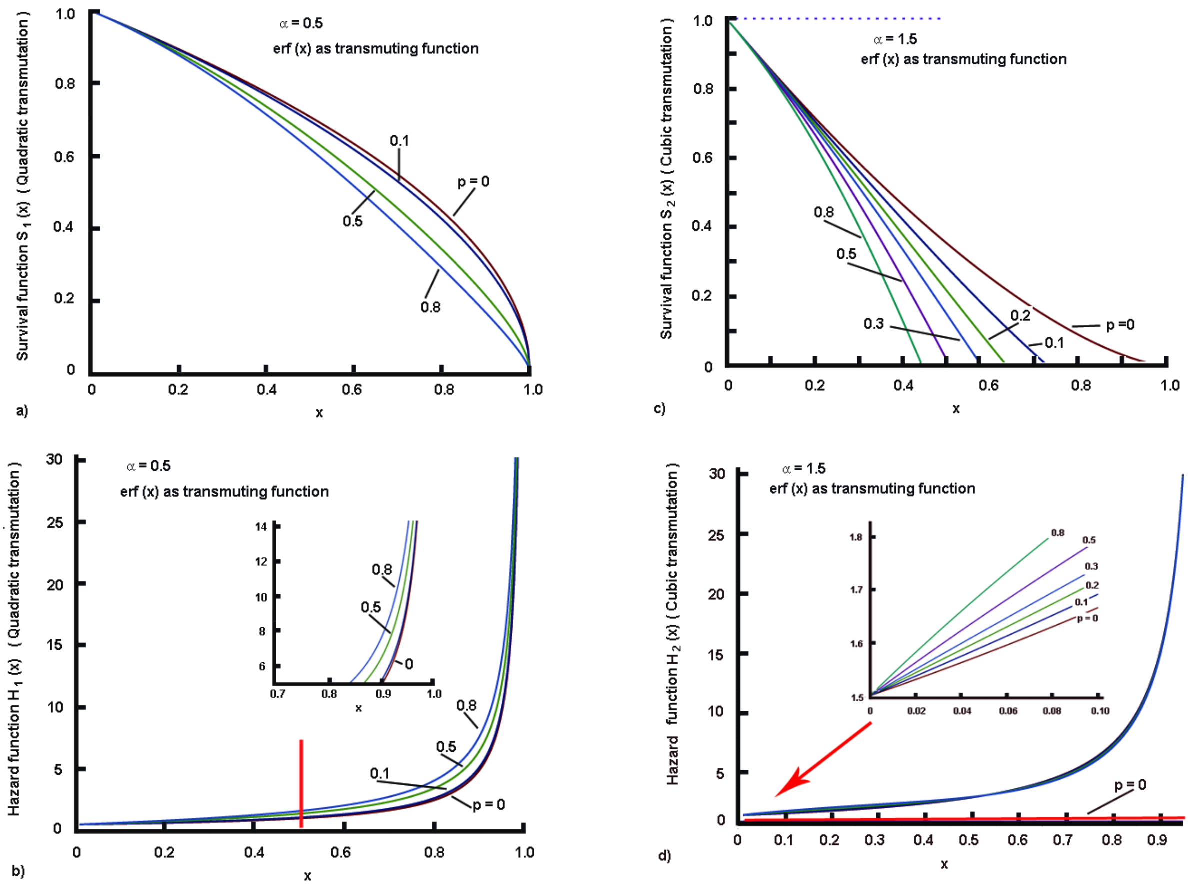

Figure 8. Survival and Hazard function with $erf(x)$ as transmutation parameter. Left column: Quadratic transmutation; Right column: Cubic transmutation

the shifts are again towards the case corresponding to $\lambda \to 1$.

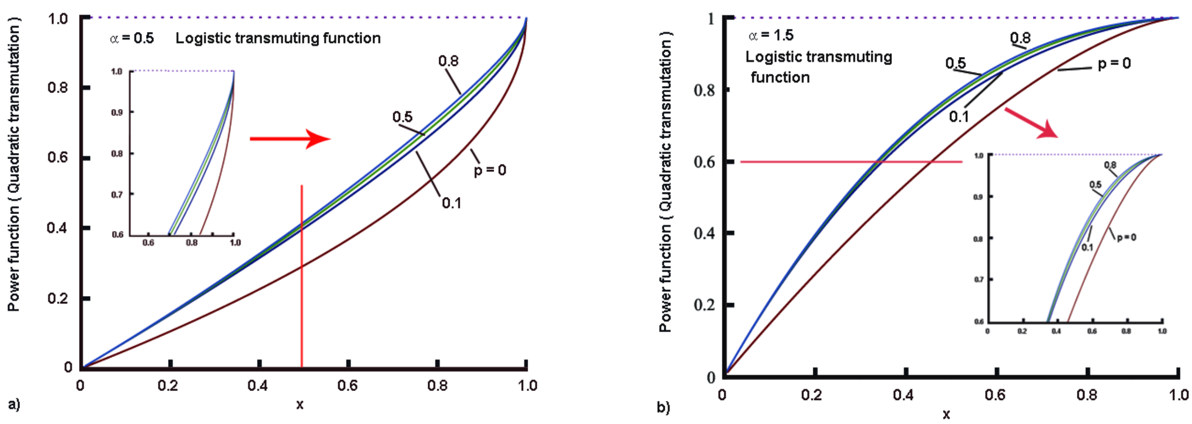

Figure 9. Two examples of distributions with variable function $V(x)$ (Standard Logistic Function) to the quadratic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$

Cubic transmutation

The cubic transmuted profiles ($k = 2$) of the cumulative power distribution in figure 11 have almost the same behaviour as the quadratic counterparts ($k = 1$) but now they are located too close and the effect on the the skewness and kurtosis is not so distinguished as in the case with $k = 1$. Only in the central zone, we can see some differences (see the insert).

Reliability analysis: Survival and Hazard functions of transmuted distributions

The survival function $S_k(x)$ of F_k and Hazard function $H_k(x)$ are shown in Figures 12 and 13.

We can see again that the effect of the cubic transmutation with the standard logistic function as transmuting function is practically negligible in contrast to the case when the error function is used.

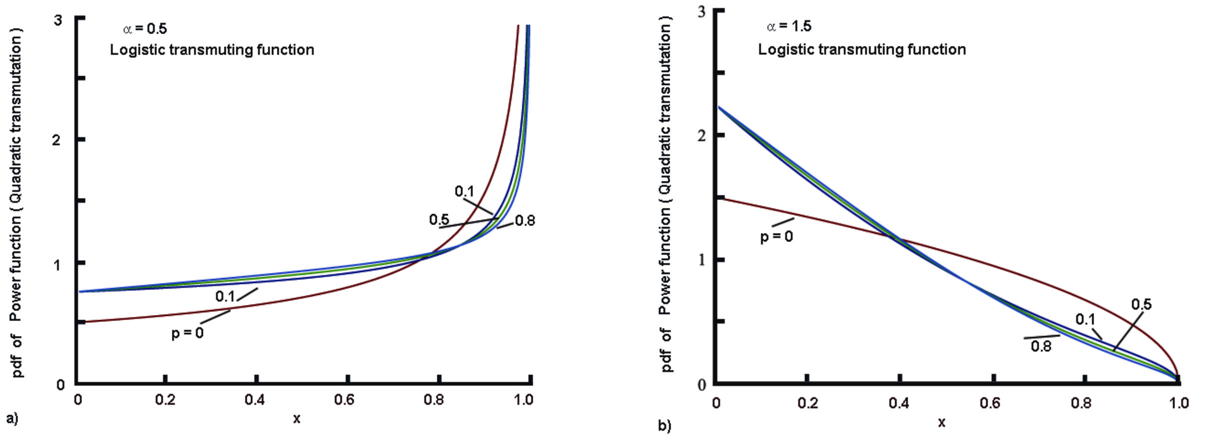

Figure 10. Two examples of *pdf*s with variable function $V(x)$ (Standard Logistic Function) to the quadratic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$

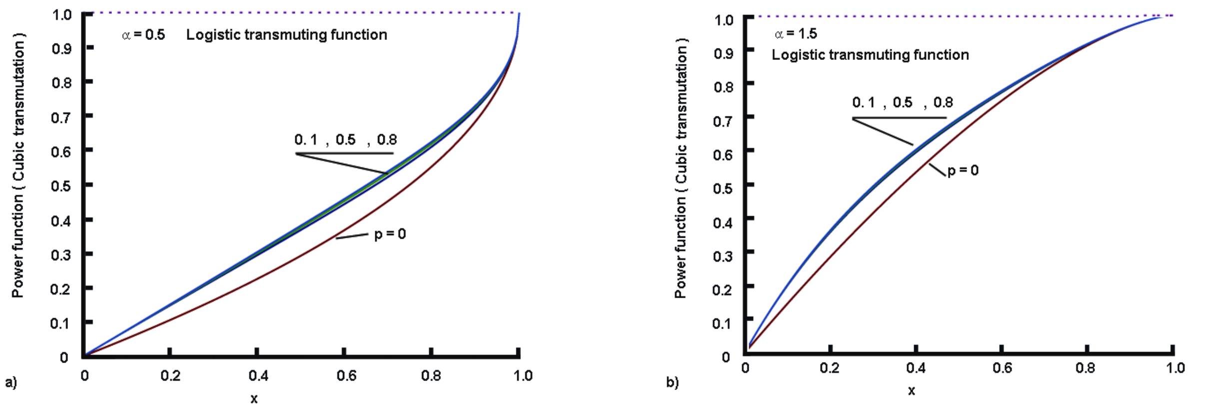

Figure 11. Two examples of *pdf*s with variable function $V(x)$ (Standard Logistic Function) to the cubic transmuted power function. Left: $\alpha = 0.5$; Right: $\alpha = 1.5$

Some briefs on the examples demonstrating the new concept

The idea developed here results in a new type of distribution which to a greater extent are similar to the mixed distribution [30, 31, 32], where the constructions of the cumulative distributions $C(x)$ as a combination of distribution functions $G_i(x)$ follow the rule [32, 33]

$$C(x) = \sum_{i=1}^{N} \omega_i G_i(x), \tag{28}$$

where $\omega_i > 0$ are mixture weights obeying the condition $\sum_{i=1}^{N} \omega_i = 1$. Moreover, it is not necessary that $G_i(x)$ belong to one and the same distribution family on have the same number of parameters [32, 33].
In the idea developed here, even though this aspect is not developed and draws future research. Moreover, this approach, to some extent, resembles the idea of transmutation (2) but bears in mind the significant differences between the two approaches. We have to stress the attention on the fact that in the approach developed here the weighting coefficients follow two main conditions: $\lambda \in [-1, 1]$ coming from the constructions of the transmutation theory, and they are dependent on the variable x , but their variations are within the range $[-1, 1]$. The numerical experiments reveal two basic issues, based on the experiments performed with the power function as a test distribution, namely:

- The quadratic transmutation provides more distinguishable distributions with both the error function and the standard logistic function as transmuting variable parameters.
- The increase in the transmutation rank, i.e. the application of the cubic transmutation, does not lead to significant changes in either the skewness or kurtosis of the new distributions. The shape parameter of the basic distribution has practically no effect on this.
- The error function is more suitable, as a variable transmuting parameter, than the standard logistic function, irrespective of the rank of transmutation, since it is allowing more distinguished new distributions to be generated.

These are comments relevant to the particular case of transmutations of the power functions. Some effects are strong and distinguishable, others not, to some extent. This cannot be considered discouraging because the next example will show how the new approach can be applied to another distribution.

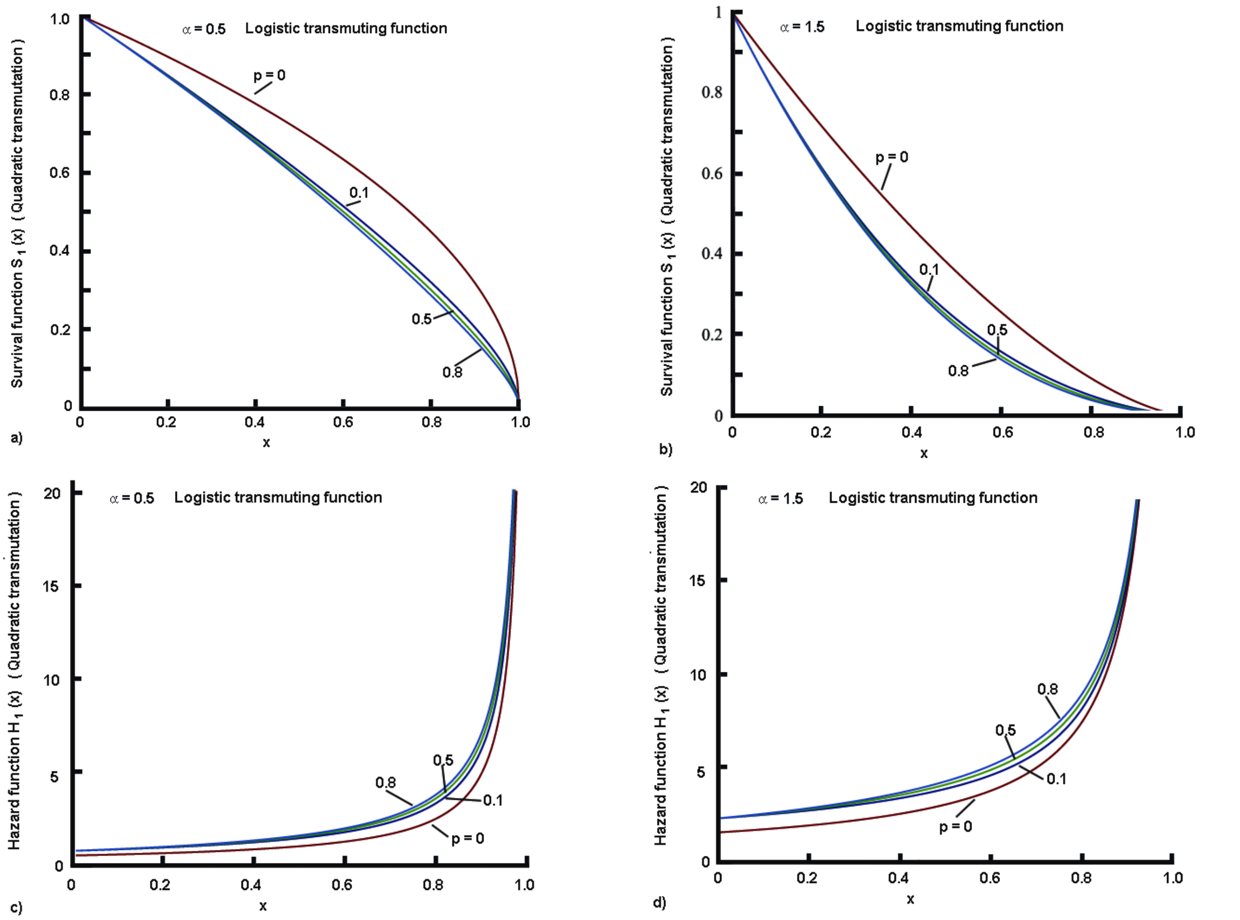

Figure 12. Survival and Hazard function with Standard Logistic Function as a variable transmutation parameter.Quadratic transmutations. Left column: $\alpha = 0.5$; Right column: $\alpha = 1.5$

7 Distributions with variable transmuting parameter: Additional example with the exponential distribution

Here we demonstrate how the quadratic transmutation, with error-function as transmuting parameter, can be applied to the exponential distribution [20, 22, 34]

$$G_{e1}(x) = 1 - \exp\left(-\frac{x}{\beta}\right), \quad x \in [0, \infty), \quad \lambda \in [0, 1], \tag{29}$$

with a quadratic transmuted *cdf*

$$F_{e1} = \left[1 - \exp\left(-\frac{x}{\beta}\right)\right]\left[1 + \lambda \exp\left(-\frac{x}{\beta}\right)\right]. \tag{30}$$

The effect of the rate parameter β on the development of the exponential distribution is shown in Figure 14. The effect of the *scale parameter* β, which may be termed as a *rate constant* of the exponential growth is stronger when $\beta < 1$ since we have $1/\beta \gg 1$ resulting in rapid saturation of the distribution. In contrast, for $\beta > 1$, the distributions are smoother. The following examples use $\beta = 0.5$ and $\beta = 1.5$, similar to the values of the shape parameter α of the power function. Moreover, $\beta = 1.5$ is used in the study of Rahman et al. [20] that allows comparing the results developed by the new approach.

Exponential distribution: Quadratic transmutation with fixed transmuting parameters

Examples of the classical transmutation approach with fixed values of λ are shown in Figure 15, thus demonstrating the effect the scale (rate) parameter β and the values of λ.

Exponential distribution: Quadratic transmutation with Error-function as a variable transmuting parameter

Now, applying the transmutation technology with the variable transmuting parameter we get flexible *cdf* and *pdf* shown in Figure 16. The generated distribution through the quadratic transmutations demonstrated the effect of the transmutation function which, to a greater extent, is similar to that of the discrete transmuting parameters. In contrast to the previous example with the power function, now we can

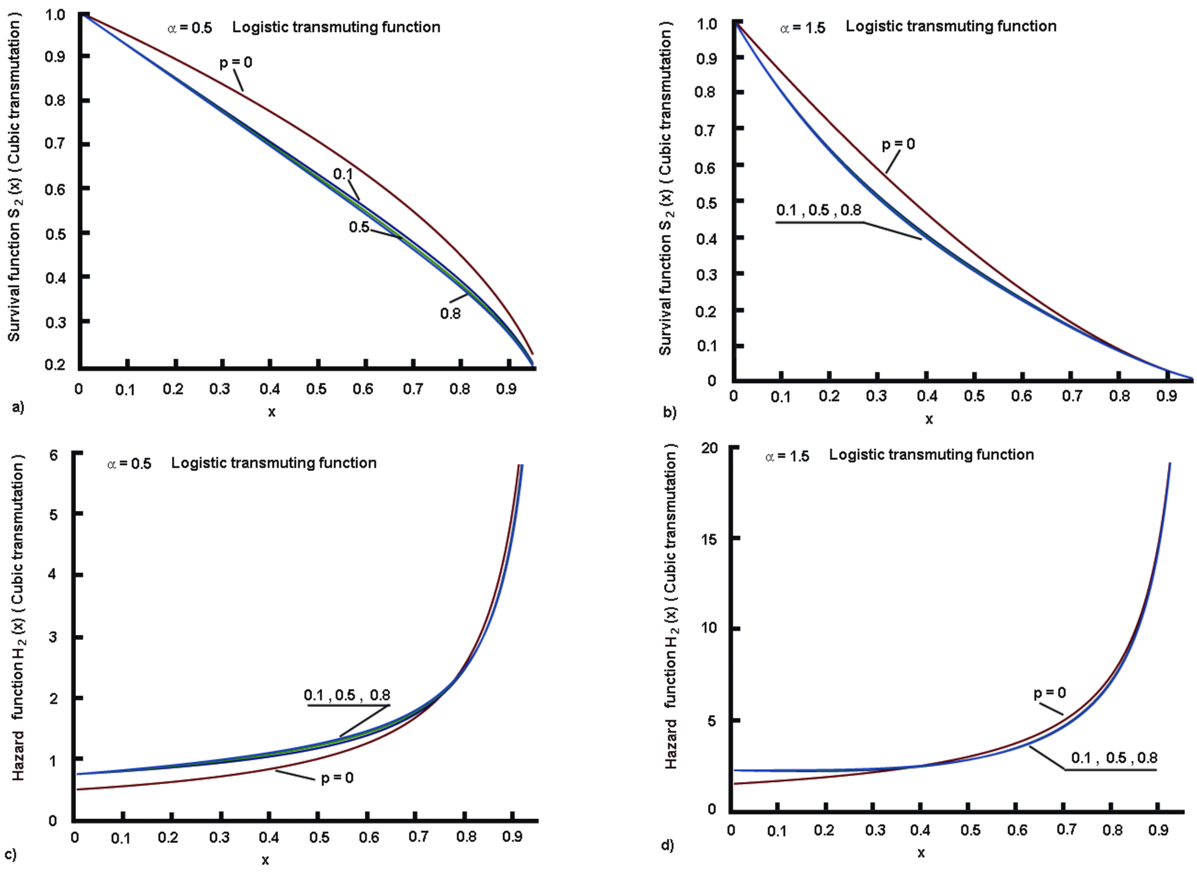

Figure 13. Survival and Hazard function with Standard Logistic Function as a variable transmutation parameter. Cubic transmutations. Left column: $\alpha = 0.5$; Right column: $\alpha = 1.5$

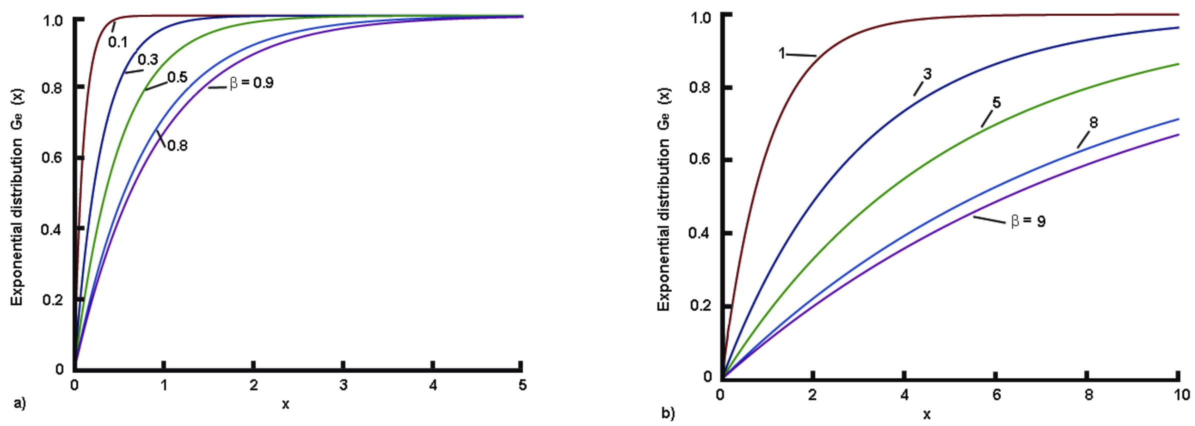

Figure 14. Two cases of basic exponential distribution: Left- with $\beta = 0.5$; Right- with $\beta = 1.5$

see that the transmuted profiles are well distinguished that could be attributed to both the rank of transmutation and the type of variable transmuting function chosen. Moreover, this can be related to the type of the basic exponential distribution which has an important control on its behaviour through the rate parameter β.

The Survival and the Hazard functions of the transmuted (quadratic) exponential distribution are shown in Figure 17.

8 Final comments and some emerging problems

This work conceived and explored tough examples of transmutations of distributions through a variable transmuting parameter (function) depending on the independent variable. The numerical experiments demonstrate the effect of the new approach is successful but at the same time formulate new problems and raise questions that should be answered through new studies, among them:

- The inverse (backward) problems are related to the determination of the rate parameter p because actually, the use of a variable transmuting function generates new basic distributions. This task is strongly dependent on the type of both the baseline distribution

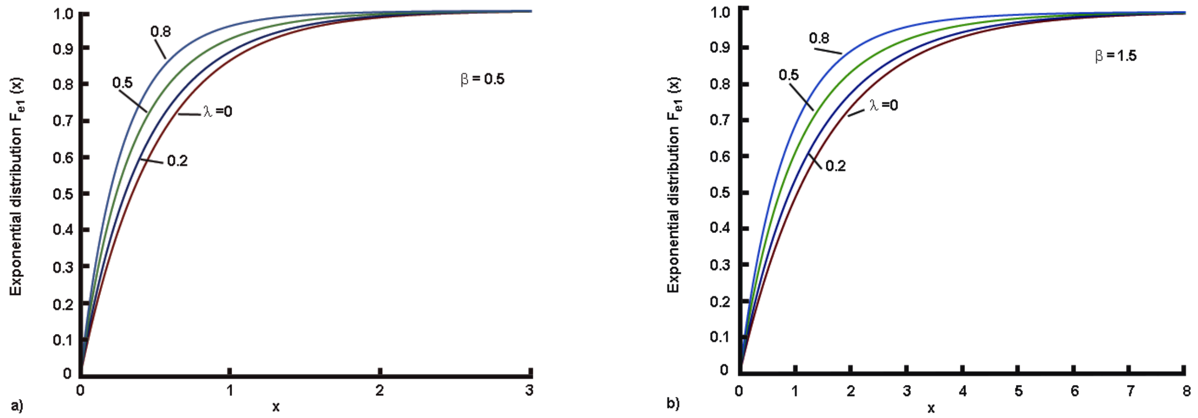

Figure 15. Two cases of quadratic transmutation of the exponential distribution (*cdf*) with fixed values of λ: Left- with $\beta = 0.5$; Right-with $\beta = 1.5$

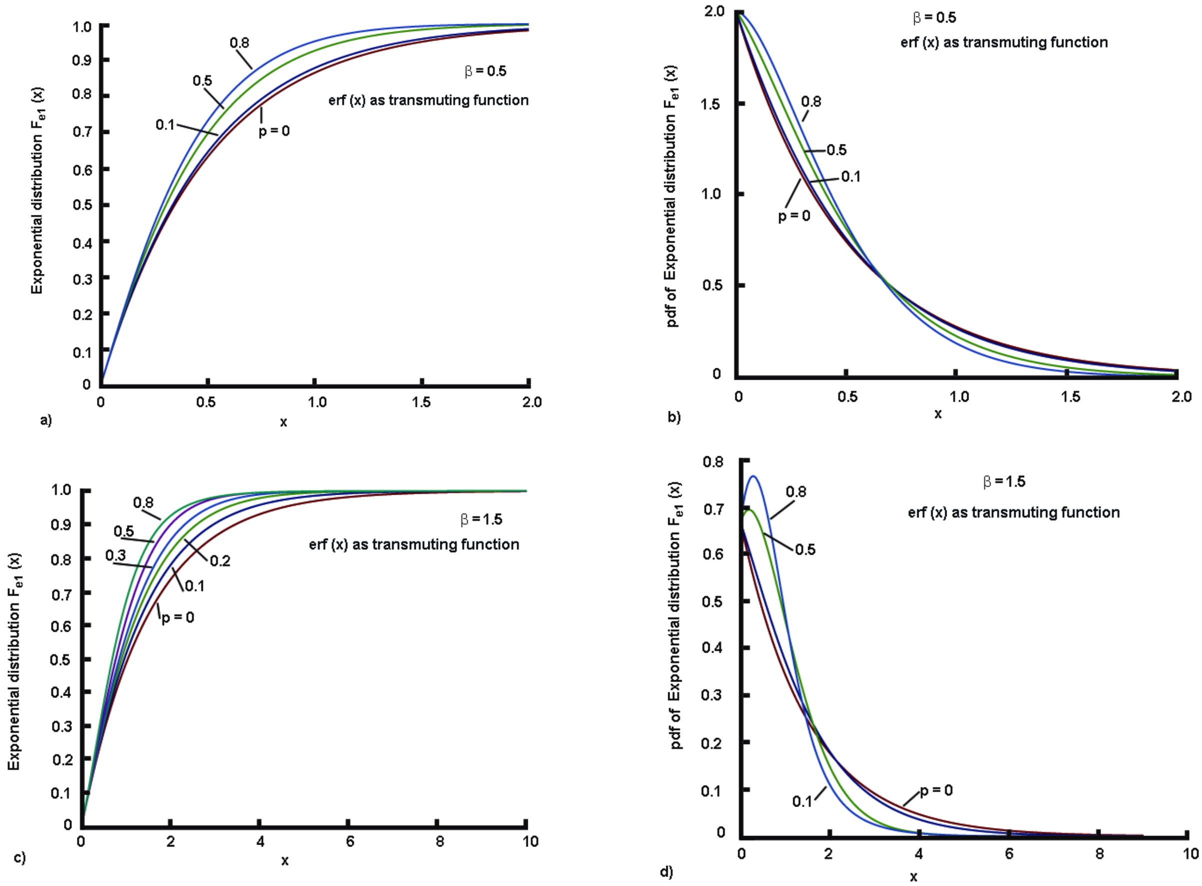

Figure 16. Cumulative and probability density functions of quadratic transmuted exponential distribution. Error-Function as a variable transmutation parameter: Left column: *cdf*s; Right column: *pdf*s

and the activation function and might be solved either analytically or numerically.

- Development of moments, quantile functions, random number generations, and many other related functions and parameters such as the ones well known from the cases when discrete transmuting parameters are applied. These are directions towards new studies beyond the scope of the present investigation.

- The new problems emerging in this study need the development of new analytical and numerical techniques for resolving the problem mentioned above and this draws new challenging areas for investigation.

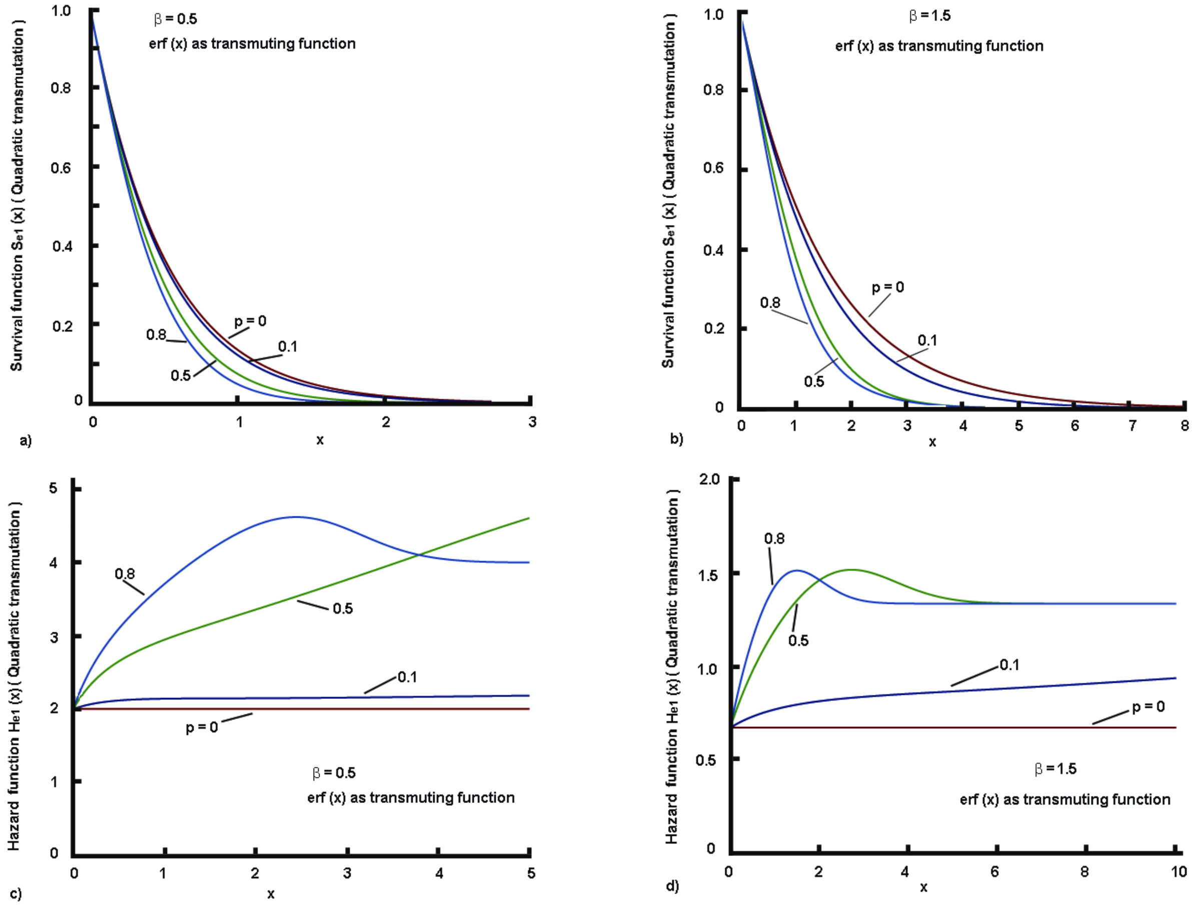

Figure 17. Survival and Hazard function of the quadratic transmuted exponential distribution. Error-Function as a variable transmutation parameter: Left column: $\beta = 0.5$; Right column: $\beta = 1.5$

9 Conclusions

A new concept in the transmutation of distributions applying variable transmuting (activation) function was conceived in this study. The idea of a variable transmuting parameter, dependent on the independent variable, was tested with the power distribution applying quadratic and cubic transmutations. This was performed through applications of two transmuting activation functions: the *error-function* and *standard logistic function*, and obeying the conditions imposed on the transmuting parameters imposed on it in the original concept of the transmutation mapping. Additional numerical experiments with the exponential distribution demonstrate the feasibility of the new approach and elucidate the fact that the effect of the transmutation strongly depends on the type of the function (distribution) to which it is applied. This is just the beginning and new tests with experimental data and available baseline distributions will allow elucidating the position of the distributions generated by the new concept among the well-know families of transformed functions. It is word remarking that the transmutation mapping can be applied not only to statistical distributions but to any other functions [35] thus allowing more flexibility in modelling of and approximate solutions.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding

The author declares that there is no funding source for the reported research.

Author's contributions

The research was carried out by the author and he accepts that the contributions and responsibilities belong to the author.

Acknowledgements

Not applicable.

References

[1] Dey, S., Kumar, D., Anis, M.Z., Nadarajah, S., & Okorie, I. A review of transmuted distributions. *Journal of the Indian Society for Probability and Statistics*, 22(1), 47-111, (2021). [CrossRef]

[2] Cordeiro, G.M., Ortega, E.M.M., & Popovic, B.V. The gamma-Lomax distribution. *Journal of Statistical computation and Simulation*, 85(2), 305-319, (2015). [CrossRef]

[3] Tahir, M.H., & Cordeiro, G.M. Compounding of distributions : a survey and new generalized classes. *Journal of Statistical Distributions and Applications*, 3(1) 1-35, (2016). [CrossRef]

[4] Afuecheta, E., Semeyutin, A., Chan, S., Nadarajah, S., & Ruiz, D.A.P. Compound distributions for financial returns. *Plos one*, 15(10), e0239652. [CrossRef]

[5] Al-Hussaini, E.K., & Ahsanullah, M. Exponentiated distributions. *Atlantis Studies in Probability and Statistics*, 21, (2015).

[6] Alexander, C. Cordeiro, G.M., Ortega, E.M., & Sarabia, J.M. Generalized beta-generated distributions. *Computational Statistics & Data Analysis*, 56(6), 1880-1897, (2012). [CrossRef]

[7] Alizadeh, M., Cordeiro, G.M., Brito, E.D., & Demetrio, C.G.B. The beta Marshal-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 2(1), 1-18, (2015). [CrossRef]

[8] Marshall, A.W., & Olkin, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652, (1997). [CrossRef]

[9] Patil, G.P., & Rao, C.R. Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics*, 34(2), 179-189, (1978). [CrossRef]

[10] Bakouch, H., Chesneau, C., & Enany, M. A weighted general family of distributions: Theory and practice. *Computational and Mathematical Methods*, 3(6), e1135, (2021). [CrossRef]

[11] Shaw W., & Buckley, I.R. The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtosis normal distribution from a rank transmutation map. Research Report. UCL discovery repository, *arXiv preprint: 0901.0434, Statistical Finance*), (2009). [CrossRef]

[12] Almalki, S.J., & Nadarajh, S. Modification of the Weibull distribution: a review. *Reliability Engineering & System Safety*, 124, 32-55, (2014). [CrossRef]

[13] Lai, M.T. Optimum number of minimal repairs for a system under increasing failure rate shock model with cumulative repair-cost limit. *International Journal of Reliability and Safety*, 7(2), 95-107, (2013).

[14] Elbatal, I. Transmuted modified inverse Weibull distribution: a generalization of the modified inverse Weibull probability distribution. *International Journal of Mathematical Archive*, 4(8), 117-129, (2013).

[15] Afify, A.Z., Nofal, Z.M., Yousuf, H.M., El Gebaly, Y.M., & Butt, N.S. The transmuted Weibull Lomax distribution: properties and application. *Pakistan Journal of Statistics and Operation Research*, 11, 135-152, (2015). [CrossRef]

[16] Granzotto, D.C.T., Louzada, F. & Balakrishnan N. Cubic rank transmuted distributions: inferential issues and applications. *Journal of statistical Computation and Simulation*, 87(14), 2760–2778, (2017). [CrossRef]

[17] Khan, M.S., & King,R. Transmuted modified Weibul distribution: a generalization of the modified Weibull probability distribution. *European Journal of pure and applied mathematics*, 6(1), 66-88, (2013).

[18] Okorie, I.E., Akpanta, A.C., Ohakwe, J., & Chikezie, D.C., The modified Power function distribution. *Cogent Mathematics*, 2017(4), 1219592, (2017). [CrossRef]

[19] Sakthivel, K.M., Rajitha, C.S., & Dhivakar, K. Two parameter cubic rank transmutation of Lindley distribution. *AIP Conference Proceedings* , 2261, 030086, (2020). [CrossRef]

[20] Rahman, M.M., Al-Zahrani, B., & Shahbaz, M.Q. A general transmuted family of distributions. *Pakistan Journal of Statistics and Operation Research*, 14(2), 451-469, (2018). [CrossRef]

[21] Ghosh, S., Kataria, K.K., & Vellaisamy, P. On transmuted generalized linear exponential distribution. *Communications in Statistics - Theory and Methods*, 50(9), 1978-2000, (2021). [CrossRef]

[22] Rahman, M.M., Al-Zahrani, B., Shahbaz, S.H., & Shahbaz, M.Q. Transmuted probability distributions: A review. *Pakistan Journal of Statistics and Operation Research*, 16(1), 83-94, (2020). [CrossRef]

[23] Celik, N. Some cubic rank transmuted distributions. *Journal of Applied Mathematics, Statistics and Informatics*, 14(2), 27-43, (2018). [CrossRef]

[24] Tian, Y., Tian, M., Zhu, Q. Transmuted Linear Exponential Distribution: A New Generalization of the Linear Exponential Distribution. *Communications in Statistics-Simulation and Computation*, 43(10), 2661-2677, (2014). [CrossRef]

[25] Alizadeh, M., Merovci, F., & Hamedani, G.G. Generalized transmuted family of distributions: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 46(4), 645-667, (2017). [CrossRef]

[26] Merovci, F., Alizadeh, M., & Hamedani, G.G. Another generalized transmuted family of distributions: properties and applications. *Austrian Journal of Statistics*, 45(3), 71-93, (2016). [CrossRef]

[27] Bourguignon, M., Gosh, I., & Cordeiro, G.M. General; results for transmuted family of distributions and new models. *Journal of Probability and Statistics*, 7208425, (2016). [CrossRef]

[28] Aryal, G.R., & Tsokos, C.P. On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods & Applications*, 71(12), e1401-e1407, (2009). [CrossRef]

[29] Aryal, G.R., & Tsokos, C.P. Transmuted Weibull distribution: A generalized of the Weibull probability distribution. *European Journal*

of pure and applied mathematics, 4(2), 89-102, (2011).
[30] Koopmans, L.H. Some simple singular and mixed probability distributions. *Some simple singular and mixed probability distributions*, 76(3), 297-299, (1969). [CrossRef]
[31] Karlis. D., & Xekalaki,E. Mixed poisson distributions. *International Statistical Review/Revue Internationale de Statistique*, 73(1), 35-58, (2005). [CrossRef]
[32] Fischer, S., Schumann, A., & Schulte, M. Characterisation of seasonal flood types according to timescales in mixed probability distributions. *Journal of Hydrology*, 539, 38-56, (2016). [CrossRef]
[33] McLachlan, G.J., & Peel, D. Finite mixture models. *John Wiley & Sons*, (2004).
[34] Owoloko, E.A., Oguntunde, P.E., & Adejumo, A.O. Performance rating of the transmuted exponential distribution: An analytical approach. *Springer Plus*, 4(1), 1-15, (2015). [CrossRef]
[35] Hristov, J. Integral-balance method with transmuted profiles: Concept, examples and emerging problems. *Journal of Computational and Applied Mathematics*, in press, (2022).

10 Appendix

The kurtosis (*Kurt*) is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution and is given as

$$Kurt = \frac{\mu_4}{\sigma^4},$$

and the skewness ($\tilde{\mu}^3$) is a measure of the asymmetry of distribution and is given as

$$\tilde{\mu}^3 = \frac{\sum_{i}^{N}\left(X_i - \bar{X}\right)^3}{(N-1)\,\sigma^3}.$$