Mathematical Modelling and Numerical Simulation with Applications, 2022, 2(3), 177–186

https://www.mmnsa.org
ISSN Online: 2791-8564 / Open Access
https://doi.org/10.53391/mmnsa.2022.014

RESEARCH PAPER

Laplace transform collocation method for telegraph equations defined by Caputo derivative

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Abstract

The purpose of this paper is to find approximate solutions to the fractional telegraph differential equation (FTDE) using Laplace transform collocation method (LTCM). The equation is defined by Caputo fractional derivative. A new form of the trial function from the original equation is presented and unknown coefficients in the trial function are computed by using LTCM. Two different initial-boundary value problems are considered as the test problems and approximate solutions are compared with analytical solutions. Numerical results are presented by graphs and tables. From the obtained results, we observe that the method is accurate, effective, and useful.

Key words: Caputo fractional derivative; collocation method; telegraph equation; approximation solution; error analysis
AMS 2020 Classification: 26A33; 35R11; 44A10; 65G99; 65N99

1 Introduction

Differential equations are a powerful tool for modeling, analyzing, and considering many physical and engineering problems and are an important branch of applied mathematics. In particular, they occur in network design, fluid dynamics, wave motion, telecommunications, electromagnetic, wave distribution, and electronic dynamics (see [1], [2] and the references therein). They are used not only in engineering and physical systems, but also in economics, risk theory, and many other social sciences. On the other hand, telegraph equation, a special kind of hyperbolic equations, is a partial differential equation that frequently appears in electrical engineering. In particular, power transmission lines are defined and designed using telegraph equations [3], [4], [5]. Many different problems in electric, electronics and communication engineering can be modeled by telegraph equations (see [4], [6] and the references therein). Mathematical modelling of problems in communication systems and transmission lines and their solvability (analytic or most of the time approximate) have great importance in today's world in which technology and communication tools regarding them have developed and spread with and increasing velocity. Depending on whether the terminations are short or open circuits and whether they are fed by current or voltage sources, there are many forms of this equation, including local or nonlocal boundary conditions.

Many of physical systems exhibit intrinsic behavior of fractional order. Therefore, fractional calculus provides more accurate models for such systems than classical calculus [7], [8], [9], [10], [11]. A significant advantage of fractional modeling is seen in systems where inheritance and memory behavior play a role, since the fractional derivative also accounts for the past. Another advantage arises in the analysis of porous and/or self-similar structures, where the theory of fractals plays a role. A great number of papers has been studied on the numerical solution methods of different types of telegraph partial differential equations. Finite

➤ **Received:** 18.08.2022 ➤ **Revised:** 23.09.2022 ➤ **Accepted:** 26.09.2022 ➤ **Published:** 28.09.2022

difference methods [1], [4], [12], [13], [14], [15] are used mostly in the literature. Less frequently, there variation methods using differential quadrature algorithm [16], Radial basis function [17], Chebyshev cardinal function [18], interpolation scaling functions [19], Chebyshev Tau method [20], Galerkin method [21].

In 2014, weighted residuals method was applied to numerical solutions of hyperbolic telegraph equations [22]. Then, LTCM was firstly implemented for the same equation in [23] in 2017 and the results were compared by weighted residuals method. From the numerical results published in the literature, it was observed that LTCM method is more convenient and effective comparing to weighted residuals method. In [6], LTCM was successfully applied to some nonlinear fractional differential equations.

This paper examines numerical solutions of the following fractional differential equation:

$$\begin{cases} \frac{\partial^{2\alpha} y(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}} + y(t,x) = \frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}} + f(t,x), \\ \text{where } x \in (0,L),\ t \in (0,T),\ \alpha \in (0,1], \\ \\ y(0,x) = \phi(x),\ y_t(0,x) = \psi(x),\ \text{where } x \in [0,L], \\ \\ y(t,0) = y(t,L) = 0,\ \text{where } t \in [0,T]. \end{cases} \tag{1}$$

Here, ϕ, ψ, f and y are known and unknown continuous functions, respectively. The term ${}_0^C D_t^{\alpha} y(t,x) = \frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}}$ is Caputo fractional derivative. If $\alpha = 1$, then, the main equation in (1) is called a telegraph partial differential equation. LTCM method is used for finding numerical solutions of problem (1). Approximate solutions are compared to the exact solution found by LT method. Then, numerical solutions are shown by both graph and table and errors in numerical solutions are analysed.

2 LTCM for fractional-order telegraph equation

To clarify the essential mathematical details of LTCM, we consider a FTDE using a similar method in [6].

Taking the LT of problem (1), we get

$$s^{2\alpha} y(s,x) - s^{2\alpha-1} y(0,x) - s^{2\alpha-2} y_t(0,x) = -L\left[\frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}}\right] + L\left[\frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}}\right] - L\left[y(t,x)\right] + L\left[f(t,x)\right]. \tag{2}$$

After simple algebraic simplification and using initial conditions in (2), we have

$$y(s,x) = \frac{1}{s^{2\alpha}}\left\{ s^{2\alpha-1}\phi(x) + s^{2\alpha-2}\psi(x) - L\left[\frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}}\right] + L\left[\frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}}\right] - L\left[y(t,x)\right] + L\left[f(t,x)\right]\right\}. \tag{3}$$

The function $y(t,x)$ and its derivative in (3) are replaced with a trial function of the form

$$y = y_0 + \sum_{j=1}^{n} c_j y_j. \tag{4}$$

In the above equation, c_j is the constant coefficient and it is determined to satisfy initial conditions given in (1). Then, $y(s,x)$ is found as follows:

$$\begin{aligned} y(s,x) &= \frac{1}{s^{2\alpha}}\left\{ s^{2\alpha-1}\left(y_0(0,x) + \sum_{j=1}^{n} c_j y_j(0,x)\right) + \frac{\partial}{\partial t}\left[s^{2\alpha-2}\left(y_0(0,x) + \sum_{j=1}^{n} c_j y_j(0,x)\right)\right] \right. \\ &\quad \left. -L\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(y_0(t,x) + \sum_{j=1}^{n} c_j y_j(t,x)\right)\right] + L\left[\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(y_0(t,x) + \sum_{j=1}^{n} c_j y_j(t,x)\right)\right] - L\left[y(t,x)\right] + L\left[f(t,x)\right] \right\}. \end{aligned} \tag{5}$$

Taking the inverse LT of Eq. (5), we get

$$\begin{aligned} y_{new}(t,x) &= L^{-1}\left\{ \frac{1}{s^{2\alpha}}\left[s^{2\alpha-1}\left(y_0(0,x) + \sum_{j=1}^{n} c_j y_j(0,x)\right) \right.\right. \\ &\quad + \frac{\partial}{\partial t}\left(s^{2\alpha-2} y_0(0,x) + \sum_{j=1}^{n} c_j y_j(0,x)\right) - L\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(y_0(t,x) + \sum_{j=1}^{n} c_j y_j(t,x)\right)\right] \\ &\quad \left.\left. + L\left[\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(y_0(t,x) + \sum_{j=1}^{n} c_j y_j(t,x)\right)\right] - L\left(y_0(t,x) + \sum_{j=1}^{n} c_j y_j(t,x)\right) + L\left[f(t,x)\right] \right]\right\}. \end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq. (1), we obtain new collocating at points $x = x_k$ as follows:

$$\frac{\partial^{2\alpha} y_{new}(t,x_k)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} y_{new}(t,x_k)}{\partial t^{\alpha}} + y(t,x) - \frac{\partial^{2\alpha} y_{new}(t,x_k)}{\partial x^{2\alpha}} = f(t,x_k),\ \text{where } x_k = \frac{L-0}{n+1}, k = 1,2,\cdots,n. \tag{7}$$

Then, we can define the residual function by the following formula

$$R_n(t,x) = L[y_{new}(t,x)] - f(t,x), \tag{8}$$

where $y_n(t,x)$ and $y(t,x)$ demonstrate approximate and exact solutions, respectively and

$$L[y_n(t,x)] = \frac{\partial^{2\alpha} y_{new}(t,x)}{\partial t^{2\alpha}} + y(t,x) + \frac{\partial^{\alpha} y_{new}(t,x)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} y_{new}(t,x)}{\partial x^{2\alpha}}. \tag{9}$$

From the above formula, we can write

$$\frac{\partial^{2\alpha} y_{new}(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} y_{new}(t,x)}{\partial t^{\alpha}} + y(t,x) - \frac{\partial^{2\alpha} y_{new}(t,x)}{\partial x^{2\alpha}} = f(t,x) + R_n(t,x). \tag{10}$$

3 Numerical implementations

For the application of LTCM, we consider two different test problems in this section and compare approximate solutions with exact solutions.

Example 1 *As the first example, consider the following initial-boundary value problem for FTDE*

$$\begin{cases} \frac{\partial^{2\alpha} y(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}} + y(t,x) - \frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}} \\ = 6\left[\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right] x^3 + t^3\left[x^2 - x^3 + 6\frac{x^{3-2\alpha}}{\Gamma(4-2\alpha)} - 2t^3\frac{x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right], \\ \text{where } x,t \in (0,1),\ \alpha \in (0,1], \\ y(0,x) = y_t(0,x) = 0,\ \text{where } x \in [0,1], \\ y(t,0) = y(t,1) = 0,\ \text{where } t \in [0,1]. \end{cases} \tag{11}$$

First, we calculate (11) by LTCM.

From the formula of the trial function (Eq. (4)), approximate solution can be written as:

$$y_{app}(t,x) = c_1 x^2(x-1)t^3 + c_2 x(x-1)^2 t^3. \tag{12}$$

Taking the LT of the main equation of (11) and using Eq. (5), we get

$$\begin{aligned} s^{2\alpha}y(s,x) - s^{2\alpha-1}y(0,x) - s^{2\alpha-2}y_t(0,x) &= -L\left[\frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}}\right] - L\left[y(t,x)\right] + L\left[\frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}}\right] \\ &+L\left\{6\left[\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right]x^3 + t^3(x^2 - x^3) + \frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right\}. \end{aligned} \tag{13}$$

Then, using zero initial conditions, the above formula can be simplified and written as:

$$\begin{aligned} y(s,x) &= \frac{1}{s^{2\alpha}}\left\{-L\left[\frac{\partial^{\alpha} y(t,x)}{\partial t^{\alpha}}\right] - L\left[y(t,x)\right] + L\left[\frac{\partial^{2\alpha} y(t,x)}{\partial x^{2\alpha}}\right]\right. \\ &\left.+L\left[\left[\frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}\right]x^3 + t^3\left(x^2 - x^3\right) + \frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right]\right\}. \end{aligned} \tag{14}$$

From the formulas (12) and (14), we have

$$\begin{aligned} y(s,x) &= \frac{1}{s^{2\alpha}}L\left\{\left[-\frac{6(x^3-x^2)t^{3-\alpha}}{\Gamma(4-\alpha)} + \left(-x^3 + x^2 + \frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right)t^3\right]c_1\right. \\ &-6\left[\frac{x(x-1)^2 t^{3-\alpha}}{\Gamma(4-\alpha)} + \left(-x(x-1)^2 + \frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{4x^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right)t^3\right]c_2 \\ &\left.+\ 6\left(\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right)x^3 + t^3(x^2 - x^3) + \frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right\}. \end{aligned} \tag{15}$$

From the formula (15), $y(s,x)$ is found as follows:

$$y(s,x) = \left\{-\frac{6\left(x^3 - x^2\right)}{s^{4+\alpha}} + \frac{6}{s^{4+2\alpha}}\left[-\left(x^3 - x^2\right) + \frac{6x^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}\right]\right\}c_1 \tag{16}$$

$$
\begin{aligned}
&+\left\{-\frac{6\left(x^3-2x^2+x\right)}{s^{4+\alpha}}+\frac{6}{s^{4+2\alpha}}\left[-x^3+2x^2-x+\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right]\right\}c_2\\
&+\left(\frac{6}{s^{4+\alpha}}+\frac{6}{s^4}\right)x^3+\frac{6}{s^{4+2\alpha}}\left[x^2-x^3+\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right].
\end{aligned}
$$

Taking the inverse LT of (16), we obtain the following trial solution:

$$
\begin{aligned}
y_{new}(t,x) \;=\; &\left\{-6\left[\frac{t^{3+\alpha}}{\Gamma(4+\alpha)}+\frac{t^{3+2\alpha}}{\Gamma(4+2\alpha)}\right](c_1+c_2)+t^3+\frac{6t^{3+\alpha}}{\Gamma(4+\alpha)}-\frac{6t^{3+2\alpha}}{\Gamma(4+2\alpha)}\right\}x^3\\
&+\left\{6\left[\frac{t^{3+\alpha}}{\Gamma(4+\alpha)}+\frac{t^{3+2\alpha}}{\Gamma(4+2\alpha)}\right](c_1+2c_2)+\frac{t^{3+2\alpha}}{\Gamma(4+2\alpha)}\right\}x^2+\left\{-6\left[\frac{t^{3+\alpha}}{\Gamma(4+\alpha)}+\frac{t^{3+2\alpha}}{\Gamma(4+2\alpha)}\right]c_2\right\}x \qquad (17)\\
&+6\left[\frac{x^{3-\alpha}}{\Gamma(4-\alpha)}-2\frac{x^{2-\alpha}}{\Gamma(3-\alpha)}\right]c_1+\frac{6t^{3+2\alpha}}{\Gamma(4+2\alpha)}\left[\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right]c_2\\
&+\frac{6t^{3+2\alpha}}{\Gamma(4+2\alpha)}\left[\frac{x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right].
\end{aligned}
$$

Substituting (17) into Eq. (11), we have the following residual formula:

$$
\begin{aligned}
R(t,x,c_1,c_2) \;=\; &\frac{\partial^{2\alpha}y_{new}(t,x)}{\partial t^{2\alpha}}+\frac{\partial^{\alpha}y_{new}(t,x)}{\partial t^{\alpha}}+y(t,x)-\frac{\partial^{2\alpha}y_{new}(t,x)}{\partial x^{2\alpha}}\\
&-6\left[\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}+\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right]x^3-t^3\left[x^2-x^3+\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{2x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right]. \qquad (18)
\end{aligned}
$$

Taking the derivatives of Eq. (17) with respect to x and t, and writing in (18), we obtain

$$
\begin{aligned}
R(t,x,c_1,c_2) \;=\; &(Ax^3-Ax^2+Bx^3-Bx^2+C+D)c_1+\left(Ax^3-2Ax^2+Ax+Bx^3-2Bx^2+Bx+E+F+C+G-\frac{G}{4x}+R\right)c_2\\
&+K+L+M+N+S=0, \qquad (19)
\end{aligned}
$$

where

$$A=-t^3-\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha},\quad B=-t^3-\frac{6}{\Gamma(4+\alpha)}t^{3+\alpha},$$

$$C=36\left[\frac{1}{\Gamma(4+\alpha)}t^{3+\alpha}+\frac{1}{\Gamma(4+2\alpha)}t^{3+2\alpha}\right]\frac{1}{\Gamma(4-2\alpha)}x^{3-2\alpha},\quad D=-\frac{6}{\Gamma(4-3\alpha)}x^{3-3\alpha}-\frac{2}{\Gamma(3-2\alpha)}x^{2-2\alpha},$$

$$E=t^3\left[\frac{6}{\Gamma(4-3\alpha)}x^{3-3\alpha}-\frac{4}{\Gamma(3-2\alpha)}x^{2-2\alpha}+\frac{1}{\Gamma(2-2\alpha)}x^{1-2\alpha}\right],$$

$$F=\frac{6}{\Gamma(4+\alpha)}t^{3+\alpha}\left[\frac{6}{\Gamma(4-2\alpha)}x^{3-2\alpha}-\frac{4}{\Gamma(3-2\alpha)}x^{2-2\alpha}+\frac{1}{\Gamma(2-2\alpha)}x^{1-2\alpha}\right],$$

$$G=-24\left[\frac{1}{\Gamma(4+\alpha)}t^{3+\alpha}+\frac{1}{\Gamma(4+2\alpha)}t^{3+2\alpha}\right]\frac{1}{\Gamma(3-2\alpha)}x^{2-2\alpha},$$

$$R=-\frac{6}{\Gamma(4+2\alpha)}t^{3+2\alpha}\left[\frac{6}{\Gamma(4-4\alpha)}x^{3-4\alpha}-\frac{4}{\Gamma(3-4\alpha)}x^{2-4\alpha}+\frac{1}{\Gamma(2-4\alpha)}x^{1-4\alpha}\right],$$

$$K=x^3\left[t^3+\frac{12}{\Gamma(4-2\alpha)}t^{3-2\alpha}+\frac{18}{\Gamma(4-\alpha)}t^{3-\alpha}-\frac{6}{\Gamma(4+\alpha)}t^{3+\alpha}\right],$$

$$L=x^2\left[\frac{6}{\Gamma(4+\alpha)}t^{3+\alpha}-\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha}-\frac{6}{\Gamma(4-2\alpha)}t^{3-2\alpha}\right],\quad M=-\frac{6}{\Gamma(3-2\alpha)}x^{2-2\alpha}t^3,$$

$$N=\frac{6t^{3+\alpha}}{\Gamma(4+\alpha)}\left[-\frac{2}{\Gamma(3-2\alpha)}x^{2-2\alpha}-\frac{30}{\Gamma(4-2\alpha)}x^{3-2\alpha}\right],$$

$$S=\frac{1}{\Gamma(4+2\alpha)}t^{3+2\alpha}\left[\frac{36}{\Gamma(4-2\alpha)}x^{3-2\alpha}-\frac{12}{\Gamma(3-2\alpha)}x^{2-2\alpha}+\frac{12}{\Gamma(3-4\alpha)}x^{2-4\alpha}-\frac{36}{\Gamma(4-4\alpha)}x^{3-4\alpha}\right].$$

Then, from (19), we obtain

$$c_1=-\frac{K+L+M}{Ax^3-Ax^2+Bx^3-Bx^2+C+D},$$

$$c_2=\frac{N+S}{Ax^3-2Ax^2+Ax+Bx^3-2Bx^2+Bx+E+F+C+G-\frac{G}{4x}+R}.$$

Example 2 *As the second example, we consider the following initial-boundary value problem for FTDE*

$$\begin{cases}\frac{\partial^{2\alpha}y(t,x)}{\partial t^{2\alpha}}+6\frac{\partial^{\alpha}y(t,x)}{\partial t^{\alpha}}+2y(t,x)-\frac{\partial^{2\alpha}y(t,x)}{\partial x^{2\alpha}}=\left[-\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)}-6\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+2e^{-t}\right]\sin x-e^{-t}\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)},\\ \text{where } x\in(0,\pi),\ t\in(0,1),\ \alpha\in(0,1],\\ y(0,x)=\sin x,\ y_t(0,x)=-\sin x,\ \text{where } x\in[0,\pi],\\ y(t,0)=y(t,\pi)=0,\ \text{where } t\in[0,1].\end{cases} \tag{20}$$

By following the similar manner of the previous example, we now calculate (20) by LTCM.

From Eq. (4), approximate solution can be written as:

$$y_{app}(t,x)=(1-t)\sin x+c_1x^2(x-\pi)t^2+c_2x(x-\pi)^2t^2. \tag{21}$$

Taking the LT of Eq. (20) and using the formula (5), we obtain

$$\begin{aligned}s^{2\alpha}y(s,x)-s^{2\alpha-1}y(0,x)-s^{2\alpha-2}y_t(0,x) &= -6L\left[\frac{\partial^{\alpha}y(t,x)}{\partial t^{\alpha}}\right]-2L\left[y(t,x)\right]+L\left[\frac{\partial^{2\alpha}y(t,x)}{\partial x^{2\alpha}}\right]\\ &+L\left\{\left[-\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)}-6\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+2e^{-t}\right]\sin x-e^{-t}\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right\}.\end{aligned} \tag{22}$$

Using initial condition of (20), y(s, x) is obtained as:

$$\begin{aligned}y(s,x) &= \left(\frac{1}{s}-\frac{1}{s^2}\right)\sin x+\frac{1}{s^{2\alpha}}\left\{-6L\left[\frac{\partial^{\alpha}y(t,x)}{\partial t^{\alpha}}\right]-2L\left[y(t,x)\right]+L\left[\frac{\partial^{2\alpha}y(t,x)}{\partial x^{2\alpha}}\right]\right.\\ &\left.+L\left\{\left[-\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)}-6\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}+2e^{-t}\right]\sin x-e^{-t}\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right\}\right\}.\end{aligned} \tag{23}$$

From the formulas (20) and (23), we have

$$\begin{aligned}y(s,x) &= \left[\frac{1}{s}-\frac{2}{s^2}-\frac{2}{s^{2\alpha+1}}+\frac{2}{s^{2+2\alpha}}+\frac{2}{s^{2\alpha}(s+1)}\right]\sin x\\ &+\left\{\left(-\frac{12}{s^{3+\alpha}}-\frac{4}{s^{3+2\alpha}}\right)x^2(x-\pi)+\frac{2}{s^{3+2\alpha}}\left[\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{2\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right]\right\}c_1\\ &+\left\{\left(-\frac{12}{s^{3+\alpha}}-\frac{4}{s^{3+\alpha}}\right)x^2(x-\pi)+\frac{2}{s^{3+2\alpha}}\left[\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{\pi^2x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right]\right\}c_2\\ &+\left(\frac{1}{s^{2\alpha+1}}-\frac{1}{s^{2\alpha+2}}-\frac{1}{s^{2\alpha}(s+1)}\right)\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}.\end{aligned} \tag{24}$$

Taking the inverse LT of (24), the following new trial solution is obtained:

$$\begin{aligned}y_{new}(t,x) &= \left[-\frac{12t^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{4t^{2+2\alpha}}{\Gamma(3+2\alpha)}\right](c_1+c_2)x^3+\left[\pi\left(\frac{12t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{4t^{2+2\alpha}}{\Gamma(3+2\alpha)}\right)(c_1+2c_2)\right]x^2\\ &+\left[-\pi^2\left(\frac{12t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{4t^{2+2\alpha}}{\Gamma(3+2\alpha)}\right)c_2\right]x+\frac{t^{2+2\alpha}}{\Gamma(3+2\alpha)}\left[\frac{12x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right]c_1\\ &+\left[1-2t-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{2t^{1+2\alpha}}{\Gamma(2+2\alpha)}+\frac{2e^{-t}t^{2\alpha-1}}{\Gamma(2\alpha)}\right]\sin x\\ &+\frac{t^{2+2\alpha}}{\Gamma(3+2\alpha)}\left[\frac{12x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{8\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{2\pi^2x^{1-2\alpha}}{\Gamma(2-2\alpha)}\right]c_2\end{aligned} \tag{25}$$

$$+\frac{x^{1-2\alpha}}{\Gamma(2-2\alpha)}\left[\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}-\frac{e^{-t}t^{2\alpha-1}}{\Gamma(2\alpha)}\right].$$

Substituting (25) into Eq. (20), we have the following residual formula:

$$\begin{aligned} R(t,x,c_1,c_2) &= \frac{\partial^{2\alpha}y(t,x)}{\partial t^{2\alpha}}+6\frac{\partial^{\alpha}y(t,x)}{\partial t^{\alpha}}+2y(t,x)-\frac{\partial^{2\alpha}y(t,x)}{\partial x^{2\alpha}} \\ &\quad -\left[-\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)}-\frac{6t^{1-\alpha}}{\Gamma(2-\alpha)}+2e^{-t}\right]\sin x+\frac{e^{-t}x^{1-2\alpha}}{\Gamma(2-2\alpha)}. \end{aligned} \tag{26}$$

Taking the derivatives of Eq. (25) with respect to x and t, and writing in Eq. (26), we obtain the formula of $R(t,x,c_1,c_2)$ as

$$\begin{aligned} R(t,x,c_1,c_2) &= \left[-9ax^3+9a\pi x^2+\frac{6ax^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{2a\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}-\frac{dt^{2+2\alpha}}{\Gamma(3+2\alpha)}+bk\right]c_1 \\ &+\left[-9ax^3+18a\pi x^2+\frac{4a\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{6ax^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4a\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}-9a\pi^2x+\frac{a\pi^2x^{1-2\alpha}}{\Gamma(2-2\alpha)}+ck-\frac{et^{2+2\alpha}}{\Gamma(3+2\alpha)}\right]c_2 \\ &+f\sin x+\frac{hx^{1-2\alpha}}{\Gamma(2-2\alpha)}-\frac{gx^{1-4\alpha}}{\Gamma(2-4\alpha)}=0, \end{aligned} \tag{27}$$

where,

$$a=-\frac{12\left(t^{2-\alpha}-4t^2\right)}{\Gamma(3-\alpha)},\quad b=\frac{12x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{4\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)},\quad c=\frac{12x^{3-2\alpha}}{\Gamma(4-2\alpha)}-\frac{8\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{2\pi^2x^{1-2\alpha}}{\Gamma(2-2\alpha)},$$

$$d=\frac{12x^{3-4\alpha}}{\Gamma(4-4\alpha)}-\frac{4\pi x^{2-4\alpha}}{\Gamma(3-4\alpha)},\quad e=\frac{12x^{3-4\alpha}}{\Gamma(4-4\alpha)}-\frac{8\pi x^{2-4\alpha}}{\Gamma(3-4\alpha)}+\frac{2\pi^2x^{1-4\alpha}}{\Gamma(2-4\alpha)},$$

$$f=-2t-\frac{8t^{1-\alpha}}{\Gamma(2-\alpha)}-\frac{12t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{1-2\alpha}t^{2\alpha-1}}{\Gamma(2-2\alpha)\Gamma(2\alpha)}+\frac{12t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{12t^{1-\alpha}t^{2\alpha-1}}{\Gamma(2-\alpha)\Gamma(2\alpha)}-\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)}+e^{-t}\left[\frac{2}{t}+\frac{12t^{\alpha-1}}{\Gamma(\alpha)}+\frac{4t^{2\alpha-1}}{\Gamma(2\alpha)}-2\right],$$

$$h=t+\frac{t^{1-2\alpha}t^{2\alpha-1}}{\Gamma(2-2\alpha)\Gamma(2\alpha)}+\frac{6t^{\alpha}}{\Gamma(\alpha+1)}-\frac{6t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{6t^{1-\alpha}t^{2\alpha-1}}{\Gamma(2-\alpha)\Gamma(2\alpha)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)}+e^{-t}\left[-\frac{1}{t}-\frac{6t^{\alpha-1}}{\Gamma(\alpha)}-\frac{4t^{2\alpha-1}}{\Gamma(2\alpha)}-2\right],$$

$$g=\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{e^{-t}t^{2\alpha-1}}{\Gamma(2\alpha)},k=t^2+\frac{6t^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{2t^{2+2\alpha}}{\Gamma 2\alpha+3)}.$$

From (27), we obtain

$$c_1=-\left[f\sin x+\frac{hx^{1-2\alpha}}{\Gamma(2-2\alpha)}\right]\left[-9ax^3+9a\pi x^2-\frac{2a\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{6ax^{3-2\alpha}}{\Gamma(4-2\alpha)}+bk-\frac{dt^{2+2\alpha}}{\Gamma(3+2\alpha)}\right]^{-1},$$

$$c_2=\frac{gx^{1-4\alpha}}{\Gamma(2-4\alpha)}\left[-9ax^3+18a\pi x^2-\frac{4a\pi x^{2-2\alpha}}{\Gamma(3-2\alpha)}+\frac{6ax^{3-2\alpha}}{\Gamma(4-2\alpha)}+\frac{a\pi^2x^{1-2\alpha}}{\Gamma(2-2\alpha)}-9\pi^2ax+ck-\frac{et^{2+2\alpha}}{\Gamma(3+2\alpha)}\right]^{-1}.$$

4 Error analysis

Errors in numerical solutions are computed by the following error formula

$$Error = \max\left|y_{exact}-y_{app}\right|,$$

where $y_{exact}(t,x)$ represents the exact solution and $y_{app}(t,x)$ represents the approximate solution obtained by using LTCM. Exact solutions of the first and second examples are $t^3(x^2-x^3)$ and $e^{-t}\sin x$ respectively. Approximate solution for the first example is $c_1x^2(x-1)t^3+c_2x(x-1)^2t^3$. For Example 2, it is equal to $(1-t)\sin x+c_1x^2(x-\pi)t^2+c_2x(x-\pi)^2t^2$.

As seen from the yellow and greenish region of Figs. 1 and 2, when t changes between 0.2 and 0.8, and x approaches to 1 for $\alpha=0.5$, the difference between exact solution and approximate solution increases. For the other values, the difference between exact and approximate solution is not obvious. Moreover, when $t=0.6$ and $x=1$, exact solution is almost 3 times greater than approximate solution. When t changes between 0.2 and 0.8, and x approaches to 1 for 0.99, the difference between exact solution and approximate solution increases as a similar result for $\alpha=0.5$, but when $t=0.6$ and $x=1$, exact solution is almost 10 times greater than approximate solution as shown in Figs. 3 and 4. For the other values, the difference between exact and approximate

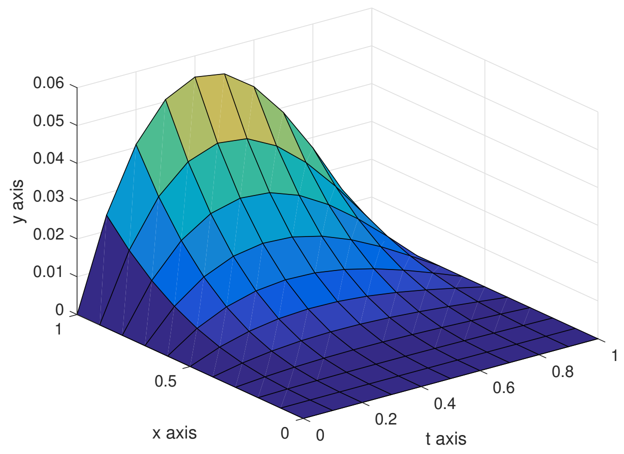

Figure 1. Approximate solution of Example 1 for $\alpha = 0.5$.

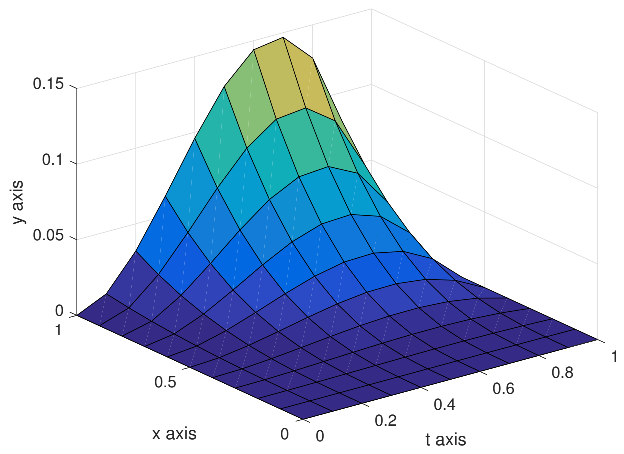

Figure 2. Exact solution of Example 1 for $\alpha = 0.5$.

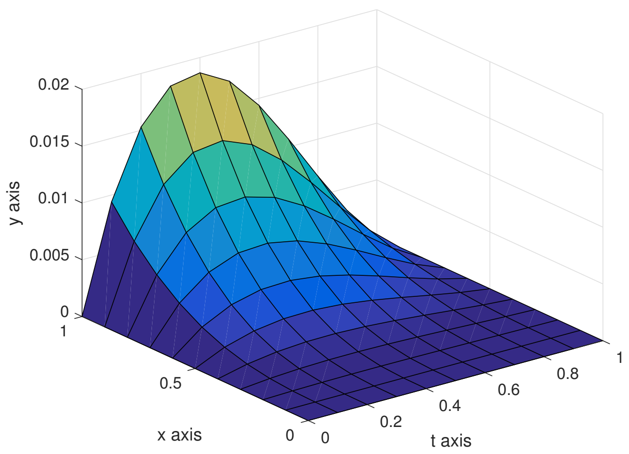

Figure 3. Approximate solution of Example 1 for $\alpha = 0.99$.

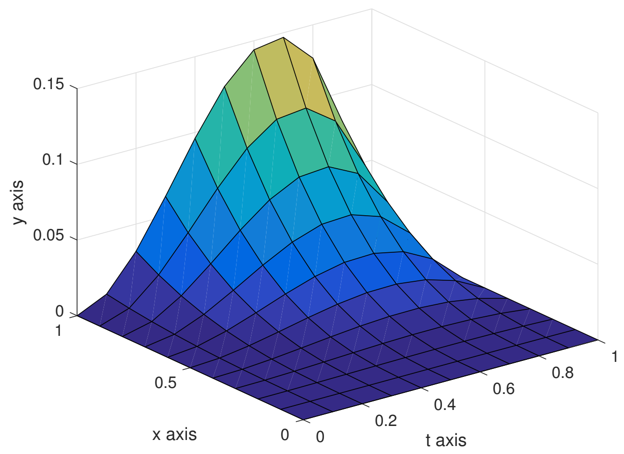

Figure 4. Exact solution of Example 1 for $\alpha = 0.99$.

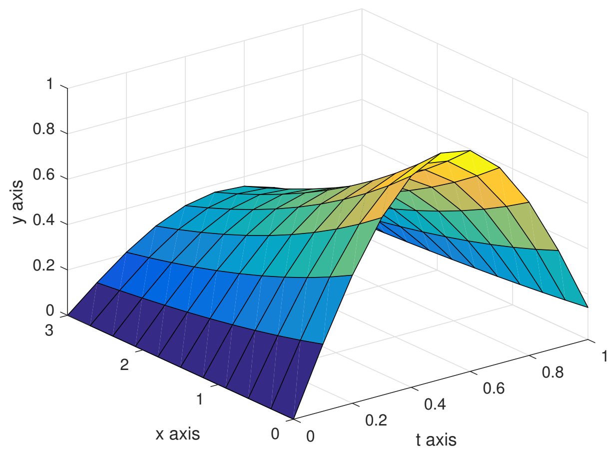

Figure 5. Exact solution of Example 2 for $\alpha = 1$.

solution is not obvious.

For Example 2, when t changes between 0.2 and 0.8, there is a great difference between the exact solution and approximate solution at the boundary point $x = 3$ as shown in Figs. 5 and 6. For the other values, the difference between exact and approximate solution is not obvious. For the better comparison of exact solution and approximate solution, we need to present results by Tables. Errors in the numerical solutions for different values of x, t, α for Examples 1 and 2 are presented in Tables 1 and 2, respectively.

Table 1. Error values for Example 1

x	t	α	**Errors**
0.01	0.01	0.01	9.9000×10^{-7}
0.01	0.01	0.5	9.9000×10^{-7}
0.01	0.01	0.99	1.7161×10^{-10}
0.1	0.591	0.01	0.00125
0.1	0.591	0.5	0.00183
0.1	0.591	0.99	8.1658×10^{-5}
0.248	0.9	0.01	0.03256
0.248	0.9	0.5	9.1222×10^{-6}
0.248	0.9	0.99	0.10107

When $x, t = 0.01$ and α changes from 0.01 to 0.5 for Example 1 in Table 1, there is no difference in the error but when α changes from 0.5 to 0.99, the error in the numerical solution decreases about 1000 times. When $x = 0.1, t = 0.591$ and α changes from 0.01 to 0.5, there is not much difference in the error but when α changes from 0.5 to 0.99, the error in the numerical solution decreases about 22 times. Lastly, when $x = 0.248, t = 0.9$ and α changes from 0.01 to 0.5, the error in the numerical solution decreases about 3570 times, but α changes from 0.01 to 0.99, the error increases about 7 times.

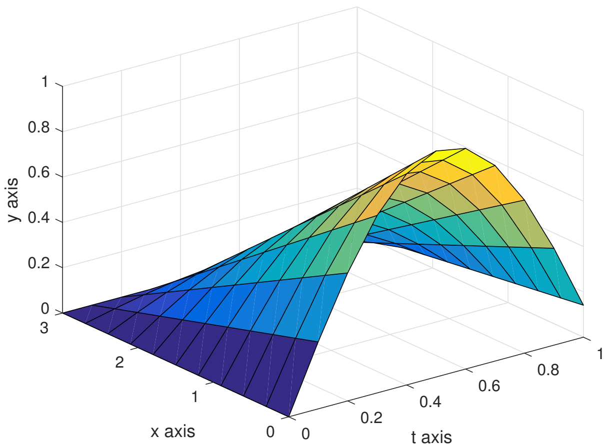

Figure 6. Approximate solution of Example 2 for $\alpha = 1$.

Table 2. Error values for Example 2

x	t	α	**Errors**
0	0.01	1	0
11/7	0.01	1	0.017
22/7	0.01	1	2.7494×10^{-7}
π	0.01	1	6.1029×10^{-21}
0.1	0.1	0.01	0.0043
0.1	0.1	0.5	0.0062
0.01	0.1	0.01	4.4652×10^{-4}
0.01	0.1	0.5	8.6324×10^{-5}

When $t = 0.01$ and $\alpha = 1$, x changes from 0 to 11/7 for Example 2, the error increases to 0.017 from 0, but when x changes from 11/7 to π, the error decreases to 6.1029×10^{-21} from 0.017. When $x, t = 0.1$ and α changes from 0.01 to 0.5, the error increases about 1.5 times, but when $x = 0.01, t = 0.1$ and α changes from 0.01 to 0.5, the error decreases about 5 times.

5 Conclusion

A combination of LTCM to develop approximate methods for fractional order telegraph partial differential equation has been adopted in this paper. The exact solution has been compared with approximate solutions in two different test problems. Error analysis has been done and it has been seen that the results were effective. However, due to the solution method for the approximate solution, when $x \to 1$, the solution goes to zero, causing the simulations to look far from each other. This deficiency due to the comparison of the simulations has been eliminated by giving the exact and approximate solutions in Table 1. As a future problem for the further developments of the present work, higher dimensional FTDEs can be studied. Moreover, this method can be also applied to nonlinear FTDEs by developing an algorithm due to the difficulty of processing.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interests.

Funding

Not applicable.

Author's contributions

M.M.: Conceptualization, Methodology, Software, Data Curation, Writing-Original draft preparation. M.E.K.: Visualization, Investigation, Supervision, Validation, Writing-Reviewing and Editing. All authors discussed the results and contributed to the final manuscript.

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