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RESEARCH PAPER

On the relations between a singular system of differential equations and a system with delays

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Abstract

In this article, we consider a class of systems of differential equations with multiple delays. We define a transform that reformulates the system with delays into a singular linear system of differential equations whose coefficients are non-square constant matrices, and the number of their columns is greater than the number of their rows. By studying only the singular system, we provide a form of solutions for both systems.

Key words: Delays; differential; singular systems; underdetermined systems
AMS 2020 Classification: 93B30; 93B60; 93C05

1 Introduction

In this article, we consider a class of systems of differential equations with multiple delays and define a transform that reformulates the system with delays into an underdetermined singular linear system of differential equations. The importance of this article is to develop a new idea and method that can bring new insight to researchers between systems with delays and singular systems of differential equations.
We are interested in the following system of differential equations with delays:

$$A_{n+1}x'(t) = A_0x(t) + A_1x(t+\tau_1) + A_2x(t+\tau_2) + \ldots + A_nx(t+\tau_n) + V(t), \tag{1}$$

where $\tau_i > 0$ is constant time delay, $V \in \mathbb{C}^{r\times 1}$, $A_i \in \mathbb{C}^{r\times r}$, and $x : [0, +\infty) \to \mathbb{C}^{r\times 1}$.
Systems of differential equations with delays have become more and more important nowadays. In the past few years, there have been lots of papers concerned with delays, see [18, 25, 26, 30], and their applications in macroeconomics, engineering, etc, see [4, 19, 27, 22, 23, 24, 31, 35].
We now consider the following singular system of differential equations:

$$EY'(t) = AY(t) + V(t), \tag{2}$$

where $E, A \in \mathbb{C}^{r\times m}$, $r < m$, $V \in \mathbb{C}^{r\times 1}$, and $Y : [0, +\infty) \to\in \mathbb{C}^{m\times 1}$.
Singular systems of differential equations, see [2, 3, 5, 7, 12, 17, 20], and difference equations, see [8, 21] have attracted the interest of several researchers in the last few decades. Some interesting results have also been obtained for singular systems of equations evolving fractional operators, see [1, 6, 9, 11, 13, 15, 29, 34]. This type of systems appear in control theory, see [3, 10, 33], and in several applications in electrical engineering such as the modeling of electrical circuits, see [20], electricity markets, see [14], and power system dynamics, see [28, 32].

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The article provides mainly two results. For the first result we consider the system with delays (1) and construct an equivalent singular system with a singular pencil without delays in the form of (2). This result can also be seen as a transform that connects these two systems. In the second result we consider the singular system (2) and by using matrix theory we provide a form of its solutions. Then, by using this formula and by taking into account that (2) is equivalent to (1), without any further computations we obtain a form of solutions also for the system with delays (1). Through this method we aim to connect two systems of different nature and have an alternative way to discuss and study the system with delays (1).

2 Main results

We, firstly, state the following theorem:

Theorem 1 *Consider a system with delays in the form of* (1)*. Then there exists a singular system of differential equations in the form of* (2) *that is equivalent to* (1)*.*

Proof 1 *System* (1) *has the form*

$$A_{n+1}x'(t) = A_0x(t) + A_1x(t+\tau_1) + A_2x(t+\tau_2) + \ldots + A_nx(t+\tau_n) + V(t).$$

We adopt the following notation:

$$\begin{aligned} y_1(t) &= x(t), \\ y_2(t) &= x(t-\tau_1), \\ y_3(t) &= x(t-\tau_2), \\ &\vdots \\ y_n(t) &= x(t-\tau_n). \end{aligned}$$

Furthermore

$$\begin{aligned} y_1'(t) &= x'(t), \\ y_2'(t) &= x'(t-\tau_1), \\ y_3'(t) &= x'(t-\tau_2), \\ &\vdots \\ y_n'(t) &= x'(t-\tau_n). \end{aligned}$$

Equivalently, we get:

$$\begin{bmatrix} A_{n+1} & 0_{r,r} & 0_{r,r} & \ldots & 0_{r,r} \end{bmatrix} \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & A_2 & \ldots & A_n \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \end{bmatrix} + V(t),$$

or, equivalently, in matrix form

$$EY'(t) = AY(t) + V(t),$$

where $m = r \cdot n$, $Y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, *and* $E = \begin{bmatrix} A_{n+1} & 0_{r,r} & 0_{r,r} & \ldots & 0_{r,r} \end{bmatrix}$, $A = \begin{bmatrix} A_{n+1} & 0_{r,r} & 0_{r,r} & \ldots & 0_{r,r} \end{bmatrix}$.

The proof is completed.

■

The importance of Theorem 1 is that we may study a system in the form of (1) which has delays through a system that may be singular and underdetermined, but it is linear and without delays. This system has a pencil that is singular because the matrix coefficients are non-square with $r < m$.

Despite several studies, most articles in the literature deal with singular systems that have regular pencils. The regularity of the pencil means that the matrices are square $r = m$, while the pencil formed has a determinant not identically equal to zero. Focus is then given in studying the solutions and stability of such a system through the eigenvalues of this pencil.

Singular systems with singular pencils are usually avoided. There are two types of singular pencils. A first case is the matrix coefficients of the system to be square but with a pencil that has a determinant identically zero. Meaning that the pencil is not invertible, something that is crucial for the existence of solutions of the system that appears in the frequency domain after the Laplace transform is applied to the system in the time domain. The other type of singular pencil is the matrix coefficients to be non-square. In this case, the determinant of the pencil cannot be defined.

In this article, as already mentioned, we study the system with delays (1) through the system (2) which is a singular underdetermined system of linear differential equations. The pencil of this type of system is singular. Unlike the regular pencil which may have finite eigenvalues & an infinite eigenvalue, a singular pencil has additional invariants the minimal column and row minimal indices. This type of invariant for such a pencil is not always easy to be obtained. It becomes even more complicated when dealing with large-scale systems. Another important characteristic of this case considered is that the existence of solutions for a system with a singular pencil is not automatically satisfied. This is very important for many applications for which the model is significant only for a certain range of its parameters. In these cases, a careful interpretation of results or even a redesign of the system may be needed.

In general, the pencil of (2) is characterized by a uniquely defined element, known as the complex Kronecker canonical form, see [3], [16], specified by the complete set of invariants of the singular pencil $sE - A$. This is the set of the finite–infinite eigenvalues and the minimal column–row indices. In the case of $r < m$ there exist only column minimal indices. Let $\mathcal{N}_r$ be the right null space of a matrix respectively. Then the equations $(sE - A)U(s) = 0_{r,1}$, have solutions in $V(s)$, which are vectors in the rational vector spaces $\mathcal{N}_r(sE - A)$. The binary vectors $U(s)$ express dependence relationships among the rows of $sE - A$. Note that $U(s) \in \mathbb{C}^{r\times 1}$ are polynomial vectors. Let d=dim$\mathcal{N}_r(sE-A)$. It is known, that $\mathcal{N}_r(sE-A)$ as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees

$$\epsilon_1 = \epsilon_2 = \ldots = \epsilon_g = 0 < \epsilon_{g+1} \leq \ldots \leq \epsilon_{g+h},$$

which is the set of *column minimal indices* of $sE - A$. This means there are $g + h = d$ column minimal indices. We are interested only in the h non-zero minimal indices. To sum up the invariants of a singular pencil with $r < m$ is the finite – infinite eigenvalues of the pencil and the minimal column indices as described above. Following the above given analysis, there exist non-singular matrices P, Q with $P \in \mathbb{C}^{r\times r}$, $Q \in \mathbb{C}^{m\times m}$, such that

$$\begin{aligned} PEQ &= E_K = I_p \oplus H_q \oplus E_\epsilon, \\ PAQ &= A_K = J_p \oplus I_q \oplus A_\epsilon, \end{aligned} \tag{3}$$

where J_p is the Jordan matrix for the finite eigenvalues, H_q is a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalues of the pencil $sA - E$. The matrices E_ϵ, A_ϵ are defined as

$$E_\epsilon = blockdiag\left\{L_{\epsilon_{g+1}}, L_{\epsilon_{g+2}}, \ldots, L_{\epsilon_d}\right\}, \tag{4}$$

where $L_\epsilon = \begin{bmatrix} I_\epsilon & \vdots & 0_{\epsilon,1} \end{bmatrix}$, for $\epsilon = \epsilon_{g+1}, \ldots, \epsilon_d$

$$A_\epsilon = blockdiag\left\{\bar{L}_{\epsilon_{g+1}}, \bar{L}_{\epsilon_{g+2}}, \ldots, \bar{L}_{\epsilon_d}\right\},$$

where $\bar{L}_\epsilon = \begin{bmatrix} 0_{\epsilon,1} & \vdots & I_\epsilon \end{bmatrix}$, for $\epsilon = \epsilon_{g+1}, \ldots, \epsilon_d$. Finally, the matrices P, Q can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q & Q_\epsilon \end{bmatrix}, \tag{5}$$

and by substituting the transformation $Y(t) = QZ(t)$ into (2) we obtain

$$EY'(t)QZ(t) = AQZ(t) + V(t),$$

whereby, multiplying by P, using (3)–(5) and setting $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \\ Z_\epsilon(t) \end{bmatrix}$, we arrive at at the subsystems

$$Z_p'(t) = J_p Z_p(t) + P_1 V(t), \tag{6}$$

$$H_q Z_q'(t) = Z_q(t) + P_2 V(t), \tag{7}$$

and

$$E_\epsilon Z_\epsilon'(t) = A_\epsilon Z_\epsilon(t) + P_3 V(t). \tag{8}$$

The subsystems (6), (7) have the following solutions, respectively:

$$Z_p(t) = e^{J_p t}c + \int_0^{\infty} e^{J_p(t-u)} P_1 V(u) du\,,$$

and

$$Z_q(t) = -\sum_{i=0}^{q_*-1} H_q^i P_2 \frac{d^i}{dt^i} V(t)\,.$$

The third subsystem has infinite solutions which can be taken arbitrarily as $Z_\epsilon = C(t)$. This can be proved as follows: Let $P_3 V = \begin{bmatrix} V_{\epsilon_{g+1}} \\ V_{\epsilon_{g+2}} \\ \vdots \\ V_{\epsilon_d} \end{bmatrix}$. If we set $Z_\epsilon = \begin{bmatrix} Z_{\epsilon_{g+1}} \\ Z_{\epsilon_{g+2}} \\ \vdots \\ Z_{\epsilon_d} \end{bmatrix}$, then we get

$$[L_{\epsilon_{g+1}} \oplus \ldots \oplus L_{\epsilon_d}] \begin{bmatrix} Z'^{\epsilon_{g+1}} \\ Z'_{\epsilon_{g+2}} \\ \vdots \\ Z'_{\epsilon_d} \end{bmatrix} = [\bar{L}_{\epsilon_{g+1}} \oplus \ldots \oplus \bar{L}_{\epsilon_d}] \begin{bmatrix} Z_{\epsilon_{g+1}} \\ Z_{\epsilon_{g+2}} \\ \vdots \\ Z_{\epsilon_d} \end{bmatrix} + \begin{bmatrix} V_{\epsilon_{g+1}} \\ V_{\epsilon_{g+2}} \\ \vdots \\ V_{\epsilon_d} \end{bmatrix}.$$

For the non-zero blocks, an arbitrary equation can be written as

$$L_{\epsilon_i} Z'_{\epsilon_i} = \bar{L}_{\epsilon_i} Z_{\epsilon_i} + V_{\epsilon_i} \quad , \quad i = g+1, g+2, \ldots, d \ ,$$

or, equivalently,

$$\begin{bmatrix} I_{\epsilon_i} & \vdots & 0_{\epsilon_i,1} \end{bmatrix} Z'_{\epsilon_i} = \begin{bmatrix} 0_{\epsilon_i,1} & \vdots & I_{\epsilon_i} \end{bmatrix} Z_{\epsilon_i} + V_{\epsilon_i},$$

or, equivalently,

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} z'_{\epsilon_i,1} \\ z'_{\epsilon_i,2} \\ \vdots \\ z'_{\epsilon_i,\epsilon_i} \\ z'_{\epsilon_i,\epsilon_i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{\epsilon_i,1} \\ z_{\epsilon_i,2} \\ \vdots \\ z_{\epsilon_i,\epsilon_i} \\ z_{\epsilon_i,\epsilon_i+1} \end{bmatrix} + \begin{bmatrix} v_{\epsilon_i,1} \\ v_{\epsilon_i,2} \\ \vdots \\ v_{\epsilon_i,\epsilon_i} \\ v_{\epsilon_i,\epsilon_i+1} \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} z'_{\epsilon_i,1} &= z_{\epsilon_i,2} + v_{\epsilon_i,1}, \\ z'_{\epsilon_i,2} &= z_{\epsilon_i,3} + v_{\epsilon_i,2}, \\ &\vdots \\ z'_{\epsilon_i,\epsilon_i} &= z'_{\epsilon_i,\epsilon_i+1} + v_{\epsilon_i,\epsilon_i}. \end{aligned}$$

It is clear from the above analysis that there is the number of unknown functions is $\epsilon_i + 1$ while the number of equations is ϵ_i. Hence by setting $C := C(t)$, the solutions of Z_ϵ can only be taken arbitrary as:

$$Z_\epsilon = C.$$

To conclude, in the case of a singular pencil with $r < m$, system (2) has the solution

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q & Q_\epsilon \end{bmatrix} \begin{bmatrix} Z_p(t)\Phi_0(t)C + \int_0^{\infty} \Phi(t-\tau)V(\tau)d\tau \\ -\sum_{i=0}^{q_*-1} H_q^i P_2 V^{(i)}(t) \\ Z_\epsilon \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_p \left[\Phi_0(t)C + \int_0^{\infty} \Phi(t-\tau)V(\tau)d\tau\right] - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V^{(i)}(t) + Q_\epsilon Z_\epsilon. \tag{9}$$

To sum up, the solution of (2) can be written in the form:

$$Y(t) = QZ(t),$$

or, equivalently,

$$Y = \begin{bmatrix} Q_p & Q_q & Q_\epsilon \end{bmatrix} \begin{bmatrix} Z_p(t) \\ Z_q(t) \\ Z_\epsilon(t) \end{bmatrix},$$

or, equivalently,

$$Y = Q_p[e^{J_p t}c + \int_0^\infty e^{J_p(t-u)} P_1 V(u)du] - Q_q[\sum_{i=0}^{q_*-1} H_q^i P_2 \frac{d^i}{dt^i} V(t)] + Q_\epsilon C(t),$$

or, equivalently,

$$Y = Q_p e^{J_p t} c + QK(t),$$

where

$$K(t) = \begin{bmatrix} \int_0^\infty e^{J_p(t-u)} P_1 V(u)du \\ -Q_q[\sum_{i=0}^{q_*-1} H_q^i P_2 \frac{d^i}{dt^i} V(t) \\ Q_\epsilon C(t) \end{bmatrix},$$

and $Q_p \in \mathbb{C}^{m\times p}$, and $Q \in \mathbb{C}^{m\times m}$. Hence by setting

$$Q_p = \begin{bmatrix} Q_p^1 \\ Q_p^2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix},$$

where $Q_p^1 \in \mathbb{C}^{r\times p}$, and $Q^1 \in \mathbb{C}^{r\times m}$, we arrive at the solution of the system with delays (1):

$$x(t) = Q_p^1 e^{J_p t} c + Q^1 K(t).$$

3 Conclusions

In this article, we considered a class of systems with delays in the form of (1). We proved that the system with delays can be studied through an equivalent singular system of differential equations whose coefficients are non-square constant matrices and the number of their columns is greater than the number of their rows. By taking into consideration that the relevant pencil is singular, we provided a formula for solutions. The importance of this result is that we may study system (1) which has delays through system (2), which may be singular and underdetermined, but it is linear and without delays.
As a future direction, we aim to further extend these theoretical results and examine promising relevant applications where delays appear, i.e. dynamics of electrical power systems, macroeconomic models, electricity market models, etc. In addition, we aim to extend our results to other types of systems where the memory effect appears such as systems of fractional differential equations, and systems of fractional nabla difference equations. For all this, there is already some research in progress.

Declarations

Consent for publication

Not applicable.

Conflicts of interest

The author declares that he has no conflict of interest.

Funding

Not applicable.

Author's contributions

The research was carried out by the author and he accepts that the contributions and responsibilities belong to the author.

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