ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

2023, 11(2), pp. 167-181, DOI: 10.20290/estubtdb.1278467

RESEARCH ARTICLE

FIVE POINT METRIC SPACES: GROMOV PRODUCT STRUCTURES, QUADRANGLE STRUCTURES AND EXPLICIT PARAMETERIZATIONS

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ABSTRACT

Let (X,d) be a finite metric space with elements $P_i, i=1,\dots,n$ and with distances $d_{ij}:=d(P_i,P_j)$ for $i,j=1,\dots,n$. The "Gromov product" Δ_{ijk}, is defined as $\Delta_{ijk}=\frac{1}{2}(d_{ij}+d_{ik}-d_{jk})$. (X,d) is called Δ-generic, if for each fixed i, the set of Gromov products has a unique least element, $\Delta_{ij_ik_i}$. The Gromov product structure on a Δ-generic finite metric space (X,d) is the map that assigns the edge $E_{j_ik_i}$ to P_i. A finite metric space is called "quadrangle generic", if for all 4-point subsets $\{P_i,P_j,P_k,P_l\}$, the set $\{d_{ij}+d_{kl}, d_{ik}+d_{jl}, d_{il}+d_{jk}\}$ has a unique maximal element. We define the "quadrangle structure" on a quadrangle generic finite metric space (X,d) as the map that assigns to each 4-point subset of X, the pair of edges corresponding to the maximal element of the sums of the distances. Two metric spaces (X,d) and (X,d') are said to be Δ-equivalent (Q-equivalent), if the corresponding Gromov product (quadrangle) structures are the same up to a permutation of X.

In this paper, Gromov product structures, quadrangle structures, optimal reductions and explicit parameterizations for 5-point spaces are obtained and compared with previous results in the literature. In the final part of this paper, we have used the Monte Carlo method to obtain the relative volume of each of the 5-point metric types inside the corresponding metric cone for 5-point spaces, meanwhile 102 different partitions of metric cone for 5-point spaces are derived, considering Gromov product structures. These 102 partitions, come in three symmetric classes forming three types of metrics for 5-point spaces. Thus, one can say that all the methods of classification given here or given before in the literature of finite metric spaces, give 3 types of metrics for 5-point spaces.

Keywords: Finite metric spaces, Split metric decompositions, Gromov products, Quadrangle structures

1. INTRODUCTION

The notions of Gromov product structures, Δ-equivalence, quadrangle structures and Q-equivalence have been defined in previous work [1]. Here, we present the applications of these notions to 5-point spaces. Basic definitions are quoted from [1].

Notation: Let (X,d) be a finite metric space with n elements P_i, $i=1,\dots,n$ $(n\geq 3)$ and let d_{ij} be the distance between P_i and P_j. The elements of X are also referred to as "vertices" or "nodes". E_{ij} and T_{ijk} denote respectively an edge and a triangle with corresponding vertices.

Gromov products: The quantity Δ_{ijk}, defined as

$$\Delta_{ijk}=\Delta_{ikj}=\frac{1}{2}(d_{ij}+d_{ik}-d_{jk})$$

is called the Gromov product of the triangle T_{ijk} at the vertex P_i [2].

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Received: 12.05.2023 Published: 28.08.2023

Δ-generic metrics: A metric space is called Δ-generic, if for each P_i the set of Gromov products Δ_{ijk} has a unique smallest element.

Gromov product structures: Let (X, d) be a Δ-generic finite metric space. Let $P_i \in X$, and let Δ_{ijk} be the minimal Gromov product at P_i, $(i = 1, \dots, n)$. The function that assigns the edge E_{jk} to the vertex P_i is called the Gromov product structure on X. Two Δ-generic metric spaces (X, d) and (X, d') are Δ-equivalent, if the corresponding Gromov product structures are the same up to a permutation of X.

The metric cone: The set C_n of all pseudo-metrics $d = (d_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ on a given n-point set X, is called the metric cone.

The metric fan: A decomposition of metric cone C_n into some sub-cones defined as below is called the metric fan [3]. Consider the $\binom{n}{2} \times$ n matrix $\mathcal{A}$ where the rows are numbered by the edges as

$$(1,2), (1,3), \dots, (1,n), (2,3), (2,4), \dots, (2,n), \dots, (n-1,n)$$

and the (i,j)-row $(i < j)$ is given by $e_i + e_j = (0, \dots, 1, \dots, 1, \dots, 0) \in \mathbb{R}^n$. Let $\mathcal{B}$ be an invertible $n \times n$ submatrix of $\mathcal{A}$ and denote the $\left[\binom{n}{2} - n\right] \times n$ matrix obtained by deleting $\mathcal{B}$ from $\mathcal{A}$ by $\mathcal{B}'$. Likewise, define $d_{\mathcal{B}} \in \mathbb{R}^n$ by choosing the components of $d \in \mathbb{R}^{\binom{n}{2}}$ corresponding to $\mathcal{B}$ and $d_{\mathcal{B}'} \in \mathbb{R}^{\binom{n}{2}-n}$ corresponding to $\mathcal{B}'$. Now consider the following system of equations and inequalities for $x \in \mathbb{R}^n$:

$$\mathcal{B}x = d_{\mathcal{B}} \text{ and } \mathcal{B}'x > d_{\mathcal{B}'}.$$

If this system has a solution we say that the matrix $\mathcal{B}$ is a "cell" or a "thrackle" for the metric d. The collection of cells of a metric d is denoted by $Cell(d)$. Two metrics d and d' on an n-point metric space X are said to be equivalent in the metric-fan sense, if they have the same collection of cells or what amounts to the same collection of sub-graphs, i.e. $Cell(d) = Cell(d')$. The equivalence class of a metric d is a sub-cone of the metric cone and these sub-cones constitute altogether the metric fan.

The classification of 6-point spaces with respect to Gromov product structures (Δ-equivalence) is obtained in [4]. In that work it is shown that there are 26 Δ-equivalence classes and also presented their correspondences to the classification by the decomposition of the metric fan. The list of Gromov product structures and the corresponding metric fan types for the 26 Δ-generic metrics are given in [4].

In [5], the Gromov classification of 7-point spaces has been obtained and shown that there are 431 equivalence classes. For 8-point metric spaces, we have obtained the Δ-equivalence classifications and found 11470 equivalence classes in the work on our website:
http://finitemetricspaces.khas.edu.tr/118F412_webpage_8pointspaces.pdf

The metric fan classification of n-point spaces for $n > 6$ is not known. It looks like the number of classes will be increasingly large and such a classification would not be practical. Even the Gromov product classification is becoming impractical for $n > 8$. Thus, we are looking for coarser equivalences that would reflect essential properties of a finite metric space.

Quadrangle generic metric spaces: An n-point finite metric space X is called "quadrangle generic", or Q-generic, if for every 4-point subset $\{P_a, P_b, P_c, P_d\} \subseteq X$, the set of distances

$$\{d_{ab} + d_{cd}, d_{ac} + d_{bd}, d_{ad} + d_{bc}\}$$

has a unique maximal element.

Quadrangle Structures: A quadrangle structure on a Q-generic finite metric space (X,d) is a map which assigns to any 4-point subset $\{P_a,P_b,P_c,P_d\}$ of X, the pair of edges corresponding to the maximal element of the set $\{d_{ab}+d_{cd},d_{ac}+d_{bd},d_{ad}+d_{bc}\}$. We denote the 4-point subset $\{P_a,P_b,P_c,P_d\}$ without any restriction on the sides by $Q(a,b,c,d)$ in which the ordering of the indices is irrelevant. If $d_{ac}+d_{bd}$ is maximal, the vertices are ordered as (P_a,P_b,P_c,P_d) and we denote this structured quadrangle by $Q(abcd)$ in which the cyclic permutations and reversal of the order of the indices give equivalent quadrangles.

***Q*-equivalence:** Two Q-generic metric spaces (X,d) and (X,d') are called Q-equivalent, if the corresponding quadrangle structures are the same up to a permutation of X.

Parameterization of 4-point spaces: Let the set of minimal Gromov products of the quadrangle $Q(abcd)$ be $\{\Delta_{abd},\ \Delta_{bac},\ \Delta_{cbd},\ \Delta_{dac}\}$ and let α and β be defined as

$$\alpha=\Delta_{\mathrm{abc}}-\Delta_{\mathrm{abd}},\qquad \beta=\Delta_{\mathrm{adc}}-\Delta_{\mathrm{adb}},$$

then, one has the following equalities between Gromov products

$$\alpha=\Delta_{abc}-\Delta_{abd}=\Delta_{bad}-\Delta_{bac}=\Delta_{cda}-\Delta_{cdb}=\Delta_{dcb}-\Delta_{dca},$$
$$\beta=\Delta_{adc}-\Delta_{adb}=\Delta_{bcd}-\Delta_{bca}=\Delta_{cba}-\Delta_{cbd}=\Delta_{dab}-\Delta_{dac},$$

and the distances are expressed as

$$d_{ab}=\Delta_{abd}+\Delta_{bac}+\alpha,\qquad d_{cd}=\Delta_{cbd}+\Delta_{dac}+\alpha,$$
$$d_{bc}=\Delta_{bac}+\Delta_{cbd}+\beta,\qquad d_{ad}=\Delta_{abd}+\Delta_{dac}+\beta,$$
$$d_{ac}=\Delta_{abd}+\Delta_{cbd}+\alpha+\beta,\quad d_{bd}=\Delta_{bac}+\Delta_{dac}+\alpha+\beta.$$

This is shown in Figure 1 below.

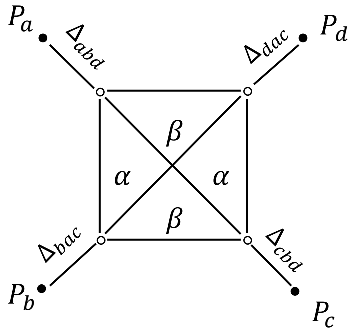

Figure 1. A quadrangle with the set of minimal Gromov products $\{\Delta_{abd},\Delta_{bac},\ \Delta_{cbd},\Delta_{dac}\}$.

Matrix representation of Gromov product structures: Gromov product structures on an n-point space are represented by the $n\times n$ matrix M_Δ defined by $M_\Delta(i,j)=1$ and $M_\Delta(i,k)=1$ if Δ_{ijk} is the minimal Gromov product at P_i and 0 otherwise [6].

Matrix representation of quadrangle structures: The matrix of a quadrangle structure Q, M_Q on an n-point space is an $n_d\times n_d$ matrix ($n_d=\frac{n(n-1)}{2}$) such that $M_Q(ab,cd)=1$ if the edges E_{ab} and E_{cd} are diagonals in $\{P_a,P_b,P_c,P_d\}$ and $M_Q(ab,cd)=0$ otherwise.

These matrix representations proved to be useful in determining equivalences/inequivalences of Δ- and Q-equivalence classes. We recall that two structures are equivalent if their matrices can be mapped to each other by a permutation of indices. Similarity and isospectrality of matrices leads to coarser classifications [6].

Split pseudo-metrics: A "split" $S=\{A,B\}$ of a finite set X is a partition of X into two non-empty subsets A and B. For simplicity we often identify the set of points of A with its index set. For each

$P_a \in X$, we denote by $S(a)$ the subset A or B that contains P_a. Corresponding to each split S we define the pseudo-metric δ_S by

$$\delta_S(a,a') = \begin{cases} 1, & if \;\; S(a) \neq S(a'), \\ 0, & if \;\; S(a) = S(a'). \end{cases}$$

If the number of elements of A or B is equal to k, the split is referred to as a k-split.

Totally split decomposable metrics: A metric on X is called totally split decomposable if it can be expressed as a linear combination (with non-negative coefficients) of the split metrics [7].

The isolation index of a split: The isolation index of a split $S = \{A, B\}$ is defined as

$$\alpha_{\{A,B\}} = \frac{1}{2} \min_{\{a,a' \in A, b, b' \in B\}} \{ \max\{ d_{ab} + d_{a'b'} \;, d_{ab'} + d_{a'b} \;, d_{aa'} + d_{bb'} \} - (d_{aa'} + d_{bb'}) \}.$$

Split prime: A pseudo-metric is called a split prime if all of its isolation indices are equal to zero [7].

Lemma 1: Let (X, d) be a finite metric space with n elements P_i $(i = 1, \dots, n)$ and let $S = \{A, B\}$ be a split for X. Then,

i. The isolation index for the 1-split with $A = \{P_a\}$ is the minimal Gromov product at P_a,
ii. If (X, d) is Q-generic, then the isolation index for the k-split with $A = \{P_{i_1}, \dots, P_{i_k}\}$ is non-zero if and only if for no pair of indices $a, a' \in A$, $E_{aa'}$ is a diagonal of the quadrangles $Q(a, a', b, b')$ where $b, b' \in B$.

Proof: See [1].

In [1], we have shown that the number of 2-splits in an n-point space is at most n. We have discussed the case $n = 6$ in terms of 3-splits, relating to the results of [7].

2. PARAMETERIZATION OF 5-POINT METRIC SPACES

In this section we will give an explicit parameterization of 5-point spaces using Gromov product structures, quadrangle structures and partial orders on Gromov products at each P_a. This parameterization coincides with the parameterization given in [8].

It is known that the Gromov product equivalence gives the known classification of 5-point Δ-generic metric spaces [4].

A: $\{\Delta_{125}, \Delta_{213}, \Delta_{324}, \Delta_{435}, \Delta_{514}\}$
B: $\{\Delta_{125}, \Delta_{213}, \Delta_{325}, \Delta_{425}, \Delta_{514}\}$,
C: $\{\Delta_{125}, \Delta_{213}, \Delta_{325}, \Delta_{425}, \Delta_{513}\}$.

Note that, if say Δ_{ijk} is minimal in the metric space X, then it is also minimal in every quadrangle $Q = \{P_i, P_j, P_k, P_l\}$. In a graphical presentation we indicate this by marking the corresponding angle by a filled arc as shown in Figure 2. For a 5-point metric space X, at least one of the Gromov products in any quadrangle belongs to the list of minimal Gromov products. It follows that for a 5-point space, the Gromov product structure determines the quadrangle structure. The determination of the parameters displayed in the quadrangles will be explained below.

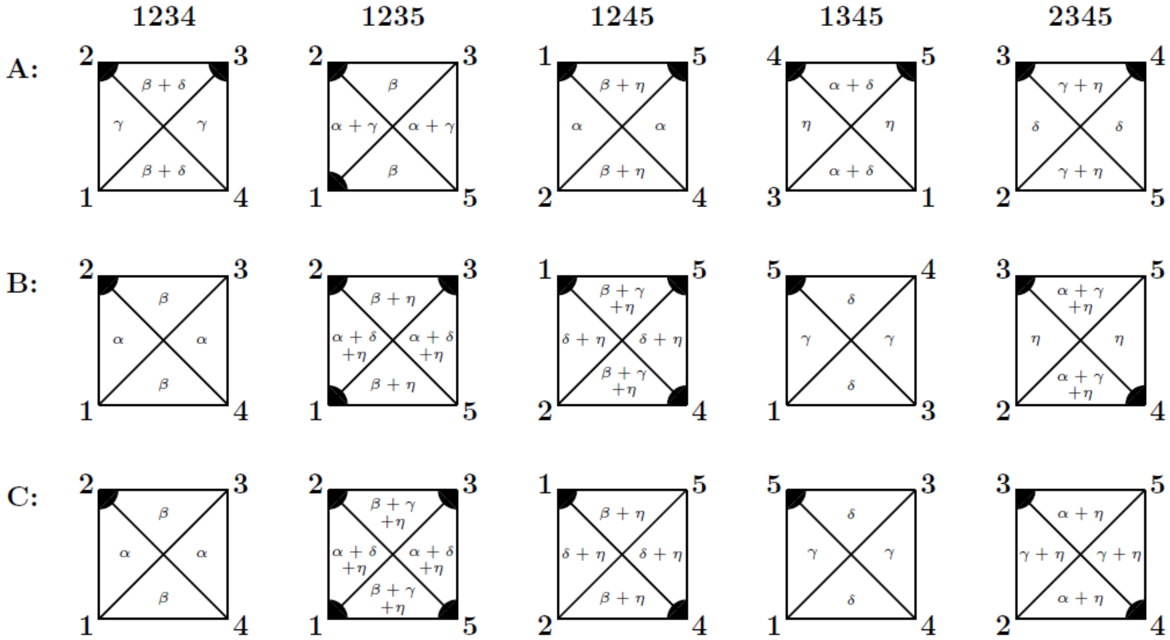

Figure 2. The structure of the 4-point subsets for the three types of 5-point metric spaces.

From Figure 2, we can see that, in Type A, the edges $E_{12}, E_{23}, E_{34}, E_{45}$ and E_{15} are "sides" in all quadrangles, hence Type A metrics are totally split-decomposable by Lemma 1. For Type B, there are 4 edges E_{45}, E_{15}, E_{12} and E_{23} that occur as sides in all quadrangles. Therefore it is not totally split-decomposable. Similarly for Type C, the edges that occur as "sides" in all quadrangles are E_{12}, E_{23}, E_{34} and E_{45}, hence it is not totally split-decomposable.

In order to obtain an explicit parameterization of these metrics, we will use the quadrangle structure to obtain partial order relations among the Gromov products, then use the relations $d_{ij} = \Delta_{ijk} + \Delta_{jik}$. The structure of the quadrangles in Figure 2 lead to the following order relations for each of the types A, B, C in the following way: Take quadrangle $Q(1234)$ of Type A for instance. Since $d_{12} + d_{34} < d_{13} + d_{24}$, equivalently $\frac{1}{2}(d_{12} + d_{14} - d_{24}) < \frac{1}{2}(d_{13} + d_{14} - d_{34})$ which is to say $\Delta_{124} < \Delta_{134}$; we can also say that since $d_{14} + d_{23} < d_{13} + d_{24}$ is equivalent to $\frac{1}{2}(d_{12} + d_{14} - d_{24}) < \frac{1}{2}(d_{12} + d_{13} - d_{23})$ which means $\Delta_{124} < \Delta_{123}$. Thus for each vertex of a quadrangle, two inequalities among three Gromov products could be derived by similar algebraic manipulations. The list of these inequalities for each type is given below. These order relations are used to determine isolation indices for 2-splits and the split primes.

From quadrangles of Type A, we have the following relations among Gromov products:

$$
\begin{array}{llll}
Q(1234): \Delta_{124} < \Delta_{123}, \Delta_{134}, & \Delta_{213} < \Delta_{214}, \Delta_{234}, & \Delta_{324} < \Delta_{312}, \Delta_{314}, & \Delta_{413} < \Delta_{412}, \Delta_{423}, \\
Q(1235): \Delta_{125} < \Delta_{123}, \Delta_{135}, & \Delta_{213} < \Delta_{215}, \Delta_{235}, & \Delta_{325} < \Delta_{312}, \Delta_{315}, & \Delta_{513} < \Delta_{512}, \Delta_{523}, \\
Q(1245): \Delta_{125} < \Delta_{124}, \Delta_{145}, & \Delta_{214} < \Delta_{215}, \Delta_{245}, & \Delta_{425} < \Delta_{412}, \Delta_{415}, & \Delta_{514} < \Delta_{512}, \Delta_{524}, \\
Q(1345): \Delta_{135} < \Delta_{134}, \Delta_{145}, & \Delta_{314} < \Delta_{315}, \Delta_{345}, & \Delta_{435} < \Delta_{413}, \Delta_{415}, & \Delta_{514} < \Delta_{513}, \Delta_{534}, \\
Q(2345): \Delta_{235} < \Delta_{234}, \Delta_{245}, & \Delta_{324} < \Delta_{325}, \Delta_{345}, & \Delta_{435} < \Delta_{423}, \Delta_{425}, & \Delta_{524} < \Delta_{523}, \Delta_{534}.
\end{array}
$$

which lead to the following Hasse diagrams given in Figure 3.

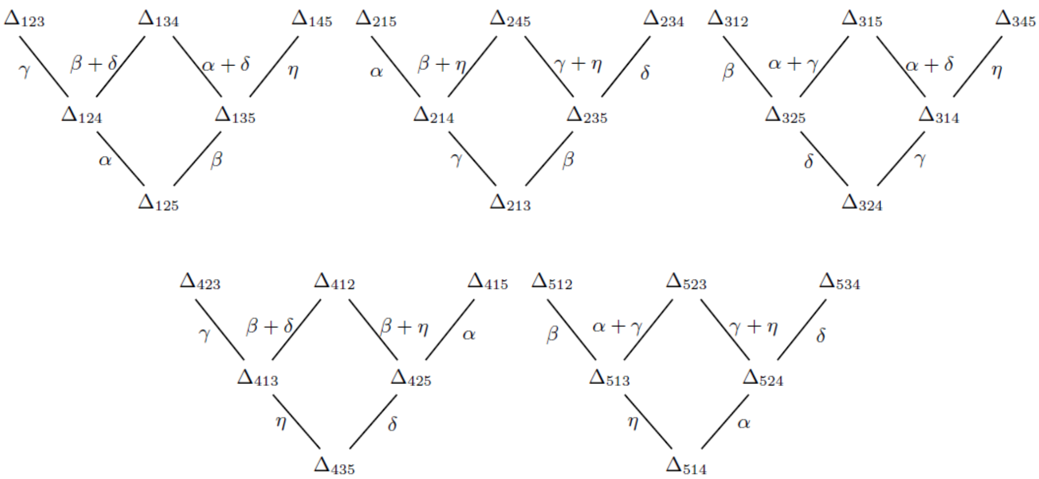

Figure 3. The partial order diagrams for the Type A.

For Type B, we have:

$Q(1234):\ \Delta_{124} < \Delta_{123}, \Delta_{134},\quad \Delta_{213} < \Delta_{214}, \Delta_{234},\quad \Delta_{324} < \Delta_{312}, \Delta_{314},\quad \Delta_{413} < \Delta_{412}, \Delta_{423},$

$Q(1235):\ \Delta_{125} < \Delta_{123}, \Delta_{135},\quad \Delta_{213} < \Delta_{215}, \Delta_{235},\quad \Delta_{325} < \Delta_{312}, \Delta_{315},\quad \Delta_{513} < \Delta_{512}, \Delta_{523},$

$Q(1245):\ \Delta_{125} < \Delta_{124}, \Delta_{145},\quad \Delta_{214} < \Delta_{215}, \Delta_{245},\quad \Delta_{425} < \Delta_{412}, \Delta_{415},\quad \Delta_{514} < \Delta_{512}, \Delta_{524},$

$Q(1345):\ \Delta_{135} < \Delta_{134}, \Delta_{145},\quad \Delta_{314} < \Delta_{315}, \Delta_{345},\quad \Delta_{435} < \Delta_{413}, \Delta_{415},\quad \Delta_{514} < \Delta_{513}, \Delta_{534},$

$Q(2354):\ \Delta_{234} < \Delta_{235}, \Delta_{245},\quad \Delta_{325} < \Delta_{324}, \Delta_{345},\quad \Delta_{425} < \Delta_{423}, \Delta_{435},\quad \Delta_{534} < \Delta_{523}, \Delta_{524}.$

which give the following Hasse diagrams given in Figure 4.

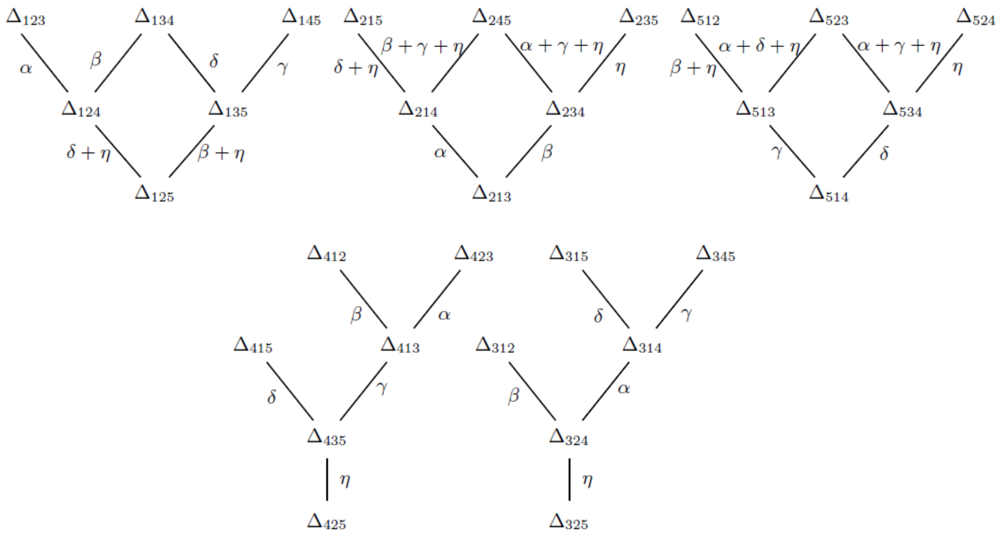

Figure 4. The partial order diagrams for the Type B.

For Type C, the quadrangles give the following relations:

$Q(1234):\ \Delta_{124} < \Delta_{123}, \Delta_{134},\quad \Delta_{213} < \Delta_{214}, \Delta_{234},\quad \Delta_{324} < \Delta_{312}, \Delta_{314},\quad \Delta_{413} < \Delta_{412}, \Delta_{423},$

$Q(1235):\ \Delta_{125} < \Delta_{123}, \Delta_{135},\quad \Delta_{213} < \Delta_{215}, \Delta_{235},\quad \Delta_{325} < \Delta_{312}, \Delta_{315},\quad \Delta_{513} < \Delta_{512}, \Delta_{523},$

$Q(1245):\ \Delta_{125} < \Delta_{124}, \Delta_{145},\quad \Delta_{214} < \Delta_{215}, \Delta_{245},\quad \Delta_{425} < \Delta_{412}, \Delta_{415},\quad \Delta_{514} < \Delta_{512}, \Delta_{524},$

$Q(1435):\ \Delta_{145} < \Delta_{134}, \Delta_{135},\quad \Delta_{345} < \Delta_{314}, \Delta_{315},\quad \Delta_{413} < \Delta_{415}, \Delta_{435},\quad \Delta_{513} < \Delta_{514}, \Delta_{534},$

$Q(2354):\ \Delta_{234} < \Delta_{235}, \Delta_{245},\quad \Delta_{325} < \Delta_{324}, \Delta_{345},\quad \Delta_{425} < \Delta_{423}, \Delta_{435},\quad \Delta_{534} < \Delta_{523}, \Delta_{524}.$

which lead to the Hasse diagrams given in Figure 5.

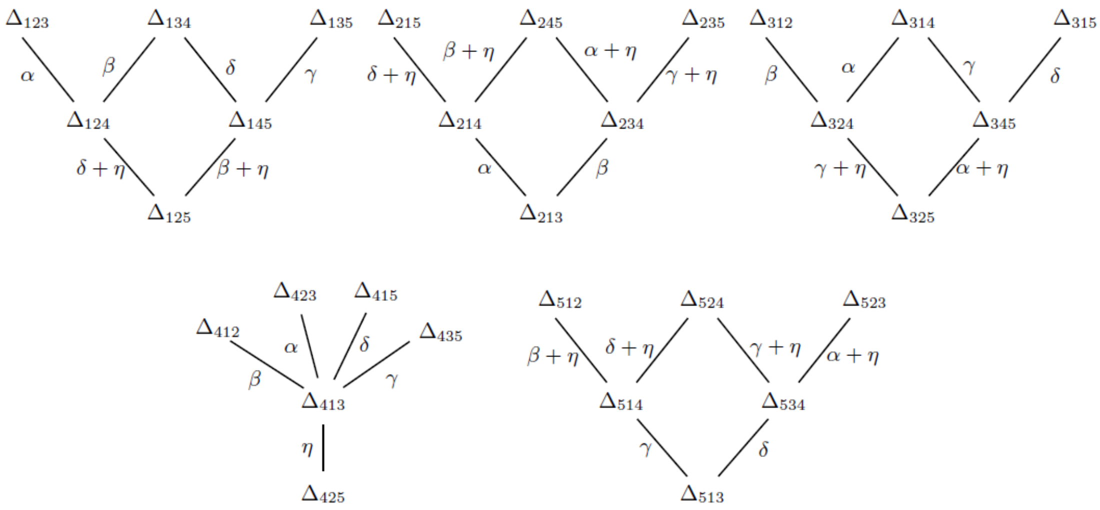

Figure 5. The partial order diagrams for the Type C.

Remark 1 For Types A and B, the quadrangle structure determines the Gromov product structure, in the sense that the partial order relations deduced from the quadrangle structure determine the smallest Gromov product at each P_i. On the other hand, for Type C, the partial order relations imply that both Δ_{413} and Δ_{425} are smaller than $\Delta_{412}, \Delta_{415}, \Delta_{423}$ and Δ_{435}, but the order relation between Δ_{413} and Δ_{425} is not determined by the quadrangle structure. This is an example for the case where the quadrangle structure does not determine the Gromov product structure.

Recall that the minimal Gromov products at each P_a are the isolation indices of 1-splits. In what follows, we assume that minimal Gromov products are zero.
The isolation indices for 2-splits will serve as free variables for the parameterization of the distances. For example, for Type A,

$$\begin{aligned}\alpha_{12} = \frac{1}{2}\min\{ &\max\{d_{13}+d_{24}, d_{14}+d_{23}, d_{12}+d_{34}\} - (d_{12}+d_{34}),\\ &\max\{d_{13}+d_{25}, d_{15}+d_{23}, d_{12}+d_{35}\} - (d_{12}+d_{35}),\\ &\max\{d_{14}+d_{25}, d_{15}+d_{24}, d_{12}+d_{45}\} - (d_{12}+d_{45})\}\\ = \min\{ &d_{13}+d_{24}-d_{12}-d_{34},\ d_{13}+d_{25}-d_{12}-d_{35},\ d_{14}+d_{25}-d_{12}-d_{45}\}.\end{aligned}$$

Which reformulating by using Gromov products gives:

$$\alpha_{12} = \min\{\Delta_{134}-\Delta_{124} = \Delta_{234}-\Delta_{213}, \Delta_{135}-\Delta_{125} = \Delta_{235}-\Delta_{213}, \Delta_{145}-\Delta_{125} = \Delta_{245}-\Delta_{214}\}.$$

Finally since $\Delta_{125} = \Delta_{213} = 0$ we may write it as:

$$\alpha_{12} = \min\{\Delta_{234}, \Delta_{135} = \Delta_{235}, \Delta_{145}\}.$$

From the partial order relations it is clear that α_{12} cannot be equal to Δ_{145}. Similarly, as $\Delta_{234} > \Delta_{235}$, we choose Δ_{135} as a free variable for the parameterization. By similar arguments and what is given when discussing "Parameterization of 4-point spaces" and Figure 1 in the introduction, the parameterization of the Gromov products and of the distance functions can be obtained as given below.

Type A: $\Delta_{125} = \Delta_{213} = \Delta_{324} = \Delta_{435} = \Delta_{514} = 0.$

$$\begin{array}{lllll}\Delta_{124} = \alpha, & \Delta_{135} = \beta, & \Delta_{123} = \alpha+\gamma, & \Delta_{145} = \beta+\eta, & \Delta_{134} = \alpha+\beta+\delta,\\ \Delta_{214} = \gamma, & \Delta_{235} = \beta, & \Delta_{215} = \alpha+\gamma, & \Delta_{234} = \beta+\delta, & \Delta_{245} = \beta+\gamma+\eta,\\ \Delta_{314} = \gamma, & \Delta_{325} = \delta, & \Delta_{312} = \beta+\delta, & \Delta_{345} = \gamma+\eta, & \Delta_{315} = \alpha+\delta+\gamma,\end{array}$$

$$\Delta_{413} = \eta,\quad \Delta_{425} = \delta,\quad \Delta_{415} = \alpha + \delta,\quad \Delta_{423} = \gamma + \eta,\quad \Delta_{412} = \beta + \delta + \eta,$$
$$\Delta_{513} = \eta,\quad \Delta_{524} = \alpha,\quad \Delta_{512} = \beta + \eta,\quad \Delta_{534} = \alpha + \delta,\quad \Delta_{523} = \alpha + \gamma + \eta.$$

$$d_{12} = \alpha + \gamma,\quad d_{13} = \alpha + \beta + \delta + \gamma,\quad d_{14} = \alpha + \beta + \delta + \eta,\quad d_{15} = \beta + \eta,\quad d_{23} = \beta + \delta,$$
$$d_{24} = \beta + \delta + \gamma + \eta,\ d_{25} = \alpha + \beta + \gamma + \eta,\ d_{34} = \gamma + \eta,\ d_{35} = \alpha + \delta + \gamma + \eta,\ d_{45} = \alpha + \delta.$$

Type B: $\Delta_{125} = \Delta_{213} = \Delta_{325} = \Delta_{425} = \Delta_{514} = 0.$

$\Delta_{124} = \delta + \eta$, $\Delta_{135} = \beta + \eta$, $\Delta_{134} = \beta + \delta + \eta$, $\Delta_{123} = \alpha + \delta + \eta$, $\Delta_{145} = \beta + \gamma + \eta$,
$\Delta_{214} = \alpha$, $\Delta_{234} = \beta$, $\Delta_{235} = \beta + \eta$, $\Delta_{215} = \alpha + \delta + \eta$, $\Delta_{245} = \alpha + \beta + \gamma + \eta$,
$\Delta_{324} = \eta$, $\Delta_{312} = \beta + \eta$, $\Delta_{314} = \alpha + \eta$, $\Delta_{315} = \alpha + \delta + \eta$, $\Delta_{345} = \alpha + \gamma + \eta$,
$\Delta_{435} = \eta$, $\Delta_{413} = \gamma + \eta$, $\Delta_{415} = \delta + \eta$, $\Delta_{423} = \alpha + \gamma + \eta$, $\Delta_{412} = \beta + \gamma + \eta$,
$\Delta_{534} = \delta$, $\Delta_{513} = \gamma$, $\Delta_{524} = \delta + \eta$, $\Delta_{512} = \beta + \gamma + \eta$, $\Delta_{523} = \alpha + \delta + \gamma + \eta$.

$$d_{12} = \alpha + \delta + \eta,\ d_{13} = \alpha + \beta + \delta + 2\eta,\ d_{14} = \beta + \delta + \gamma + 2\eta,\ d_{15} = \beta + \gamma + \eta, d_{23} = \beta + \eta,$$
$$d_{24} = \alpha + \beta + \gamma + \eta,\ d_{25} = \alpha + \beta + \delta + \gamma + 2\eta,\ d_{35} = \alpha + \delta + \gamma + \eta,\ d_{45} = \delta + \eta.$$

Type C : $\Delta_{125} = \Delta_{213} = \Delta_{325} = \Delta_{425} = \Delta_{513} = 0.$

$\Delta_{123} = \alpha + \delta + \eta$, $\Delta_{124} = \delta + \eta$, $\Delta_{134} = \beta + \delta + \eta$, $\Delta_{135} = \beta + \gamma + \eta$, $\Delta_{145} = \beta + \eta$,
$\Delta_{214} = \alpha$, $\Delta_{215} = \alpha + \delta + \eta$, $\Delta_{234} = \beta$, $\Delta_{235} = \beta + \gamma + \eta$, $\Delta_{245} = \alpha + \beta + \eta$,
$\Delta_{312} = \beta + \gamma + \eta$, $\Delta_{314} = \alpha + \gamma + \eta$, $\Delta_{315} = \alpha + \delta + \eta$, $\Delta_{324} = \gamma + \eta$, $\Delta_{345} = \alpha + \eta$,
$\Delta_{412} = \beta + \eta$, $\Delta_{413} = \eta$, $\Delta_{415} = \delta + \eta$, $\Delta_{423} = \alpha + \eta$, $\Delta_{435} = \gamma + \eta$,
$\Delta_{512} = \beta + \gamma + \eta$, $\Delta_{514} = \gamma$, $\Delta_{523} = \alpha + \delta + \eta$, $\Delta_{524} = \delta + \gamma + \eta$, $\Delta_{534} = \delta$.

$$d_{12} = \alpha + \delta + \eta,\ d_{13} = \alpha + \beta + \delta + \gamma + 2\eta, d_{14} = \beta + \delta + 2\eta,\ d_{15} = \beta + \gamma + \eta,\ d_{23} = \beta + \gamma + \eta,$$
$$d_{24} = \alpha + \beta + \eta,\ d_{25} = \alpha + \beta + \delta + \gamma + 2\eta,\ d_{34} = \alpha + \gamma + 2\eta,\ d_{35} = \alpha + \delta + \eta,\ d_{45} = \delta + \gamma + \eta.$$

These parameterizations are exactly the ones given by Koolen, Lesser and Moulton [8]. In the paper [8], the classes obtained via the decomposition of the metric cone are denoted as Type I, Type II and Type III. These correspond respectively to our equivalence classes denoted by Type A, Type C and Type B. The metrics of Type I, II and III are defined by their split decompositions, given as below. For simplicity we consider the pendant free case, i.e, we take the coefficients of the 1-splits as zero, equivalently the minimal Gromov products at each node are zero.

We use the labeling of the nodes by $\{x, y, u, v, w\}$.
(Type I): $d = \alpha_{xy}\delta_{xy} + \alpha_{yu}\delta\_yu + \alpha_{uv}\delta_{uv} + \alpha_{vw}\delta_{vw} + \alpha_{wx}\delta_{wx}$,
(Type II): $d = \alpha_{xu}\delta_{xu} + \alpha_{xv}\delta_{xv} + \alpha_{uy}\delta_{uy} + \alpha_{vy}\delta_{vy} + c\ d'$,
(Type III): $d = \alpha_{xu}\delta_{xu} + \alpha_{xv}\delta_{xv} + \alpha_{wy}\delta_{wy} + \alpha_{vy}\delta_{vy} + c\ d'$,
where $d'(a,b) = 0\ if\ a = b$, $d'(x,y) = d'(u,v) = d'(u,w) = d'(v,w) = 2$ and $d'(a,b) = 1$ for all other cases.

We identify the indices x, y, u, v, w with our notation. For example, for Type I, i.e, our Type A, x, y, u, v, w correspond to $1, 2, 3, 4, 5$ respectively and the correspondence of the parameters are
$$\alpha_{xy} = \beta,\qquad \alpha_{yu} = \gamma,\qquad \alpha_{uv} = \delta,\qquad \alpha_{vw} = \eta,\qquad \alpha_{wx} = \alpha.$$
For Type II, i.e, our Type C, x, y, u, v, w correspond to $5, 2, 1, 3, 4$ respectively and the correspondence of the parameters are
$$\alpha_{xu} = \delta,\qquad \alpha_{xv} = \gamma,\qquad \alpha_{uy} = \beta,\qquad \alpha_{vy} = \alpha,\qquad c = \eta.$$
For Type III, i.e, our Type B, x, y, u, v, w correspond to $2, 5, 3, 1, 4$ respectively and the correspondence of the parameters are

$$\alpha_{xu} = \alpha, \qquad \alpha_{xv} = \beta, \qquad \alpha_{wy} = \gamma, \qquad \alpha_{vy} = \delta, \qquad c = \eta.$$

Explicit parametrizations for certain 6-point spaces have been also obtained via partial order relations and quadrangle classifications. It is available on http://finitemetricspaces.khas.edu.tr/Optimal%20Realizations,%20h-optimal%20Realizations%20and%20Tight%20Spans%20of%20Metric%20Spaces.pdf}.

3. OPTIMAL REDUCTIONS OF 5-POINT METRIC SPACES

Optimal realizations of 5-point metric spaces for three types are given in [8], in what follows we will give underlying graphs for each metric type and will drive their optimal reductions.

The weighted graph $G = (V, E, w)$ is called a realization of the finite metric space (X, d) if there is a labeling function $\phi : X \to V$ such that for all $x, y \in X$ the weight of any path between $\phi(x)$ and $\phi(y)$ is equal to $d(x, y)$. Any such realization is called optimal if $||G||$, the total edge weight of the graph G , is minimal among all realizations of the metric space (X, d) [8].

As it is clear from the definition above that a finite metric space can have many realizations. In the following, we will start with the pendant free reductions and use certain "moves" as defined in [9] to reduce the total weight and reach the optimal representation. This kind of operations are generally done by adjoining new vertices to the original graph, which in this case the added vertices are called secondary vertices and the original vertices as primary, discarding some edges or adding new edges between the enlarged set of vertices and assigning weights to the new edges in a way that the distance between primary nodes are unchanged but the weight of the graph, namely $||G||$, is reduced.

The first move, which is called *joining edges*, is done in the following way: Consider a vertex u and all (or some) of the other nodes $v_1, v_2, \ldots, v_k$ of G, which are neighbors of u. Calculate the Gromov products of all triangles $T_{uv_iv_j}$ with $1 \leq i, j \leq k$ at vertex u and call the minimum m_u. Now delete all the edges between u and v_i's, introduce a new vertex v and connect v_i's to v by edges of weight $w_{uv_i} - m_u$ for $1 \leq i \leq k$ and also u to v by an edge of weight m_u; hence the nodes v_i become connected to u by two edges through v and the total weight of the graph is reduced by an amount of $(k-1)m_u$.

The second move, which is called *edge removing*, is done by deleting the edge between two nodes u and v if it can be avoided by a shortest path. This move reduces $||G||$ by an amount of the weight of the deleted edge.

The "$\Delta - Y$" transform is a consequence of the above moves and can be applied to any triangle with 1-connected vertices in G. It is called a $\Delta - Y$ transform, because a triangle shape (Δ) turns to a Y shape after the operation.

We should also note that what we mean by *underlying graph of a metric*, is the complete graph with the same set of vertices as the metric space and all the edges with weight d_{ij} removed for which there is a point in space p_k such that $d_{ij} = d_{ik} + d_{kj}$.

For Type A with the Gromov product structure as $\{ \Delta_{125}, \Delta_{213}, \Delta_{324}, \Delta_{435}, \Delta_{514} \}$, when edge removing operations are applied and passed to pendant-free reduction, a 5-cycle given in Figure 6 is obtained. The optimal realization given in [8] is a 5-cycle with edges connected to each of its nodes (Type (a) of [8]).

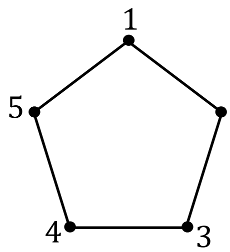

Figure 6. Optimal reduction of metric Type A.

For Type B with the Gromov product structure as $\{ \Delta_{125}, \Delta_{213}, \Delta_{325}, \Delta_{425}, \Delta_{514}\}$, the underlying graph is given in Figure 7:

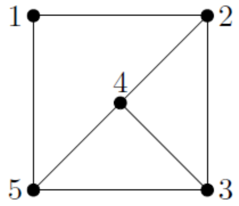

Figure 7. Underlying graph of metric Type B.

By applying a $\Delta - Y$ transform to T_{345} we have Figure 8

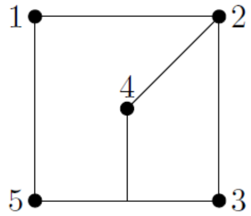

Figure 8. Graph with $\Delta - Y$ transformed.

In this step, one can follow two different approaches which reduce the metric to Type (b) or (c) of [8]. To observe the process closely we need to point out that the parameterization of Type B is given in Figure 9:

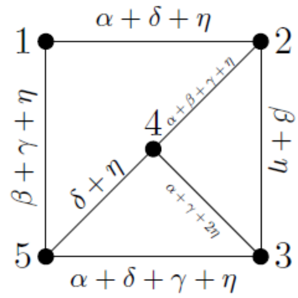

Figure 9. Underlying graph of metric Type B with distances parameterized.

Here we have $\Delta_{345} = \alpha + \gamma \; + \eta, \; \Delta_{435} = \eta$ and $\Delta_{534} = \delta$, and applying a $\Delta - Y$ transform to T_{345} will be as in Figure 10:

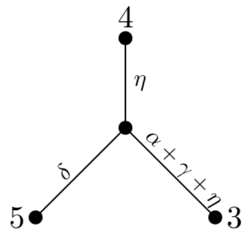

Figure 10. T_{345} of Type B after $\Delta - Y$ transform.

So the Type B with parameters are as following:

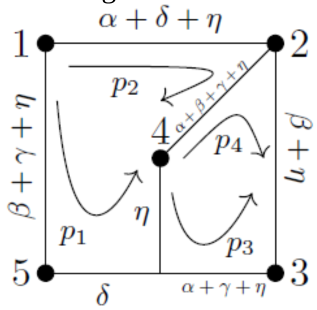

Figure 11. Metric Type B with the parameters.

Now according to the graph above, we have d_{14} equal to $\beta + \delta + \gamma + 2\eta$ (path p_1) or equal to $2\alpha + \beta + \gamma + \delta + 2\eta$ (path p_2). Path p_2 is longer than path p_1 by an amount of 2α. Likewise d_{34} is equal to $\alpha + \gamma + 2\eta$ (path p_3) or equal to $\alpha + 2\beta + \gamma + 2\eta$ (path p_4). Here path p_4 is longer than path p_3 by a difference of 2β. It should be noted that $\alpha = \Delta_{214}$ and $\beta = \Delta_{234}$ and two scenarios are possible: either $\alpha > \beta$ or $\beta > \alpha$. If $\alpha > \beta$, in order to decrease the total weight of the graph, we will introduce a new node called v on the edge joining 1 to 2 as shown below:

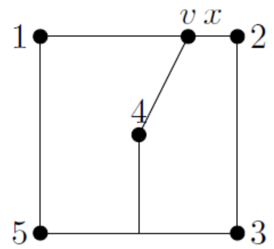

Figure 12. Reduction of Type B to (b).

This will reduce the total weight as $x = \Delta_{214}$ and that results the Type B to reduced into (b) of [8] and the metric will be as following:

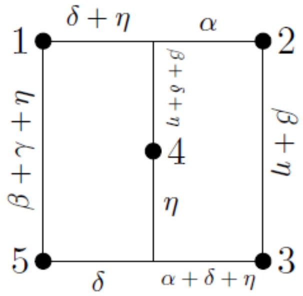

Figure 13. Reduction of Type B to (b) when $\alpha > \beta$ (parameters given).

In the other case, when $\beta > \alpha$, if we do the same operation as before, but this time for the edge joining 2 to 3 we will have the following reduction:

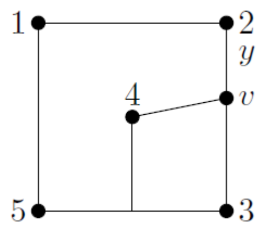

Figure 14. Reduction of Type B to (c).

This reduces the weight of graph as $y = \Delta_{234}$ and turns it into Type (c) given as below:

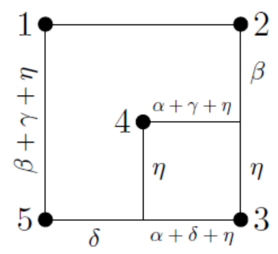

Figure 15. Reduction of Type B to (c) when $\beta > \alpha$ (parameters given).

For Type C which the underlying graph with the parameters given is depicted below, the following can be done:

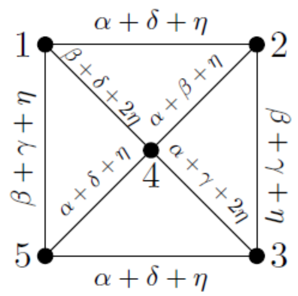

Figure 16. Underlying graph of metric Type C with the metric parameterized.

Since $\Delta_{124} = \delta + \eta$, $\Delta_{214} = \alpha$ and $\Delta_{412} = \beta + \eta$, applying a $\Delta - Y$ transform to T_{124} will result in the following:

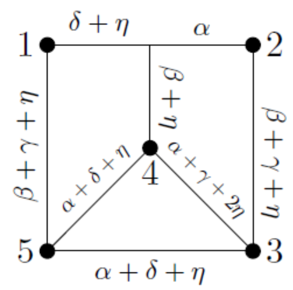

Figure 17. Type C with a $\Delta - Y$ transform applied to T_{124}.

Finally considering that $\Delta_{345} = \alpha + \eta$, $\Delta_{435} = \gamma + \eta$ and $\Delta_{534} = \delta$, applying another $\Delta - Y$ transform to T_{345} will result in the following:

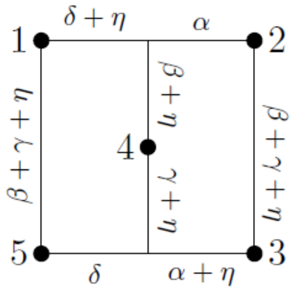

Figure 18. Type C with a second $\Delta - Y$ transform applied to T_{345} and reduced to (b).

4. VOLUMES OF GROMOV METRIC TYPES

One of the ways to study the stability of Δ-equivalence classes under small numerical perturbations on components of metric represented as the vector $d = (d_{ij})$, is to consider the relative volume of each class inside the metric cone. To estimate these relative sizes of Δ-equivalence classes in an n-point space, we generate random points that lie in the intersection of the metric cone with unit ball in $\mathbb{R}^{\frac{n(n-1)}{2}}$ and then count the occurrence of points in each class.

We note that the volume of unit ball in $\mathbb{R}^N$ is equal to $V_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$, where Γ is the Gamma function. It should also be noted that since the rate of growth of Gamma function is greater than the exponentials, as the dimension of space increases this volume decreases. It is known that the maximum volume is obtained for $N = 6$ and for the values of N greater than 6, V_N starts to decrease. On the other hand by keeping in mind that a metric d on an n-point space can be shown by a vector of positive coordinates in $\mathbb{R}^N$ where $N = \frac{n(n-1)}{2}$, we need to work with the intersection of unit ball with the orthant in which all the coordinates are positive (the first orthant in higher dimensions). Both of these issues leave us with only a few samples to work with.

To deal with the problem of generating a statistically significant number of points in the metric cone in $\mathbb{R}^{10}$ (since every metric on a 5-point space can be shown by a vector in $\mathbb{R}^{10}$) on a standard computer, we generate 10^7 random points $P = (x_1, x_2, \dots, x_{10})$, $0 < x_i < 1$ and accumulate these points from 10 such runs to get 10^8 points. Each of these points has 10 positive coordinates that are uniformly distributed random numbers in the range $(0,1)$. Then the points that fall inside the unit ball are chosen and in the next step by checking which points satisfy the triangle inequalities, we select the points inside the metric cone. Finally, for each of these points (metrics) we calculate the Gromov product structure in order to determine the metric type. This process is repeated 30 times and some of the results are given in Table \ref{table:random} below. The Matlab code for this program is available at http://finitemetricspaces.khas.edu.tr/Volume_of_Metric_Cone_n=5.m.

Table 1. Sample results of accumulating 10^8 points in $\mathbb{R}^{10}$. Each row is a single run of the program and shows how many points fall inside the unit ball, metric cone, and each type.

points in unit ball	points in metric cone	Type A	Type B	Type C
274578	705	142	360	203
273136	735	186	351	198
273891	716	161	362	193
273426	733	170	376	187
272959	721	167	363	191

As shown in Table 1, from 10^8 points in the cube, around 2.7×10^5 points (0.275%) fall inside the unit ball and around 0.25% of these points fall inside the metric cone. To understand why these small amounts of points in unit cube of $\mathbb{R}^{10}$ fall inside the unit ball, it should be noted that the volume of unit ball V_{10} in $\mathbb{R}^{10}$ is equal to $\frac{\pi^5}{120}$ and we work only with the portion of unit ball intersecting the first orthant. This volume is approximately 0.00249 which is 0.24% of the volume of the unit cube.

In order to interpret the data given in Table 1, some clarifications must be made. 5-point metrics inside the metric cone in $\mathbb{R}^{10}$, when the Gromov product structure is considered, fall into 102 classes. Under permutation of the points of underlying metric space, these 102 classes form 3 families. In a family which is the orbit of the Gromov product structure $\{\Delta_{125}, \Delta_{213}, \Delta_{324}, \Delta_{435}, \Delta_{514}\}$ under the action of the permutation group S_5, there are 12 elements. The metrics that have a Gromov product structure in this family are called Type A metrics. Furthermore, the orbit of the Gromov product structure $\{\Delta_{125}, \Delta_{213}, \Delta_{325}, \Delta_{425}, \Delta_{514}\}$ and $\{\Delta_{125}, \Delta_{213}, \Delta_{325}, \Delta_{425}, \Delta_{513}\}$ have 60 and 30 elements respectively and the metrics of these families are called Type B and Type C in this order.

For calculating the type of a metric inside the metric cone to obtain the results given in Table 1, these 102 classes are taken into consideration. With this view in hand, the volume of Type A, Type B and Type C metrics on average are 22.07%, 51.02% and 26.26 % of the metric cone (within a standard deviation of 21.1 for points inside the metric cone, 10.83 for Type A metrics, 17.03 for Type B metrics and 12.43 for Type C metrics in our runs to obtain the data given in Table 1). If we take the other view, without considering the permutations, results of Type A, B and C should be divided by 12, 60 and 30 respectively to obtain the volume of a single representative of each class. This means that within error bounds, the volumes of a single representative of Type A, B and C are respectively 1.84 %, 0.85 % and 0.87 % of the metric cone.

The results above, give us the following intuitive conclusions: first that the volume of a single representative of Type B and Type C metrics are almost equal and Type A is “thicker'” than these two types. Second, although a single representative of metric Type A is thicker than other types, these representatives are small in number (12 among 102 classes) with respect to Type B (60 among 102) and Type C (30 among 102) inside the metric cone.

ACKNOWLEDGMENTS

This work has been supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the project number 118F412 titled “Analysis of Finite Metric Spaces via Gromov Products and their Applications to Phylogenetics”.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

AUTHORSHIP CONTRIBUTIONS

All authors contributed to the theoretical part of this article. Computer aided results were carried out by Arash M. Rezaeinazhad.

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