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An Almost Complex Structure with Norden Metric on the Phase Space

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

On the total space of the cotangent bundle of a Riemannian manifold, we construct a semi-Riemannian metric G, with respect to which an almost complex structure J introduced by Oproiu and Poroșniuc is self-adjoint. The structure (J, G) turnes out to be an almost complex structure with Norden metric (this notion is known in the literature from Norden's papers). The semi-Riemannian context is different from the Riemannian one, as it is pointed out by Duggal and Bejancu in their monograph. We study this structure and provide some necessary and sufficient conditions for it to be a Kähler structure with Norden metric.

Keywords: *almost complex structure; Norden metric; cotangent bundle; vertical and horizontal lifts.*
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1. Introduction

Complex manifolds represent a field of interest for both Complex Analysis and Differential Geometry. Obviously, any manifold endowed with an almost complex structure J (i. e. $J^2 = -Id$) admits a compatible Riemannian metric ρ, with respect to which J is anti-self-adjoint and hence (J, ρ) provides an almost Hermitian structure on the manifold. Not the same thing can be done with a semi-Riemannian metric, with respect to which J is self-adjoint, because such a metric may not exist, globally defined on the whole manifold. The idea of introducing and studying such a semi-Riemannian metric belonged to Norden, who defined in [6] the so-called B-manifold, called now Kähler manifold with Norden metric, where J is self-adjoint with respect to the semi-Riemannian metric ρ and parallel with respect to the Levi-Civita connection of ρ.

The semi-Riemannian context has many differences from the Riemannian one, as it was explained in the monograph [5].

Recently, in [2], the present authors with Blaga studied Kähler manifolds with Norden metric establishing, on these manifolds, the relation between three concepts: constant totally real sectional curvatures, holomorphic Einstein and Bochner flatness. Also, in [1] almost complex and hypercomplex Norden structures induced by natural Riemann extensions were constructed by the present authors.

In our paper here, we construct an almost complex structure with Norden metric (J, G) on the total space of the cotangent bundle T^*M of a Riemannian manifold (M, g) and then we study when T^*M endowed with this structure becomes a Kähler manifold with Norden metric. The existence of such a metric depends on some topological restrictions of the manifold, but in our case, we showed how we obtained a metric with the above properties. The construction presented here provides an example for the theory developed in [2]. The reason why we chose the background to be the total space of the cotangent bundle T^*M of a manifold M is the fact that T^*M has many applications in physics as a phase space.

The tools we use in the present work are the vertical and horizontal lifts from the base manifold M to its cotangent bundle T^*M. We highlight here the almost complex structure J constructed in [7] and [8], this structure being useful in literature, as one can see from the recent paper [4].

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Now we summarize, by giving an overview of this paper: after some preliminaries which contain some basic notions and useful properties, we deal with the almost complex structure with Norden metric and we study when the manifold carrying such a structure is a Kähler manifold with Norden metric.

2. Preliminaries

Let M^n be an n-dimensional manifold and let $\pi : T^*M \longrightarrow M$ be the natural projection of its cotangent bundle T^*M to M. The total space T^*M is known in mechanics as the phase space. A local coordinate neighbourhood $(U; x^1, \dots, x^n)$ on M around any point $x \in M$ induces a local chart $(\pi^{-1}(U); x^1, \dots, x^n, \omega_1, \dots, \omega_n)$ around any point $(x, \omega) \in T^*M$. We use [9] to recall the vertical and horizontal lifts from the base manifold M to the total space of its cotangent bundle.

In what follows, we identify any smooth function $f \in \mathcal{F}(M)$, (locally) defined on M, with its vertical lift $f^v = f \circ \pi \in \mathcal{F}(T^*M)$.

For a Riemannian manifold (M, g) one has the splitting of the tangent space of T^*M into the direct sum

$$T(T^*M) = V(T^*M) \oplus H(T^*M), \tag{2.1}$$

where $V(T^*M) = \operatorname{Ker} \pi_*$ and $H(T^*M)$ is defined by the Levi-Civita connection ∇ of g. A local frame of $V(T^*M)$ and $H(T^*M)$ is respectively $\left\{\frac{\partial}{\partial \omega_i}\right\}_{i=\overline{1,n}}$ and $\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \Gamma^k_{ij}\omega_k \frac{\partial}{\partial \omega_j}\right\}_{i=\overline{1,n}}$, where Γ^k_{ij} are the Christoffel symbols of ∇.

For any vector field $X \in \chi(M)$, written in local coordinates as $X = X^i \frac{\partial}{\partial x^i}$ and for any one-form $\theta \in \Omega(M)$, written in local coordinates as $\theta = \theta_i \mathrm{d}x^i$, one has the horizontal lift $X^h \in \Gamma(H(T^*M))$ written locally as $X^h = X^i \frac{\delta}{\delta x^i}$ and the vertical lift $\theta^v \in \Gamma(V(T^*M))$ written locally as $\theta^v = \theta_i \frac{\partial}{\partial \omega_i}$.

The musical isomorphisms $\sharp : T^*M \longrightarrow TM$ and $\flat : TM \longrightarrow T^*M$ are defined by:

$$g(\theta^\sharp, Y) = \theta(Y) \quad \text{and} \quad X^\flat(Y) = g(X, Y) \tag{2.2}$$

for any $X, Y \in \chi(M)$ and any $\theta \in \Omega(M)$.

Remark 2.1. From the decomposition (2.1), it follows that the vector bundles $V(T^*M)$ and $H(T^*M)$ have the same rank n.

The energy density defined by g in a cotangent vector ω is given by:

$$t = \frac{1}{2}\|\omega\|^2 = \frac{1}{2}g(\omega^\sharp, \omega^\sharp) \geq 0, \quad \forall\, \omega \in T^*M. \tag{2.3}$$

From [9], if S is a $(1, s)$-tensor field on M, $s \geq 1$, whose components are $S^a_{i_1 \dots i_s}$ in local coordinates, then γS is a $(1, s-1)$-tensor field on the total space of T^*M, defined such that its components in local coordinates are $\omega_a S^a_{i_1 \dots i_s}$, at any point $(x, \omega) \in T^*M$. In particular, if F is a $(1, 1)$-tensor field on M, whose local coordinates are F^a_i, then γF is a vertical vector field on the total space of T^*M, whose local components in the vertical distribution are $\omega_a F^a_i$, $\forall\ (x, \omega) \in T^*M$ and the local components in the horizontal distribution vanish identically.

If R is a $(1, 3)$-tensor field on M whose components are R^a_{ijk} in local coordinates, then γR is a $(1, 2)$-tensor field on the total space of T^*M, whose local components are $\gamma R^{\widetilde{h}}_{kj} = \omega_a R^a_{kjh}$, all the others being zero.

Remark 2.2. Since for any $X, Y \in \chi(M)$, one has that $R(X, Y)$ is a $(1, 1)$-tensor field on M, it follows that the composition $\omega \circ R(X, Y)$ is a 1-form on M and we obtain:

$$\gamma R(X, Y) = (\omega \circ R(X, Y))^v. \tag{2.4}$$

Different from the almost Hermitian geometry, where the metric is compatible with the almost complex structure, A. P. Norden defined in [6] the almost complex manifolds endowed with a semi-Riemannian metric which is skew-compatible with the almost complex structure, as follows:

Definition 2.1. [6] A manifold (N, F, G) endowed with an almost complex structure F (i.e. $F^2 = -Id$) and a semi-Riemannian metric G is an almost complex manifold with Norden metric if

$$G(FX, FY) = -G(X, Y), \quad \forall\, X, Y \in \chi(N). \tag{2.5}$$

If moreover F is parallel with respect to the Levi-Civita connection ∇ of G, which means

$$\nabla F = 0, \tag{2.6}$$

i.e. $\nabla_X(FY) = F(\nabla_X Y)$, $\forall X, Y \in \chi(N)$, then we say that (N, F, G) is a Kähler manifold with Norden metric.

Remark 2.3. An almost complex manifold N with Norden metric should be of even dimension $2n$ and the metric G should be semi-Riemannian of neutral signature, which means (n, n). A classification of the almost complex manifolds with Norden metric was given in [3].

3. Kähler manifolds with Norden metrics

In this section, we denote by T^*M, the cotangent bundle of a Riemannian n-dimensional manifold (M^n, g). Let a_1, a_2, b_1, b_2 be some smooth real functions of one variable t, which is the energy density.

Proposition 3.1. *[7] Let J be the $(1,1)$-tensor field on the total space of T^*M, defined at each point $(x, \omega) \in T^*M$ by:*

$$\begin{aligned} JX^h &= a_1 (X^\flat)^v + b_1 \omega(X) \omega^v \\ J\theta^v &= -a_2 (\theta^\sharp)^h - b_2 g(\omega^\sharp, \theta^\sharp)(\omega^\sharp)^h, \end{aligned} \tag{3.1}$$

for any $X \in \chi(M)$ and $\theta \in \Omega(M)$. Then J is an almost complex structure on the total space T^*M if and only if

$$a_1 a_2 = 1; \qquad a_1 b_2 + a_2 b_1 + 2t b_1 b_2 = 0. \tag{3.2}$$

Remark 3.1. The second relation in (3.2) is equivalent to

$$(a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{3.3}$$

From (3.2) and (3.3) it follows that a_1, a_2 have the same sign and similarly for $a_1 + 2tb_1$, $a_2 + 2tb_2$. Consequently, we assume $a_1, a_2 > 0$ and $a_1 + 2tb_1, a_2 + 2tb_2 > 0$ since otherwise, we may proceed in a similar way.

Proposition 3.2. *Let G be the $(0,2)$-tensor field on the total space of T^*M, defined at each point $(x, \omega) \in T^*M$ by:*

$$\begin{aligned} G(X^h, Y^h) &= 0 = G(\theta^v, \sigma^v) \\ G(X^h, \theta^v) &= \theta(X) = G(\theta^v, X^h), \end{aligned} \tag{3.4}$$

for any $X, Y \in \chi(M)$ and $\theta, \sigma \in \Omega(M)$. Then (T^*M, J, G), with J and G defined respectively by (3.1) and (3.4), is an almost complex manifold with Norden metric if and only if (3.2) is satisfied.

Proof. Obviously, G is symmetric. Moreover, for any non-zero $X^h \in \chi(T^*M)$, with $X \in \chi(M)$, there exists $(X^\flat)^v \in \chi(T^*M)$ such that $G((X^\flat)^v, X^h) = g(X, X) \neq 0$. Similarly, for any non-zero $\theta^v \in \chi(T^*M)$, with $\theta \in \Omega(M)$, there exists $(\theta^\sharp)^h \in \chi(T^*M)$ such that $G((\theta^\sharp)^h, \theta^v) = g(\theta^\sharp, \theta^\sharp) \neq 0$. Hence, G is non-degenerate. We note that G is a metric of signature (n, n) on the total space of T^*M, by taking into account Remark 2.1. One can see that G restricted to each vertical and horizontal bundle $V(T^*M)$ and $H(T^*M)$, respectively, is identically zero. By straightforward computations the relation (2.5) is verified and from Proposition 3.1, we complete the proof. □

We recall that on a Riemannian manifold (N, g), the Levi-Civita connection ∇ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \tag{3.5}$$

for any $X, Y, Z \in \chi(N)$.

We provide some formulas which we use later on.

Let (M, g) be a Riemannian manifold, whose Levi-Civita connection is ∇ and the Riemannian curvature is R. Then:

$$\begin{aligned} &[X^h, \theta^v] = (\nabla_X \theta)^v; \quad [X^h, Y^h] = [X, Y]^h + \gamma R(X, Y); \quad [\theta^v, \sigma^v] = 0; \\ &X^h f^v = (Xf)^v; \quad (f\theta)^v = f^v \theta^v; \quad \theta^v f^v = 0, \quad \forall X, Y \in \chi(M), \ \theta, \sigma \in \Omega(M), \ f \in \mathcal{F}(M). \end{aligned} \tag{3.6}$$

By using the above formulas, by a straightforward computation, we obtain from (3.5) and Proposition 3.2, the following:

Proposition 3.3. *Let (M, g) be a Riemannian manifold, whose Levi-Civita connection is ∇ and the Riemannian curvature is R. Then the Levi-Civita connection $\widetilde{\nabla}$ of the semi-Riemannian structure G defined by (3.4) on the total space of T^*M is given by:*

$$\widetilde{\nabla}_{X^h} Y^h = (\nabla_X Y)^h + \gamma R(., Y)X; \quad \widetilde{\nabla}_{X^h} \theta^v = (\nabla_X \theta)^v;$$
$$\widetilde{\nabla}_{\theta^v} Y^h = 0 = \widetilde{\nabla}_{\theta^v} \sigma^v, \quad \forall X, Y \in \chi(M), \ \ \theta, \sigma \in \Omega(M). \tag{3.7}$$

Proof. To sketch the proof, for any $X, Y, Z \in \chi(M)$ we have:

$$2G(\widetilde{\nabla}_{X^h} Y^h, Z^h) = G(\gamma R(X, Y), Z^h) + G(\gamma R(Z, X), Y) - G(\gamma R(Y, Z), X)$$
$$= \omega(R(X, Y)Z + R(Z, X)Y - R(Y, Z)X),$$

where we have used (3.4), (3.5), (3.6), (2.4). From the first Bianchi identity we obtain:

$$G(\widetilde{\nabla}_{X^h} Y^h, Z^h) = \omega(R(Z, Y)X).$$

From the definition of γS of a $(1, s)$-tensor field S written above, we deduce:

$$\omega(R(Z, Y)X) = G(\gamma R(., Y)X, Z^h),$$

which gives

$$G(\widetilde{\nabla}_{X^h} Y^h, Z^h) = G(\gamma R(., Y)X, Z^h). \tag{3.8}$$

Then, for any $\theta \in \Omega(M)$, we have

$$G(\widetilde{\nabla}_{X^h} Y^h, \theta^v) = \theta(\nabla_X Y) = G((\nabla_X Y)^h, \theta^v). \tag{3.9}$$

From (3.8) and (3.9) we obtain the first equality in (3.7). The other equalities are similar. □

We recall the following

Theorem 3.1. *[7, 8] Let J be the almost complex structure on the total space of T^*M, given by (3.1) and satisfying (3.2). Then J is integrable if and only if (M, g) has constant sectional curvature c and one of the following conditions holds:*
(i)

$$b_1 = \frac{c - a_1 a_1'}{2ta_1' - a_1}, \quad \forall t \geq 0.$$

There is no $A \in \mathbb{R}$ such that $a_1 = A\sqrt{t}$, $t \geq 0$.
(ii)

$$c > 0, \quad a_1 = \sqrt{2ct}, \quad b_1 > -\sqrt{c/2t}, \quad \forall t > 0.$$

When (ii) holds, then J is defined on $T^*M \setminus \{0\}$, which is the total space of the non-zero cotangent vectors.

From Definition 2.1, Theorem 3.1 and (2.6), we obtain by a straightforward computation, the following:

Theorem 3.2. *Let (M, g) be a Riemannian manifold and let T^*M be the total space of its cotangent bundle endowed with the almost complex structure J given by (3.1) which satisfies (3.2) and with the semi-Riemannian structure G defined by (3.4). Then (T^*M, J, G) is a Kähler manifold with Norden metric if and only if M is flat, $a_1 \in \mathbb{R}$ and $b_1, b_2 = 0$.*

Remark 3.2. Obviously, if the conditions of Theorem 3.2 are fulfilled (i.e. M is flat, $a_1 \in \mathbb{R}$ and $b_1, b_2 = 0$), then the conditions of Theorem 3.1 are also fulfilled, which shows that if (T^*M, J, G) is a Kähler manifold with Norden metric, then J is integrable.

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Author's contributions

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