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Legendre ve Chebyshev Polinomları Çözümü ile Radyasyon Transfer Denkleminin Kesikli Özdeğerleri

Hatice Asel ZİLAYAZ [1], Halide KÖKLÜ[2]

Öne Çıkanlar:
- İntegro-Diferansiyel Denklem Sistemleri
- İzotropik Saçılma
- Küresel Harmonik Metodu

ÖZET:

Işınım transfer denklemi, homojen ortamda izotropik saçılmalı, sonlu plaka sistemi için ele alınmıştır. Legendre ve Chebyshev polinomları, çeşitli tekil saçılma albedolarıyla sistemin kesikli özdeğerlerini belirlemek üzere ışınım transfer denklemini çözmekte kullanılmıştır. Sayısal sonuçlar, P_N ve T_N yöntemlerinin yüksek seviye tekrarlama basamakları için gerçekleştirilmiştir. Sayısal değerler tablolaştırılmış ve önceki çalışmalarla karşılaştırılmıştır. Sonuçlarımızın önceki çalışmalarla çok iyi uyum içinde olduğu gösterilmiştir.

Anahtar Kelimeler:
- Özdeğer Problemi
- Işınım Transfer Denklemi
- Chebyshev Polinomları

Discrete Eigenvalues of the Radiative Transfer Equation with Legendre and Chebyshev Polynomials Solutions

Highlights:
- Integro-Differential Equation System
- Isotropic scattering
- Spherical Harmonics Method

ABSTRACT:

The radiation transfer equation has been considered for finite slab with isotropic scattering in homogeneous medium. The Legendre and Chebyshev polynomials are used to solve the radiative transfer equation to determine the discrete eigenvalues of the system for various single scattering albedo. The numerical results are performed for high level iterations of the P_N and T_N methods. The numerical values are tabulated and compared with the previous works. It is shown that our results are in very good agreement with previous studies.

Keywords:
- Eigenvalue Problem
- Radiative Transfer
- Chebyshev Polynomials

[1] Hatice Asel ZİLAYAZ (**Orcid ID:** 0009-0002-1899-0068), Halide KÖKLÜ (**Orcid ID:** 0000-0003-1787-6693), Iğdır Üniversitesi, Sağlık Hizmetleri Meslek Yüksekokulu, Tıbbi Hizmetler ve Teknikler Bölümü, Iğdır, Türkiye

***Sorumlu Yazar/*Corresponding Author*:** Halide KÖKLÜ, e-mail: halide.koklu@igdir.edu.tr

Bu çalışma Hatice Asel ZİLAYAZ'ın Yüksek Lisans tez çalışmasından üretilmiştir.

INTRODUCTION

Radiative transfer theory has wide applications in astrophysics, atmospheric optics, marine science and biomedical optics. Radiative transfer theory is very important in describing and understanding the processes related to radiation in the universe. The radiative transfer equation is an integro-differential equation that describes the variation of the number of photons in a certain volume in a medium with time, taking into account the behaviour of photons in a medium such as scattering, absorption, and re-emission as a result of interaction in the medium (Liemert and Kienle, 2011). The radiative transfer equation is a fundamental tool in astrophysics, having been developed over approximately one hundred years. (Milne, 1921; Chandrasekhar, 1934; Kosirev, 1934). The solution of the Radiative transfer equation from the family of integral differential equations is difficult. The solutions are made with special polynomials such as Legendre, Lagrange, Bessel, Chebyshev, and Hermit. In Yılmazer, A. and Kocar, C., (2009), numerical convergence of spectral polynomial approximations to the radiation transfer equation in spherical environments is demonstrated. The T_N method was used as a representative of classical polynomial approaches to the corresponding pseudo-slab problem of the spherical medium radiated transfer equation. Macchali H. M., Haggag M.H. and Al-Gorashi A. K., (2013) investigated the integral form of the radiation transfer equation in a planar plate with isotropic scattering by using the Chebyshev polynomial approach. He made the numerical calculations for the transmittance and reflectivity of the plates with various values of the single scattering albedo. Arnold (2002) studied Chebyshev spectral effects to examine radiative transmission problems. The solutions of the time dependent radiative transfer equations are performed by Chebyshev spectral methods. By expanding the Chebyshev polynomials, the spatial dependence of the density could be calculated approximately. A unified system of complete differential equations was obtained for the expansion coefficients with respect to angle and time. Legendre polynomials method called P_N and Chebyshev polynomials method called T_N method are used to solve the radiative transfer equation for the plane geometrical system for isotropic scattering case. The calculations are made for many single scattering albedo numbers.

MATERIALS AND METHODS

P_N Solution of the Theory

The time-independent radiative transfer equation for~~,~~ isotropically scattering slab is can be written as (Pomraning 1970);

$$\mu\frac{\partial I(x,\mu)}{\partial x}+I(x,\mu)=\frac{\omega}{2}\int_{-1}^{1}I(x,\mu')d\mu' \tag{1}$$

where $I(x, \mu)$ is the angular intensity μ is the cosine angle of the scattering direction, x is the optical variable, ω is the single scattering albedo. To obtain the eigenvalues for the various orders of P_N approximations, the specific intensity $I(x, \mu)$ is demonstrated a full set of spherical harmonics (Case and Zweifel 1967; Pomraning 1973; Duderstadt and Martin 1979), in the one-dimensional case to a full set of Legendre polynomials:

$$I(x,\mu)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\phi_n(x)P_n(\mu) \tag{2}$$

Legendre moments of the flux are given by

$$\phi_n(x)=\int_{-1}^{1}P_n(\mu)\,I(x,\mu)d\mu \tag{3}$$

when Eqn. (2) is put into radiative transfer equation given in Eqn. (1)

$$\mu\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{2n+1}{2}\phi_n(x)P_n(\mu)\right)+\sum_{n=0}^{\infty}\frac{2n+1}{2}\phi_n(x)P_n(\mu)=\frac{\omega}{2}\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{2n+1}{2}\phi_n(x)P_n(\mu')d\mu' \quad (4)$$

The recursion relation (Stacey, 2007)

$$\mu P_n(\mu)=\frac{1}{2n+1}[(n+1)P_{n+1}(\mu)+nP_{n-1}(\mu)] \quad (5)$$

and the orthogonality of the Legendre polynomials;

$$\int_{-1}^{1}P_n(x)P_m(x)=\begin{cases}0 & if \quad m\neq n\\ \frac{2}{2n+1} & if \quad m=n\end{cases} \quad (6)$$

are applied to Equation (4). After some algebra, one can obtain

$$n\frac{d}{dx}\phi_{n-1}(x)+\ (n+1)\frac{d}{dx}\phi_{n+1}(x)+(2n+1)\phi_n(x)=\omega\phi_0(x)\delta_{n,0}$$

$$n=0,1,2,3\ldots \quad (7)$$

The eigen-spectrum of the Eq. (1) is obtained by employing the well-known ansatz for the solution of the form (Davison & Sykes, 1957),

$$\phi_n(x)=A_n(v)e^{x/v} \quad (8)$$

where $A_n(v)$ is the eigenfunctions and ν is the eigenvalues of $A_n(v)$ function. The Eqn. (8) is substituted into general form of differential set of the angular moment into Eqn. (7) so a system of equations is obtained for the analytic expressions of $A_n(v)$; $A_0(v)=1$. $A_{n+1}=0$ relation enables to compute the discrete eigenvalues (Köklü and Özer, 2021). After all of these processes, the eigenvalues become straightforward. The eigenfunctions are obtained for n =0 and 1 in Eqn. (9 and 10);

$$A_1(v)=vA_0(v)(\omega-1) \quad (9)$$

$$A_2(v)=\frac{-A_0(v)-3v.A_1(v)}{2} \quad (10)$$

Here $A_0(v)$=1 and $A_2(v)$ is assumed as zero and therefore the eigenvalues can be found as

$$v=\pm\frac{1}{\sqrt{3(1-w)}} \quad (11)$$

The other eigenvalues can be computed by applying the same procedure. The general form of the eigenfunctions may be written by inserting Eq. (8) into Eq. (7),

$$nA_{n-1}(v)+\ (n+1)A_{n+1}(v)+(2n+1)v.A_n(v)=vA_0(v)\delta_{n0} \quad (12)$$

As shown in Equation (11), the analytical solution of the $A_n(v)$ gives the discrete eigenvalues v by solving $A_{n+1}(v)=0$ for any single scattering albedo w. Here, it has (N+1)/2 eigenvalues v, $k=1,2,3,\ldots N+1$ roots are used to find the flux moment.

T$_N$ Solution of the Theory

The angular intensity function can be written in the following expression for the first type of Chebyshev polynomial method that is T$_N$ method It is first used by Anli, F. Yaşa, F. Güngör, S. et al. (2006).

$$I(r,\mu)=\frac{\phi_0(r)T_0(\mu)}{\pi\sqrt{1-\mu^2}}+\frac{2}{\pi}\sum_{n=1}^{N}\frac{\phi_n(r)T_n(\mu)}{\sqrt{1-\mu^2}} \tag{13}$$

where $\phi_n(r)$ is called as the Chebyshev moment and $T_n(\mu)$is the term of the Chebyshev polynomial of first type. Equation (13) is substituted in Equation (1). When the equation has been solved the recurrence and the orthogonality relations are used, and then, m^{th} order Chebyshev polynomial is applied and integrated over [–1, 1] to both sides of the resultant equation to determine the infinite set of ordinary differential equations as follows,

$$\frac{d}{dr}\phi_{n-1}(r)+\frac{d}{dr}\phi_{n+1}(r)+2\phi_n(r)=\omega\phi_n(r)\delta_{n0.} \tag{14}$$

Eigen-spectrum of the radiative transfer equation for Chebyshev polynomial is obtained by seeking a solution of the form as (Yılmazer and Kocar, 2009);

$$\phi_n(r)=A_n(v)e^{r/v} \tag{15}$$

The series expression of $A_n(v)$ is found by substituting into Eq. (15) into Eqn. (14). The eigenvalue for $A_2(v)=0$ is computed as

$$v=\pm\frac{1}{\sqrt{(4-2w)}} \tag{16}$$

The eigenvalues of the higher order iterations of the solution method are found by calculating the well-known assumption as $A_{n+1}(v)=0$

RESULTS AND DISCUSSION

The radiative transfer equation is solved to obtain the discrete eigenvalues for various single scattering albedos by using the Legendre and first-type Chebyshev polynomials methods. The solutions are done up to the 13th iteration in order to observe the convergence in the results. The accuracy in the last iteration is acceptable for code verification. The single scattering albedo values are chosen from zero to two which are commonly used intervals in literature. The wide range of eigenvalues spectrum with two methods are tabulated in the tables. Wolfram Mathematica Program is used for the calculations. In Table 1, the numerical values of the discrete eigenvalues obtained from the solution of the Radiative Transfer equation with the Legendre polynomial are presented with all iteration steps. Table 2 is organized for the first type of Chebyshev polynomial method. The same iteration steps are applied for the solution of the radiative transfer equation with the same single scattering albedo numbers. As it is figured out that the radiative transfer equation solutions to calculate the discrete eigenvalues are rarely studied among researchers. Even so, in Table 3, our solutions with P_N and T_N methods are compared with the literature Taşdelen, N. (2017) and good agreement in the results is obtained.

In this paper, it is shown that the solution of the radiative transfer equation can be done with the Chebyshev polynomial of the first type. It means that the presented study can be rearranged for different problems in the radiative transfer equation for the future work. Radiative transfer equation attracts the attention of researchers working in both physics and mathematics. It is a structure of equations suitable for usage in many areas of physics. However, results can only be obtained with special solution methods. For this reason, our study is convenient in terms of approaching the radiative transfer equation differently. It is pleasing to note that the results of the Chebyshev method are in agreement with the P_N method in the literature.

Table 1: Discrete eigenvalues with P_N method for various single scattering albedo ω

ω	P_1	P_3	P_5	P_7	P_9	P_{11}	P_{13}
0.00	0.577350	0.339981	0.238619	0.183434	0.148874	0.125233	0.108054
		0.861136	0.661209	0.525532	0.433395	0.367831	0.319112
			0.932469	0.796666	0.679409	0.587317	0.515248
				0.960289	0.865063	0.769902	0.687292
					0.973906	0.904117	0.827201
						0.981560	0.928434
							0.986283
0.25	0.666666	0.370920	0.253620	0.192206	0.154610	0.129271	0.111049
		0.911411	0.696158	0.548404	0.449131	0.379216	0.327695
			0.962178	0.824300	0.701000	0.603953	0.528253
				0.980110	0.886464	0.788603	0.702900
					0.988275	0.921011	0.843117
						0.992594	0.942094
							0.995116
0.50	0.816496	0.409381	0.270885	0.201936	0.160837	0.133594	0.114223
		1.011376	0.740949	0.575296	0.466855	0.391716	0.336963
			1.036621	0.861400	0.727348	0.623233	0.542854
				1.042230	0.916237	0.812275	0.721556
					1.043745	0.945021	0.863918
						1.044189	0.961671
							1.044323
0.75	1.154700	0.455944	0.290519	0.212656	0.167568	0.138206	0.117579
		1.284234	0.785622	0.603418	0.485462	0.404806	0.346631
			1.289198	0.891430	0.752448	0.642384	0.557581
				1.289449	0.935828	0.832698	0.739002
					1.289462	0.958037	0.880136
						1.289463	0.970573
							1.289463
1.01	5.773502	5.750541	5.750539	5.750539	5.750539	5.750539	5.750539
		0.509117	0.313111	0.224794	0.175087	0.143306	0.121260
			0.817092	0.629102	0.503853	0.418174	0.356664
				0.906866	0.771095	0.658808	0.571149
					0.944113	0.845546	0.752162
						0.962924	0.889087
							0.973685
1.10	1.825741	1.757034	1.756653	1.756651	1.756651	1.756651	1.756651
		0.526921	0.321269	0.229197	0.177803	0.145139	0.122578
			0.824371	0.636523	0.509655	0.422571	0.360037
				0.910017	0.775761	0.663407	0.575208
					0.945733	0.848495	0.755538
						0.963865	0.891037
							0.974282
1.20	1.290994	1.200164	1.198298	1.198265	1.198265	1.198265	1.198265
		0.545469	0.330410	0.234184	0.180883	0.147216	0.124068
			0.830893	0.643836	0.515668	0.427253	0.363687
				0.912764	0.780103	0.667900	0.579309
					0.947136	0.851167	0.758721
						0.964680	0.892779
							0.974799
1.30	1.054092	0.950538	0.946161	0.946005	0.946000	0.946000	0.946000
		0.562336	0.339525	0.239247	0.184020	0.149331	0.125585
			0.836143	0.650229	0.521198	0.431691	0.367209
				0.914937	0.783720	0.671808	0.582994
					0.948243	0.853347	0.761407
						0.965324	0.894187
							0.975209

Continuation Table 1: Discrete eigenvalues with P_N method for various single scattering albedo ω

1.40	0.912870	0.801783	0.794225	0.793793	0.793769	0.793768	0.793768
		0.577350	0.348504	0.244358	0.187206	0.151481	0.127125
			0.840421	0.655798	0.526244	0.435864	0.370587
				0.916692	0.786756	0.675205	0.586290
					0.949136	0.855151	0.763686
						0.965844	0.895343
							0.975541
1.50	0.816496	0.701130	0.690105	0.689205	0.689136	0.689130	0.689130
		0.590531	0.357238	0.249488	0.190431	0.153662	0.128689
			0.843956	0.660647	0.530823	0.439764	0.373806
				0.918135	0.789325	0.678164	0.589234
					0.949871	0.856661	0.765634
						0.966273	0.896308
							0.975815
1.60	0.745355	0.627830	0.613356	0.611794	0.611637	0.611621	0.611619
		0.602017	0.365637	0.254601	0.193685	0.155871	0.130274
			0.846913	0.664877	0.534966	0.443390	0.376859
				0.919339	0.791520	0.680750	0.591863
					0.950486	0.857942	0.767312
						0.966632	0.897123
							0.976045
1.70	0.690065	0.571775	0.554073	0.551676	0.551377	0.551340	0.551335
		0.612001	0.373629	0.259665	0.196955	0.158103	0.131878
			0.849418	0.668581	0.538707	0.446748	0.379741
				0.920359	0.793410	0.683021	0.594213
					0.951008	0.859039	0.768768
						0.966938	0.897820
							0.976240
1.80	0.645497	0.527361	0.506762	0.503398	0.502894	0.502821	0.502810
		0.620687	0.381164	0.264647	0.200228	0.160353	0.133499
			0.851562	0.671838	0.542085	0.449850	0.382453
				0.921233	0.795051	0.685023	0.596318
					0.951455	0.859989	0.770039
						0.967200	0.898422
							0.976409
1.90	0.608580	0.491203	0.468080	0.463664	0.462891	0.462761	0.462739
		0.628267	0.388218	0.269517	0.203492	0.162615	0.135134
			0.853416	0.674715	0.545138	0.452711	0.384998
				0.921990	0.796488	0.686798	0.598208
					0.951843	0.860817	0.771159
						0.967428	0.898947
							0.976555
2.00	0.577350	0.461121	0.435850	0.430341	0.429239	0.429027	0.428987
		0.634908	0.394783	0.274249	0.206732	0.164882	0.136781
			0.855033	0.677270	0.547900	0.455348	0.387381
				0.922651	0.797755	0.688379	0.599911
					0.952183	0.861546	0.772150
						0.967628	0.899409
							0.976684

Table 2: Discrete eigenvalues with T_N method for various single scattering albedo ω

ω	T_1	T_3	T_5	T_7	T_9	T_{11}	T_{13}
0.00	0.707106	0.382683	0.258819	0.195090	0.156434	0.130526	0.111964
		0.923879	0.707106	0.555570	0.453990	0.382683	0.330279
			0.965925	0.831469	0.707106	0.608761	0.532032
				0.980785	0.891006	0.793353	0.707106
					0.987688	0.923879	0.846724
						0.991444	0.943883
							0.993712
0.25	0.816496	0.422799	0.276446	0.205056	0.162771	0.134920	0.115181
		0.965583	0.747351	0.580801	0.471292	0.394972	0.339482
			0.988005	0.861412	0.730013	0.626614	0.545822
				0.994837	0.913403	0.812691	0.723561
					0.997634	0.941182	0.862893
						0.998983	0.957672
							0.999704
0.50	1.000000	0.475087	0.296994	0.216200	0.169682	0.139639	0.118598
		1.052437	0.801739	0.610901	0.490965	0.408537	0.349458
			1.049924	0.903794	0.758457	0.647542	0.561409
				1.047160	0.945955	0.837641	0.743482
					1.045659	0.966471	0.884434
						1.044945	0.977614
							1.044626
0.75	1.414213	0.541196	0.320605	0.228565	0.177181	0.144690	0.122218
		1.306562	0.853795	0.642212	0.511632	0.422750	0.359877
			1.291605	0.933760	0.785001	0.668189	0.577087
				1.289733	0.963223	0.858411	0.761843
					1.289500	0.976866	0.900417
						1.289469	0.984197
							1.289464
1.01	7.071067	5.748472	5.750546	5.750539	5.750539	5.750539	5.750539
		0.615038	0.347816	0.242632	0.185582	0.150288	0.126196
			0.883822	0.669779	0.531780	0.437176	0.370655
				0.945815	0.803632	0.685408	0.591329
					0.969012	0.870646	0.775184
						0.980062	0.908660
							0.986147
1.10	2.236067	1.753107	1.756821	1.756645	1.756652	1.756651	1.756651
		0.637744	0.357571	0.247737	0.188620	0.152303	0.127622
			0.889885	0.677463	0.538032	0.441882	0.374263
				0.948053	0.808121	0.690116	0.595533
					0.970074	0.873364	0.778520
						0.980651	0.910408
							0.986509
1.20	1.581138	1.198087	1.198709	1.198240	1.198267	1.198264	1.198265
		0.659859	0.368410	0.253514	0.192065	0.154586	0.129235
			0.895085	0.684903	0.544443	0.446867	0.378153
				0.949956	0.812239	0.694662	0.599749
					0.970980	0.875801	0.781633
						0.981155	0.911958
							0.986821
1.30	1.290994	0.951262	0.946883	0.945959	0.946007	0.945999	0.946000
		0.678568	0.379093	0.259368	0.195572	0.156911	0.130876
			0.899128	0.691299	0.550276	0.451562	0.381894
				0.951435	0.815632	0.698575	0.603511
					0.971687	0.877775	0.784238
						0.981551	0.913204
							0.987067

Continuatio Table 2: Discrete eigenvalues with T_N method for various single scattering albedo ω

1.40	1.118033	0.805312	0.795343	0.793738	0.793783	0.793766	0.793768
		0.694161	0.389465	0.265257	0.199131	0.159276	0.132544
			0.902342	0.696792	0.555543	0.455950	0.385466
				0.952613	0.818454	0.701946	0.606854
					0.972253	0.879398	0.786433
						0.981869	0.914224
							0.987265
1.50	0.990147	0.698985	0.683056	0.680418	0.680407	0.680371	0.680375
		0.708274	0.400357	0.271725	0.203091	0.161916	0.134407
			0.905180	0.701955	0.560722	0.460414	0.389184
				0.953658	0.821043	0.705130	0.610101
					0.972758	0.880875	0.788470
						0.982154	0.915149
							0.987443
1.60	0.912870	0.635805	0.615475	0.611799	0.611679	0.611616	0.611621
		0.717885	0.408776	0.276969	0.206352	0.164102	0.135952
			0.907096	0.705605	0.564520	0.463789	0.392055
				0.954367	0.822842	0.707392	0.612461
					0.973102	0.881896	0.789898
						0.982349	0.915787
							0.987566
1.70	0.845154	0.581324	0.556742	0.551762	0.551442	0.551335	0.551338
		0.726921	0.417549	0.282704	0.209984	0.166553	0.137688
			0.908895	0.709154	0.568323	0.467256	0.395062
				0.955037	0.824571	0.709603	0.614809
					0.973427	0.882873	0.791281
						0.982534	0.916397
							0.987682
1.80	0.790569	0.538124	0.509976	0.503600	0.502993	0.502817	0.502815
		0.734559	0.425683	0.288302	0.213608	0.169020	0.139442
			0.910422	0.712255	0.571734	0.470442	0.397880
				0.955608	0.826068	0.711545	0.616904
					0.973706	0.883718	0.792486
						0.982693	0.916924
							0.987782
1.90	0.745355	0.502888	0.471811	0.464011	0.463033	0.462761	0.462746
		0.741075	0.433178	0.293729	0.217209	0.171496	0.141210
			0.911733	0.714980	0.574797	0.473367	0.400513
				0.956100	0.827374	0.713260	0.618779
					0.973948	0.884453	0.793544
						0.982831	0.917383
							0.987869
2.00	0.707106	0.473499	0.440055	0.430858	0.429434	0.429036	0.428997
		0.746682	0.440055	0.298955	0.220769	0.173975	0.142989
			0.912870	0.717390	0.577555	0.476051	0.402969
				0.956529	0.828524	0.714783	0.620464
					0.974159	0.885100	0.794478
						0.982952	0.917787
							0.987945

Table 3: Comparison of the discrete eigenvalues with literature

	Present study results		Taşdelen, N., (2017)
ω	P_9	T_9	P_9
0.00	0.183434	0.195090	0.18343
	0.525532	0.555570	0.52553
	0.796666	0.831469	0.79666
	0.960289	0.980785	0.96028
0.25	0.192206	0.205056	0.19220
	0.548404	0.580801	0.54840
	0.824300	0.861412	0.82430
	0.980110	0.994837	0.98011
0.50	0.201936	0.216200	0.20193
	0.575296	0.610901	0.57529
	0.861400	0.903794	0.86140
	1.042230	1.047160	1.04223
0.75	0.212656	0.228565	0.21265
	0.603418	0.642212	0.60341
	0.891430	0.933760	0.89143
	1.289449	1.289733	1.28944
0.99	0.223829	0.241512	0.22381
	0.627343	0.667937	0.62734
	0.906054	0.945229	0.90605
	5.796729	5.796729	5.79672

CONCLUSION

This study proved that the Chebyshev polynomials can be used to find the discrete eigenvalues of the radiation transfer equation. The solution of the radiative transfer equation has been done by Legendre polynomials by some researchers Ozisik, M. N. and Shouman S. M. (1980), Menguc M. P. and Viskanta R. (1982), Taşdelen, N. (2017) and Taşdelen, M. (2017). The results obtained from Legendre polynomial method can be considered as benchmark results. So the radiative transfer equation is solved first with the Legendre polynomial to make a comparison with the Chebyshev polynomial solution. The numerical results obtained from Legendre polynomials and Chebyshev polynomials methods are coherent with each other as it can be seen by Table 3. The discrete eigenvalues are computed with two different methods for various single-scattering albedo numbers. The discrete eigenvalues can be used for many problems in the radiative transfer equation. Such as the boundary value or albedo problems calculations are done by using the discrete eigenvalues of the angular eigenfunctions. The numerical results of the eigenvalues for the both methods are presented in tables. As can be seen from the tables, the results converge with each other in the case of isotropic scattering. It is seen that there is a complete agreement between the results. Comparison of the results obtained shows that the calculations are consistent with each other and with the literature.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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