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The universal property of commutative algebras' internal crossed modules

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Universal property, Internal crossed module, Ideal, Category

Abstract — In this work, we identify subobjects and ideals in the category of internal crossed modules, which provide a deeper understanding of the structure of these objects. Moreover, we provide several propositions through examples, which illustrate the properties and relationships between ideals and subobjects in the category of internal crossed modules. The examples and propositions provided in this work can serve as a foundation for further research in this area and may lead to new insights and discoveries in the study of these complex algebraic structures. Overall, in conclusion, we give a brief overview of the contributions and future research directions of the work presented, highlighting the significance of internal crossed modules in algebraic topology and category theory as well as making suggestions for possible areas of additional research and application.

Subject Classification (2020): 18D40,18A05

1. Introduction

Crossed modules were defined by Whitehead [1] as a tool for homotopy theory. Instead of using crossed modules in name, equivalently, commutative algebra has been used in [2]. Although specific terms are used in crossed modules, their terminology is not entirely standardized. For a crossed module $\partial : M \to N$, the algebras M and N are called the top and the base of the crossed module, respectively, while the homomorphism ∂ is referred to as the boundary. Crossed module axioms are also known by their names; CM1 is occasionally referred to as equivariance, and CM2 is known as the Peiffer identity, as explained in [3]. And a pre-crossed module is a structure that contains the same data as a crossed module, satisfies the equivariance condition, but does not satisfy the Peiffer identity.

By providing relevant objects and morphisms, groups, and other algebraic structures, an internal category can be constructed in a category with pullbacks. Thus, pullbacks enable the formulation of the notion of a category internal to any other category. Brown and Spencer noted in [4] that crossed modules are equivalent to internal categories within the category of groups. Porter [5] for the case of commutative algebra and Ellis [6] for the case of Lie algebra are two well-known examples of the analogous equivalence of this result.

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The category of internal crossed modules is equivalent to the category of crossed squares and 2-crossed modules. More information about the category of crossed squares and 2-crossed modules can be found in [7–11].

In the present paper, we begin the study of some of the more complex properties of internal crossed modules of algebras and compile basic terminology and information for future use. For more information on internal crossed modules, see [12–17]. We begin with the definition of internal crossed modules. Later, we provide some fundamental properties for a given internal crossed-module morphism, including the kernel and the image. Moreover, we define the ideals and subobjects of a given internal crossed module. Additionally, we provide the universal property of internal crossed modules.

The main ideas of this work can be given as:

i. To fully describe the subobjects and ideals within this category.

ii. To construct the quotient object by using ideals in this category.

iii. To give the universal property of internal crossed modules.

All crossed modules throughout the text will be crossed modules of commutative algebras.

2. Internal Crossed Modules

Let R and C be algebras. A *crossed module* is an R-algebra homomorphism $\partial : C \to R$ with the action R on C such that

$$CM1 \quad \partial(c \cdot r) = \partial(c)r$$

$$CM2 \quad c' \cdot \partial(c) = c'c$$

for all $r \in R$ and $c, c' \in C$. We shortly show this crossed module, such as (C, R, ∂).

A pair $\phi : C \to C'$ and $\varphi : R \to R'$ of k-algebra (or an algebra over k) homomorphism such that $\phi(c \cdot r) = \phi(c)\varphi(r)$ defines a morphism of crossed modules from (C, R, ∂) to (C', R', ∂'). Thus, we obtain the category of crossed modules of commutative algebras, denoted by XMod.

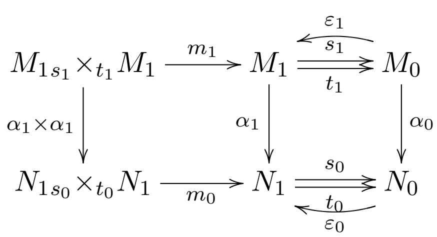
Definition 2.1. Let C be a category within the XMod category of crossed modules over algebras. Then, C consists of two crossed modules $\varkappa_1 = (M_1, N_1, \alpha_1)$ and $\varkappa_0 = (M_0, N_0, \alpha_0)$ with crossed module morphisms $s = (s_1, s_0)$, $t = (t_1, t_0) : \varkappa_1 \to \varkappa_0$ called source and target, respectively, the identity map $\varepsilon = (\varepsilon_1, \varepsilon_2) : \varkappa_0 \to \varkappa_1$ and the composition $m = (m_1, m_0) : \varkappa_1 \,{}_s{\times}_t\, \varkappa_1 \to \varkappa_1$. These data satisfy:

$$s\varepsilon = t\varepsilon = 1_{\varkappa_0} \tag{2.1}$$

$$sm = s\pi_2, tm = t\pi_1 \tag{2.2}$$

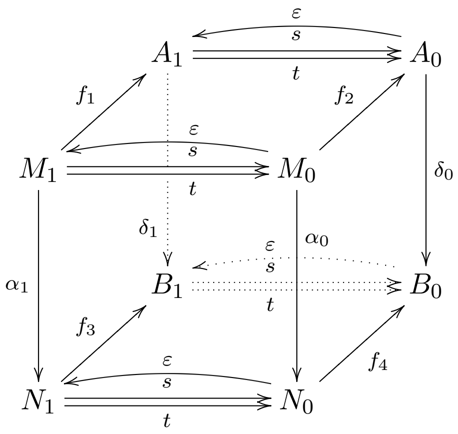
$$m\left(1_{\varkappa_1} \times m\right) = m\left(m \times 1_{\varkappa_1}\right) \tag{2.3}$$

$$m\left(\varepsilon s, 1_{\varkappa_1}\right) = m\left(1_{\varkappa_1}, \varepsilon t\right) = 1_{\varkappa_1} \tag{2.4}$$



We denote such an internal crossed module of algebras by $\left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$.

Definition 2.2. A morphism $f = (f_1, f_2, f_3, f_4)$ of internal crossed modules from $\left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$ to $\left(\begin{array}{cc} A_1 & A_0 \\ B_1 & B_0 \end{array}, \delta, s, t, e, m\right)$ consists of algebra homomorphisms $f_1 : M_1 \to A_1$, $f_2 : M_0 \to A_0$, $f_3 : N_1 \to B_1$, and $f_4 : N_0 \to B_0$, compatible with source, target, identity, and the actions.



Thus, we get the category of internal crossed modules which is denoted by IntXMod.

Definition 2.3. [18] A subcrossed module of a crossed module $\partial : C \to R$ is a crossed module $\partial' : C' \to R'$ such that C' is a subalgebra of C and ∂' is the restriction of ∂ to C'.

In a categorical sense, a subcrossed module should be a subobject. Inclusion of $\partial : C \to R$ to $\partial' : C' \to R'$ in this way results in a monomorphism of crossed modules.

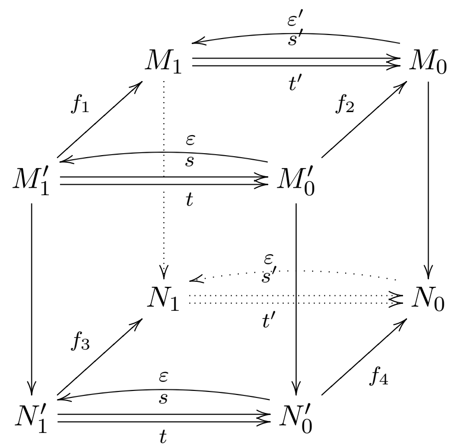
Definition 2.4. Let $I_1 = \left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$ be an object in IntXMod. Then, we say that $I_2 = \left(\begin{array}{cc} M_1' & M_0' \\ N_1' & N_0' \end{array}, \alpha', s', t', e', m'\right)$ is a subobject of I_1 if

i. M_1' and N_1' are subalgebras of M_1 and N_1, respectively and M_0' and N_0' are subrings of M_0 and N_0, respectively.

ii. the homomorphisms s', t', e', and m' of I_2 are the restrictions of the homomorphisms s, t, e, and m of I_1.

iii. I_2 is an object in IntXMod.

iv. $f : I_2 \to I_1$ is a morphism in IntXMod where for $i \in \{1, 2, 3, 4\}$ the morphisms f_i are injections.



The ideals of a crossed module $\partial : C \to R$ are subcrossed modules $\partial : C' \to R'$ contained in the kernel of ∂ [18]. We define the ideal of an object in IntXMod.

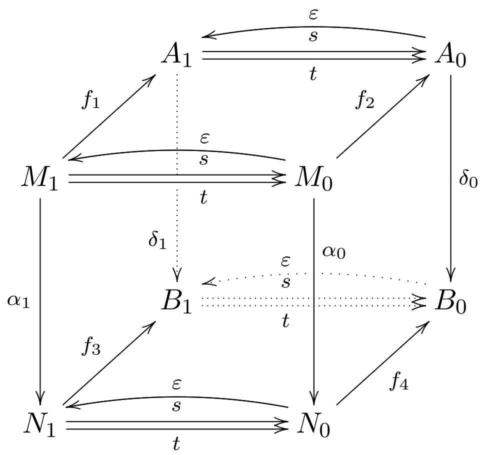
Definition 2.5. Let $I_1 = \left(\begin{matrix} M_1 & M_0 \\ N_1 & N_0 \end{matrix}, \alpha, s, t, e, m\right)$ be an object in IntXMod. Then, we say that the subobject $I_2 = \left(\begin{matrix} M'_1 & M'_0 \\ N'_1 & N'_0 \end{matrix}, \alpha', s', t', e', m'\right)$ of I_1 is an ideal if

i. $M'_i \cup M_i \subset M'_i$, for $i \in \{0,1\}$, N'_0 is an ideal of N_0 and N'_1 is an ideal of N_1.

ii. $n'_i \cdot m_i \in M_i$, for $n'_i \in N'_i$ and $m_i \in M_i$ $(i \in \{0,1\})$.

iii. $n_i \cdot m'_i \in M'_i$, for $n_i \in N_i$ and $m'_i \in M'_i$ $(i \in \{0,1\}$.

Example 2.6. Let $f = (f_1, f_2, f_3, f_4) : \left(\begin{matrix} M_1 & M_0 \\ N_1 & N_0 \end{matrix}, \alpha, s, t, e, m\right) \to \left(\begin{matrix} A_1 & A_0 \\ B_1 & B_0 \end{matrix}, \delta, s, t, e, m\right)$ be a morphism in IntXMod.



Then,

$$\left(\begin{matrix} \operatorname{Im} f_1 & \operatorname{Im} f_2 \\ \operatorname{Im} f_3 & \operatorname{Im} f_4 \end{matrix}, \delta|_{\operatorname{Im}}, s|_{\operatorname{Im}}, t|_{\operatorname{Im}}, e|_{\operatorname{Im}}, m|_{\operatorname{Im}}\right) \text{ is a subobject of } \left(\begin{matrix} A_1 & A_0 \\ B_1 & B_0 \end{matrix}, \delta, s, t, e, m\right)$$

and

$$\left(\begin{matrix} \operatorname{Ker} f_1 & \operatorname{Ker} f_2 \\ \operatorname{Ker} f_3 & \operatorname{Ker} f_4 \end{matrix} \alpha|_{\operatorname{Ker}}, s|_{\operatorname{Ker}}, t|_{\operatorname{Ker}}, e|_{\operatorname{Ker}}, m|_{\operatorname{Ker}}\right) \text{ is an ideal of } \left(\begin{matrix} M_1 & M_0 \\ N_1 & N_0 \end{matrix}, \alpha, s, t, e, m\right)$$

Proposition 2.7. Let $f = (f_1, f_2, f_3, f_4) : \left(\begin{matrix} M_1 & M_0 \\ N_1 & N_0 \end{matrix}, \partial, s, t, e, m\right) \to \left(\begin{matrix} A_1 & A_0 \\ B_1 & B_0 \end{matrix}, \delta, s, t, e, m\right)$ be a morphism in IntXMod. The image of the morphism f, $\left(\begin{matrix} \operatorname{Im} f_1 & \operatorname{Im} f_2 \\ \operatorname{Im} f_3 & \operatorname{Im} f_4 \end{matrix}, \delta|_{\operatorname{Im}}, s|_{\operatorname{Im}}, t|_{\operatorname{Im}}, e|_{\operatorname{Im}}, m|_{\operatorname{Im}}\right)$ is a subobject of $\left(\begin{matrix} A_1 & A_0 \\ B_1 & B_0 \end{matrix}, \delta, s, t, e, m\right)$.

Proof.

$\delta_1|_{\operatorname{Im}} : \operatorname{Im} f_1 \to \operatorname{Im} f_3$ and $\delta_0|_{\operatorname{Im}} : \operatorname{Im} f_2 \to \operatorname{Im} f_4$ are crossed modules [18]. $\operatorname{Im} f_1$, $\operatorname{Im} f_2$, $\operatorname{Im} f_3$, and $\operatorname{Im} f_4$ are subalgebras of A_1, A_0, B_1, and B_0, respectively. The morphisms $s|_{\operatorname{Im}}$, $t|_{\operatorname{Im}}$, $e|_{\operatorname{Im}}$ and the composition $m|_{\operatorname{Im}}$ are the restrictions and satisfy (2.1)-(2.4) in Definition 2.1. Thus, $\left(\begin{matrix} \operatorname{Im} f_1 & \operatorname{Im} f_2 \\ \operatorname{Im} f_3 & \operatorname{Im} f_4 \end{matrix}, \delta|_{\operatorname{Im}}, s|_{\operatorname{Im}}, t|_{\operatorname{Im}}, e|_{\operatorname{Im}}, m|_{\operatorname{Im}}\right)$ is a subobject of $\left(\begin{matrix} A_1 & A_0 \\ B_1 & B_0 \end{matrix}, \delta, s, t, e, m\right)$. $\square$

Proposition 2.8. Let $f = (f_1, f_2, f_3, f_4) : \left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \partial, s, t, e, m\right) \to \left(\begin{array}{cc} A_1 & A_0 \\ B_1 & B_0 \end{array}, \delta, s, t, e, m\right)$ be a morphism in IntXMod. The kernel of the morphism f, $\left(\begin{array}{cc} \mathrm{Ker}\, f_1 & \mathrm{Ker}\, f_2 \\ \mathrm{Ker}\, f_3 & \mathrm{Ker}\, f_4 \end{array} \alpha|_{\mathrm{Ker}}, s|_{\mathrm{Ker}}, t|_{\mathrm{Ker}}, e|_{\mathrm{Ker}}, m|_{\mathrm{Ker}}\right)$ is an ideal of $\left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$.

Proof.

Similarly, it is clear that $\left(\begin{array}{cc} \mathrm{Ker}\, f_1 & \mathrm{Ker}\, f_2 \\ \mathrm{Ker}\, f_3 & \mathrm{Ker}\, f_4 \end{array}, \delta|_{\mathrm{Im}}, s|_{\mathrm{Im}}, t|_{\mathrm{Im}}, e|_{\mathrm{Im}}, m|_{\mathrm{Im}}\right)$ is a subobject of $\left(\begin{array}{cc} A_1 & A_0 \\ B_1 & B_0 \end{array}, \delta, s, t, e, m\right)$. We will show the ideal conditions for a subobject in IntXMod.

i. $\mathrm{Ker}\, f_3$ is an ideal of N_1. Since

$$\begin{aligned} f_3(n_1 - n_1') &= f_3(n_1) - f_3(n_1') \\ &= 0_{N_1} - 0_{N_1} \\ &= 0_{N_1} \end{aligned}$$

for $n_1, n_1' \in \mathrm{Ker}\, f_3$ we obtain $n_1 - n_1' \in \mathrm{Ker}\, f_3$. Moreover, since

$$\begin{aligned} f_3(na) &= f_3(n) f_3(a) \\ &= f_3(n) 0_{N_1} \\ &= 0_{N_1} \end{aligned}$$

for $n \in N_1, a \in \mathrm{Ker}\, f_3$, we get $na \in \mathrm{Ker}\, f_3$. Therefore, $\mathrm{Ker}\, f_3$ is an ideal of N_1.

ii. We get $f_2(n_0' \cdot m_0) = 0_{M_0}$, for $n_0' \in \mathrm{Ker}\, f_2, m_0 \in M_0$. Thus, $n_0' \cdot m_0 \in \mathrm{Ker}\, f_2$.

iii. We get $f_1(n_1 \cdot m_1') = 0_{\mathrm{Ker}\, f_1}$, for $n_1 \in N_1, m_1' \in \mathrm{Ker}\, f_1$. Therefore, $n_0 \cdot m_1' \in \mathrm{Ker}\, f_1$.

Remaning conditions can be shown similarly. As a result, $\left(\begin{array}{cc} \mathrm{Ker}\, f_1 & \mathrm{Ker}\, f_2 \\ \mathrm{Ker}\, f_3 & \mathrm{Ker}\, f_4 \end{array} \alpha|_{\mathrm{Ker}}, s|_{\mathrm{Ker}}, t|_{\mathrm{Ker}}, e|_{\mathrm{Ker}}, m|_{\mathrm{Ker}}\right)$ is an ideal of $\left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$. □

3. Universal Property of Internal Crossed Modules

In this section, using the ideal I_2 of an object I_1 in IntXMod, we prove that the quotient I_1/I_2 is an object in IntXMod. Let $I_2 = \left(\begin{array}{cc} M_1' & M_0' \\ N_1' & N_0' \end{array} \alpha', s', t', e', m'\right)$ be an ideal of $I_1 = \left(\begin{array}{cc} M_1 & M_0 \\ N_1 & N_0 \end{array}, \alpha, s, t, e, m\right)$ in IntXMod. The action of N_0/N_0' on M_0/M_0' can be given as

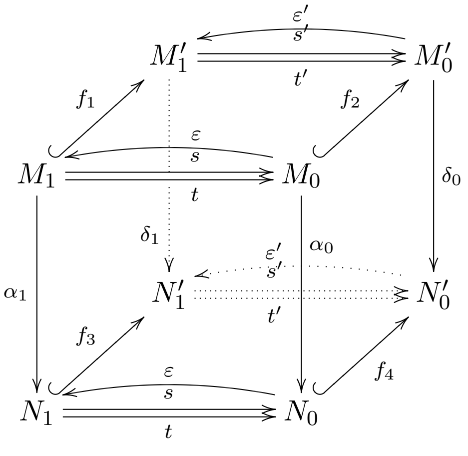
$$(x + N_0') \cdot (y + M_0') = x \cdot y + M_0'$$

and for $n_0' \in N_0'$,

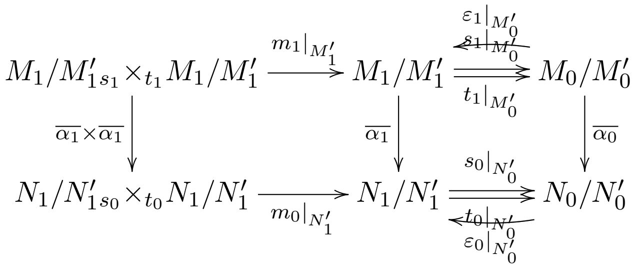
$$\begin{aligned} n_0' \cdot (y + M_0') &= n_0' \cdot y + M_0' \\ &= 0 + M_0' \qquad (\because n_0' \cdot y \in M_0') \end{aligned}$$

N_0/N_0' acts on M_0/M_0' trivially and N_1/N_1' acts on M_1/M_1' similarly.

Theorem 3.1. Let $I_2 = \left(\begin{matrix} M_1' & M_0' \\ N_1' & N_0' \end{matrix}, \alpha', s', t', e', m'\right)$ be the ideal of $I_1 = \left(\begin{matrix} M_1 & M_0 \\ N_1 & N_0 \end{matrix}, \alpha, s, t, e, m\right)$ given with the following diagram



in IntXMod. Then,



Hence, $\left(\begin{matrix} M_1/M_1' & M_0/M_0' \\ N_1/N_1' & N_0/N_0' \end{matrix}, \alpha|_{I_2}, s|_{I_2}, t|_{I_2}, e|_{I_2}, m|_{I_2}\right)$ is an object in IntXMod.

Proof.

Define $\overline{\alpha_1} : M_1/M_1' \to N_1/N_1'$, $m_1 + M_1' \mapsto \alpha_1(m_1) + N_1'$ and $\overline{\alpha_0} : M_0/M_0' \to N_0/N_0'$, $m_0 + M_0' \mapsto \alpha_0(m_0) + N_0'$. To show that $\overline{\alpha_1}$ and $\overline{\alpha_0}$ are crossed modules of algebras. We only need to show that $\overline{\alpha_1}$ and $\overline{\alpha_0}$ are well-defined. For $m_1, m_1' \in M_1$, we get

$$\begin{aligned} m_1 + M_1' = m_1' + M_1' &\Rightarrow m_1 - m_1' \in M_1' \\ &\Rightarrow \alpha_1(m_1 - m_1') \in N_1' \\ &\Rightarrow \alpha_1(m_1) - \alpha_1(m_1') \in N_1' \\ &\Rightarrow \alpha_1(m_1) + N_1' = \alpha_1(m_1') + N_1' \end{aligned}$$

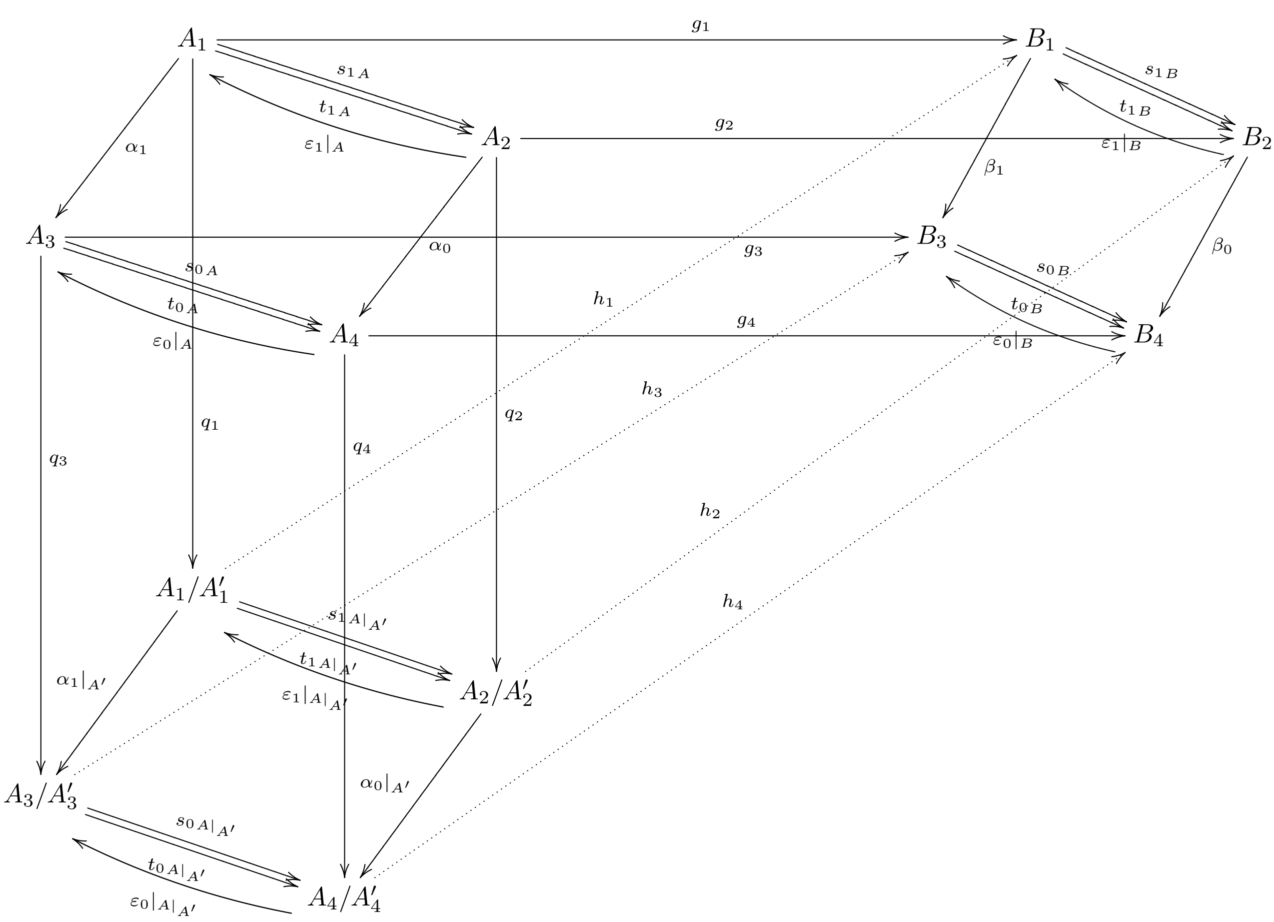
It is obvious that $\overline{\alpha_1}$ is a N_1/N_1'-algebra homomorphism. Therefore, $\overline{\alpha_1} : M_1/M_1' \to N_1/N_1'$ is a crossed module of algebras. Similar way, $\overline{\alpha_0} : M_0/M_0' \to N_0/N_0'$ is a crossed module.

Since I_1 and I_2 are objects in IntXMod, the restrictions $\varepsilon|_{I_2} = (\varepsilon_1|_{M_0'}, \varepsilon_2|_{N_0'})$, $s|_{I_2} = (s_1|_{M_0'}, s_0|_{N_0'})$, $t|_{I_2} = (t_1|_{M_0'}, t_0|_{N_0'})$ and the composition $m|_{I_2} = (m_1|_{M_0'}, m_0|_{N_0'})$ satisfy (2.1)-(2.4) in Definition 2.1. Thus, $\left(\begin{matrix} M_1/M_1' & M_0/M_0' \\ N_1/N_1' & N_0/N_0' \end{matrix}, \alpha|_{I_2}, s|_{I_2}, t|_{I_2}, e|_{I_2}, m|_{I_2}\right)$ is an object in IntXMod. □

Corollary 3.2. Let $A = \left(\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array}, \alpha, s_A, t_A, e_A, m_A \right)$ and $B = \left(\begin{array}{cc} B_1 & B_2 \\ B_3 & B_4 \end{array}, \beta, s_B, t_B, e_B, m_B \right)$ be two objects in IntXMod and $A' = \left(\begin{array}{cc} A'_1 & A'_2 \\ A'_3 & A'_4 \end{array}, \alpha', s_{A'}, t_{A'}, e_{A'}, m_{A'} \right)$ be an ideal of A. Since

$$A/A' = \left(\begin{array}{cc} A_1/A'_1 & A_2/A'_2 \\ A_3/A'_3 & A_4/A'_4 \end{array}, \alpha|_{A'}, s_A|_{A'}, t_A|_{A'}, e_A|_{A'}, m_A|_{A'} \right)$$

is an object in IntXMod, then $q : A \to A/A'$ is a morphism in IntXMod. Let $g : A \to B$ be another morphism in IntXMod such that $g_i(A'_i) = 0_{B_i}$, for $i \in \{1, 2, 3, 4\}$. For $g : A \to B$ in IntXMod, there exists a unique morphism $h : A/A' \to B$ in IntXMod making the following diagram commutative.



4. Conclusion

In this work, we defined the subobjects and ideals in the category of internal crossed modules. We provide the kernel and image of an internal crossed module in order to adapt the isomorphism theorems for ideals and ring homomorphisms found in the ring theory for internal crossed modules.

Crossed modules of groups have been defined in higher dimensions: by Conduché [19] and for commutative algebras Grandjean and Vale [20], namely 2-crossed modules. As additional work, categorical equivalences of internal crossed modules and 2-crossed modules can be investigated. In recent years, category theory has found use in programming languages [21–24]. Programming language adaptations are also possible for internal crossed-module applications. The research presented in this study has addressed the category IntXMod's basic concepts, and these offer direction for upcoming work in the following areas:

i. Utilizing group theory's isomorphism theorems for internally crossed modules.

ii. Obtaining universal constructions in IntXMod (i.e., pullback, pushout, product, etc.).

iii. Programming language adaptations of internal crossed modules.

iv. Determining whether internal crossed modules and 2-crossed modules have the same categorical equivalences.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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