Communications in Advanced Mathematical Sciences
Vol. 6, No. 3, 142-147, 2023
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1281434

Ambarzumyan-Type Theorem for a Conformable Fractional Diffusion Operator

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Abstract
In this paper, we prove an Ambarzumyan-type theorem for a Conformable fractional diffusion operator, i.e. we show that $q(x)$ and $p(x)$ functions are zero if the eigenvalues are the same as the eigenvalues of zero potentials.

Keywords: Ambarzumyan-type theorem, Conformable fractional derivative, Diffusion operator, Inverse problem
2010 AMS: 34A55, 34B24

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Received: 11 April 2023, **Accepted:** 10 September 2023, **Available online:** 15 September 2023
How to cite this article: Y. Çakmak, *Ambarzumyan-Type theorem for a conformable fractional diffusion operator*, Commun. Adv. Math. Sci., (6)3 (2023) 142-147.

1. Introduction

Inverse problems of spectral analysis consist of recovering the coefficients of an operator from their spectral characteristics. Such problems often appear in mathematics, mathematical physics, mechanics, and other branches of natural sciences.

The first result in inverse spectral problems for the Sturm-Liouville operator has been obtained by Ambarzumyan in 1929 (see [1]). He considered the boundary value problem

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad 0 < x < \pi \\ y'(0) = y'(\pi) = 0 \end{cases} \tag{1.1}$$

and proved that if the eigenvalues of (1.1) are $\lambda_n = n^2$, $n \geq 0$, then $q(x) \equiv 0$ a.e. on $(0, \pi)$.

Until now, some Ambarzumyan-type theorems for the Sturm-Liouville, Dirac and diffusion operators including classical derivatives have been studied by many authors (see [2]-[17]). Particularly, in [15], by considering a quadratic Sturm-Liouville operator called the diffusion operator, it is shown that $q(x)$ and $p(x)$ functions are zero if the spectrum is the same as the spectrum of zero potential.

In 2014, Khalil et al. gave a new fractional derivative definition called conformable derivative that extends the well-known limit definition of the classical derivative (see [18]). The conformable fractional derivative has some advantages over fractional derivatives. For instance, while some properties such as the derivative of the product of two functions, the derivative of the quotient of two functions, and the chain rule are not satisfied in all fractional derivatives, these properties are satisfied in the conformable fractional derivative. The basic properties and main results of this derivative were developed in the works in [19]-[24].

In recent years, the direct and inverse problems for the various operators which include conformable fractional derivative have been studied (see [25]-[33]). These problems appear in various branches of applied sciences (see [34]-[38]). In the current literature, there are no results related to the Ambarzumyan-type theorem for a diffusion operator with the conformable fractional derivative.

We consider a conformable fractional diffusion operator (CFDO) with Neumann boundary conditions. The operator $L_\alpha = L_\alpha(p(x), q(x))$ is the form

$$\begin{cases} -T_x^\alpha T_x^\alpha y + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, \quad 0 < x < \pi \\ T_x^\alpha y(0) = T_x^\alpha y(\pi) = 0 \end{cases} \tag{1.2}$$

where λ is the spectral parameter, $\alpha \in (0,1]$, $q(x) \in W_{2,\alpha}^1[0,\pi]$, $p(x) \in W_{2,\alpha}^2[0,\pi]$ are real-valued functions and $T_x^\alpha y$ is a conformable fractional derivative of order α of y at x.

The goal of this paper is to prove an Ambarzumyan-type theorem for the operator L_α. The result obtained can be considered as a partial $\alpha-$generalization of the result given in [15].

2. Preliminaries

Definition 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a given function. Then, the conformable fractional derivative of f of order α with respect to x is defined by*

$$T_x^\alpha f(x) = \lim_{h\to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}, \; T_x^\alpha f(0) = \lim_{x\to 0^+} T_x^\alpha f(x),$$

for all $x > 0$, $\alpha \in (0,1]$. If f is differentiable that is $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$, then, $T_x^\alpha f(x) = x^{1-\alpha} f'(x)$.

We note that more detailed knowledge about conformable fractional calculus can be seen in [18]-[24].
Let $\varphi(x,\lambda;\alpha)$ be the solution of equation (1.2) satisfying the initial conditions

$$\varphi(0,\lambda;\alpha) = 1, \; T_x^\alpha \varphi(0,\lambda;\alpha) = 0.$$

From [32], this solution can be shown with the $\alpha-$integral representation for $h = H = 0$ as

$$\varphi(x,\lambda;\alpha) = \cos\left(\lambda \tfrac{x^\alpha}{\alpha} - \theta(x)\right) + \int_0^x A\left(x, \tfrac{t^\alpha}{\alpha}\right) \cos\lambda \tfrac{t^\alpha}{\alpha}\, d_\alpha t + \int_0^x B\left(x, \tfrac{t^\alpha}{\alpha}\right) \sin\lambda \tfrac{t^\alpha}{\alpha}\, d_\alpha t, \tag{2.1}$$

where the functions $A\left(x, \frac{t^\alpha}{\alpha}\right)$ and $B\left(x, \frac{t^\alpha}{\alpha}\right)$ satisfy the following system

$$\begin{cases} T_x^\alpha T_x^\alpha A\left(x, \tfrac{t^\alpha}{\alpha}\right) - q(x) A\left(x, \tfrac{t^\alpha}{\alpha}\right) - 2p(x)\, T_t^\alpha B\left(x, \tfrac{t^\alpha}{\alpha}\right) = T_t^\alpha T_t^\alpha A\left(x, \tfrac{t^\alpha}{\alpha}\right) \\ T_x^\alpha T_x^\alpha B\left(x, \tfrac{t^\alpha}{\alpha}\right) - q(x) B\left(x, \tfrac{t^\alpha}{\alpha}\right) + 2p(x)\, T_t^\alpha A\left(x, \tfrac{t^\alpha}{\alpha}\right) = T_t^\alpha T_t^\alpha B\left(x, \tfrac{t^\alpha}{\alpha}\right). \end{cases}$$

Besides, the following relations are provided:

$$B(x,0) = 0, \;\; T_t^\alpha A\left(x, \tfrac{t^\alpha}{\alpha}\right)\Big|_{t=0} = 0,$$

$$\theta(x) = \int_0^x p(t)\, d_\alpha t,$$

$$A(0,0) = 0,$$

$$A\left(x, \frac{x^\alpha}{\alpha}\right) \cos\theta(x) + B\left(x, \frac{x^\alpha}{\alpha}\right) \sin\theta(x) = \frac{1}{2} \int_0^x \left(q(t) + p^2(t)\right) d_\alpha t,$$

$$\theta(x) = p(0) \frac{x^\alpha}{\alpha} + 2 \int_0^x \left[A\left(s, \frac{s^\alpha}{\alpha}\right) \sin\theta(s) - B\left(s, \frac{s^\alpha}{\alpha}\right) \cos\theta(s)\right] d_\alpha s.$$

The function

$$\Delta_\alpha(\lambda) = T_x^\alpha \varphi(\pi,\lambda;\alpha) \tag{2.2}$$

is entire function in λ and is called as the characteristic function of operator L_α. It is well-known that the roots of $\Delta_\alpha(\lambda)=0$ are coincide with the eigenvalues of operator L_α.

From (2.1) and (2.2), we have easily that

$$\begin{aligned}\Delta_\alpha(\lambda)= \; & -(\lambda-p(\pi))\sin\left(\lambda\tfrac{\pi^\alpha}{\alpha}-\theta(\pi)\right)+A\left(\pi,\tfrac{\pi^\alpha}{\alpha}\right)\cos\lambda\tfrac{\pi^\alpha}{\alpha}+B\left(\pi,\tfrac{\pi^\alpha}{\alpha}\right)\sin\lambda\tfrac{\pi^\alpha}{\alpha}\\ & +\int_0^\pi\left(T_x^\alpha A\left(\pi,\tfrac{t^\alpha}{\alpha}\right)\right)\cos\lambda\tfrac{t^\alpha}{\alpha}d_\alpha t+\int_0^\pi\left(T_x^\alpha B\left(\pi,\tfrac{t^\alpha}{\alpha}\right)\right)\sin\lambda\tfrac{t^\alpha}{\alpha}d_\alpha t.\end{aligned} \tag{2.3}$$

With the help of (2.3), the following theorem for the eigenvalues λ_n of operator L_α can be proved as in [32]:

Theorem 2.2. *The operator L_α has a countable set of eigenvalues $\{\lambda_n\}$ and the following asymptotic formula holds:*

$$\lambda_n=\frac{n\alpha}{\pi^{\alpha-1}}+c_{\alpha,0}+\frac{c_{\alpha,1}}{n}+o\left(\frac{1}{n}\right),\;\; |n|\to\infty, \tag{2.4}$$

where

$$c_{\alpha,0}=\frac{\alpha}{\pi^\alpha}\int_0^\pi p(x)\,d_\alpha x,\;\; c_{\alpha,1}=\frac{1}{2\pi}\int_0^\pi\left(q(x)+p^2(x)\right)d_\alpha x.$$

3. Main Result

In this section, we prove an Ambarzumyan-type theorem for the operator L_α, i.e. we show that $q(x)$ and $p(x)$ functions are zero if the eigenvalues are the same as the eigenvalues of zero potentials.

Theorem 3.1. *If the eigenvalues of the operator L_α are $\lambda_n=\dfrac{n\alpha}{\pi^{\alpha-1}}$, $n\in\mathbb{Z}$, then for each fixed α, $q(x)=0$, $p(x)=0$ a.e. on $(0,\pi)$ and $\theta(\pi)=0$.*

Proof. It follows from (2.4) that for each fixed α, $c_{\alpha,0}=0$, $c_{\alpha,1}=0$, i.e.

$$\int_0^\pi p(x)\,d_\alpha x=0=\theta(\pi)$$

and

$$\int_0^\pi q(x)\,d_\alpha x=-\int_0^\pi p^2(x)\,d_\alpha x. \tag{3.1}$$

Let $y_0(x;\alpha)=y(x,0;\alpha)$ be an eigenfunction corresponding to the eigenvalue $\lambda_0=0$ of the operator L_α. Then we can write

$$\begin{cases}-T_x^\alpha T_x^\alpha y_0(x;\alpha)+q(x)y_0(x;\alpha)=0, \quad 0<x<\pi\\ T_x^\alpha y_0(0;\alpha)=0,\; T_x^\alpha y_0(\pi;\alpha)=0\end{cases}. \tag{3.2}$$

It is clear that $y_0(0;\alpha)\neq 0$ and $y_0(\pi;\alpha)\neq 0$. Otherwise, $y_0(0;\alpha)=T_x^\alpha y_0(0;\alpha)=0$ or $y_0(\pi;\alpha)=T_x^\alpha y_0(\pi;\alpha)=0$. In both cases, we get $y_0(x;\alpha)=0$ through the uniqueness of the solution of an initial value problem, which contradicts the fact that $y_0(x;\alpha)$ is an eigenfunction.

Taking into account the relation

$$\frac{T_x^\alpha T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}=T_x^\alpha\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)+\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)^2,$$

we obtain from (3.2) that

$$T_x^\alpha\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)+\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)^2=q(x).$$

By $\alpha-$integrating of both sides of the above equation from 0 to π

$$\int_0^\pi T_x^\alpha\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)d_\alpha x+\int_0^\pi\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)^2 d_\alpha x=\int_0^\pi q(x)d_\alpha x$$

is obtained. From (3.1) and (3.2), we get

$$\left.\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right|_0^\pi+\int_0^\pi\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)^2 d_\alpha x=-\int_0^\pi p^2(x)\,d_\alpha x$$

or

$$\int_0^\pi\left[\left(\frac{T_x^\alpha y_0(x;\alpha)}{y_0(x;\alpha)}\right)^2+p^2(x)\right]d_\alpha x=0.$$

Thus, for each fixed α, $(T_x^\alpha y_0(x;\alpha))^2+p^2(x)y_0^2(x;\alpha)\equiv 0$, i.e. $p(x)=0$ a.e. on $(0,\pi)$ and $y_0(x;\alpha)\equiv k$, where $0\neq k-const$. Hence, it is concluded from (3.2) that

$$-T_x^\alpha T_x^\alpha k+q(x)k=0$$

then $q(x)=0$ a.e. on $(0,\pi)$. Therefore, the theorem is completed. □

4. Conclusion

As known, the inverse problems of spectral analysis consist in recovering operators from their spectral characteristics, and the first result in this direction belongs to Ambarzumyan for the Sturm-Liouville operator. Until today, many studies have been done on the Ambarzumyan theorem for various operators including classical derivatives. In this paper, the Ambarzumyan theorem is proved for the diffusion operator with Neumann boundary value conditions including fractional derivatives. This study will make an important contribution to the inverse problems of spectral analysis. This theorem can be proved in the future for various operators with different derivatives.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] V.A. Ambarzumian, *Uber eine frage der eigenwerttheorie*, Zeitschrift für Physik, **53** (1929), 690-695.

[2] H.H. Chern, C.K. Law, H.J. Wang, *Extensions of Ambarzumyan's theorem to general boundary conditions*, J. Math. Anal. Appl., **263** (2001), 333-342. Corrigendum: J. Math. Anal. Appl., **309** (2) (2005), 764-768.

[3] E.M. Harrell, *On the extension of Ambarzunyan's inverse spectral theorem to compact symmetric spaces*, Amer. J. Math., **109** (1987), 787-795.

[4] M. Horváth, *On a theorem of Ambarzumyan*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **131** (2001), 899-907.

[5] M. Kiss, *An n-dimensional Ambarzumyan type theorem for Dirac operators*, Inverse Problems, **20** (5) (2004), 1593-1597.

[6] C.F. Yang, X.P. Yang, *Some Ambarzumyan-type theorems for Dirac operators*, Inverse Problems, **25** (9) (2009), 095012, 13 pages.

[7] C.F. Yang, Z.Y. Huang, X.P. Yang, *Ambarzumyan's theorems for vectorial Sturm-Liouville systems with coupled boundary conditions*, Taiwanese J. Math., **14** (4) (2010), 1429-1437.

[8] C.L. Shen, *On some inverse spectral problems related to the Ambarzumyan problem and the dual string of the string equation*, Inverse Problems, **23** (6) (2007), 2417-2436.

[9] N.V. Kuznetsov, *Generalization of a theorem of V.A. Ambarzumyan*, Doklady Akademii Nauk SSSR., **146** (1962), 1259-1262, (in Russian).

[10] H.H. Chern, C.L. Shen, *On the n-dimensional Ambarzumyan's theorem*, Inverse Problems, **13** (1) (1997), 15-18.

[11] C.F. Yang, X.P. Yang, *Ambarzumyan's theorem with eigenparameter in the boundary conditions*, Acta Math. Sci., **31** B(4) (2011), 1561-1568.

[12] V.A. Yurko, *On Ambarzumyan-type theorems*, Appl. Math. Lett., **26** (4) (2013), 506-509.

[13] K. Márton, *An n-dimensional Ambarzumyan type theorem for Dirac operators*, Inverse Problems, **20** (5) (2004), 1593-1597.

[14] A.A. Kırac, *Ambarzumyan's theorem for the quasi-periodic boundary conditions*, Anal. Math. Phys., **6** (2016), 297-300.

[15] H. Koyunbakan, D. Lesnic, E.S. Panakhov, *Ambarzumyan type theorem for a quadratic Sturm-Liouville operator*, Turkish Journal of Sciences and Technology, **8** (1) (2013), 1-5.

[16] G. Freiling, V.A. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publishers, New York, 2001.

[17] E. Yilmaz, H. Koyunbakan, *Some Ambarzumyan type theorems for Bessel operator on a finite interval*, Differ. Equ. Dyn. Syst., **27** (4) (2019), 553–559.

[18] R. Khalil, M. Al Horania, A. Yousefa, et al., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65-70.

[19] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57-66.

[20] A. Atangana, D. Baleanu, A. Alsaedi, *New properties of conformable derivative*, Open Math., **13** (2015), 889-898.

[21] M. Abu Hammad, R. Khalil, *Abel's formula and Wronskian for conformable fractional differential equations*, International J. Differ. Equ. Appl., **13** (3) (2014), 177-183.

[22] O.T. Birgani, S. Chandok, N. Dedovic, S. Radenoviç, *A note on some recent results of the conformable derivative*, Advances in the Theory of Nonlinear Analysis and Its Applications, **3** (1) (2019), 11-17.

[23] D. Zhao, M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, **54** (3) (2017), 903-917.

[24] H.W. Zhou, S. Yang, S.Q. Zhang, *Conformable derivative approach to anomalous diffusion*, Physica A: Statistical Mechanics and its Applications, **491** (2018), 1001-1013.

[25] F. Jarad, E. Uğurlu, T. Abdeljawad, D. Baleanu, *On a new class of fractional operators*, Adv. Difference Equ., **2017** (2017), Article ID 247, 16 pages.

[26] H. Mortazaasl, A. Jodayree Akbarfam, *Trace formula and inverse nodal problem for a conformable fractional Sturm-Liouville problem*, Inverse Probl. Sci. Eng., **28** (4) (2020), 524-555.

[27] B. Keskin, *Inverse problems for one dimensional conformable fractional Dirac type integro differential system*, Inverse Problems, **36** (6) (2020), 065001.

[28] E. Yilmaz, T. Gulsen, E.S. Panakhov, *Existence results for a conformable type Dirac system on time scales in quantum physics*, Appl. Comput. Math., **21** (3) (2022), 279-291.

[29] I. Adalar, A.S. Ozkan, *Inverse problems for a conformable fractional Sturm-Liouville operators*, J. Inverse III Posed Probl., **28** (6) (2020), 775-782.

[30] Y. Çakmak, *Inverse nodal problem for a conformable fractional diffusion operator*, Inverse Probl. Sci. Eng., **29** (9) (2021), 1308-1322.

[31] Y. Çakmak, *Trace formulae for a conformable fractional diffusion operator*, Filomat, **36** (14) (2022), 4665-4674.

[32] E. Koç, Y. Çakmak, *α-integral representation of the solution for a conformable fractional diffusion operator and basic properties of the operator*, Cumhuriyet Science Journal, **44** (1) (2023), 170-180.

[33] B.P. Allahverdiev, H. Tuna, Y. Yalçinkaya, *Conformable fractional Sturm-Liouville equation*, Math. Methods Appl. Sci., **42** (10) (2019), 3508-3526.

[34] D. Baleanu, Z.B. Guvenc, J.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[35] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, London, 2010.

[36] C.A. Monje, Y. Chen, B.M. Vinagre, et al., *Fractional-Order Systems and Controls: Fundamentals and Applications*, Springer, London, 2010.

[37] A. Pálfalvi, *Efficient solution of a vibration equation involving fractional derivatives*, Internat. J. Non-Linear Mech., **45** (2010), 169–175.

[38] M.F. Silva, J.A.T. Machado, *Fractional order PD^{α} joint control of legged robots*, J. Vib. Control, **12** (12) (2006), 1483–1501.