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Some Approaches for Solving Multiplicative Second-Order Linear Differential Equations with Variable Exponentials and Multiplicative Airy's Equation

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Abstract: This paper offers several approaches for solving multiplicative second-order linear differential equations with variable exponentials, such as normalization and reduction to Riccati equations. In addition, in this paper, the multiplicative version of the Airy equation, which emerges in fluid mechanics, geophysics, and atomic physics, is solved using the multiplicative power series solution method.

Key words: Multiplicative calculus, transform, multiplicative Airy's equation.

İkinci Mertebeden Değişken Üslü Çarpımsal Lineer Diferansiyel Denklemlerin ve Çarpımsal Airy Denkleminin Çözümleri için Bazı Yaklaşımlar

Öz: Bu makale, normalleşme ve Riccati denklemlerine indirgenme gibi ikinci mertebeden değişken katsayılı çarpımsal lineer diferansiyel denklemleri çözmek için çeşitli yaklaşımlar sunmaktadır. Ayrıca bu makalede akışkan mekaniği, jeofizik ve atom fiziğinde ortaya çıkan Airy denkleminin çarpımsal versiyonu çarpımsal kuvvet serisi çözüm yöntemi kullanılarak çözülmüştür.

Anahtar kelimeler: Çarpımsal hesap, dönüşüm, çarpımsal Airy denklemi.

1. Introduction

As a substitute for traditional calculus, Grossman and Katz [1-2] invented multiplicative calculus in 1967. Because it differs from the traditional calculus of Newton and Leibniz, this sort of calculus is often referred to as "non-Newtonian calculus". Multiplicative calculus is a valuable addition to standard calculus since it is designed to be similar to how standard calculus is suited to cases involving linear functions and scenarios involving exponential functions. In multiplicative calculus, the functions of addition and subtraction are moved to multiplication and division. There are a lot of benefits to studying the calculation of multiplicative. It enhances the effectiveness of additive computations indirectly. Problems that are challenging to tackle in a traditional situation are solved here with amazing simplicity. Within the confines of specific constraints, multiplicative analysis may specify any attribute in the Newtonian situation.

Natural phenomena frequently change exponentially. Events that act in this way include the populations of nations and the magnitude of earthquakes, to quote Benford [3] as an example. Multiplicative analysis, as opposed to classical analysis, enables a better physical assessment of these types of occurrences. In several disciplines, including finance, economics, biology, and demography, this calculus also yields better findings than typical. Up until the beginning of the 2000s, relatively little research had been done on this analysis. Numerous studies have recently been conducted on it, and the results are of high quality and effectiveness (see [3-17]). Using the fundamental ideas of multiplicative analysis, various investigations on multiplicative ordinary differential equations have been conducted in recent years [18-22].

The Airy's equation, a classical equation in mathematical physics, has recently gained popularity among scientists because it is used to model light deflection and some optics problems. It is possible to see Airy's equation that in several solutions in the fields fluid mechanics, geophysics and atomic physics etc. Because of the need to effectively and necessity to express a physical phenomenon and Airy's equation resulting from the need to express it in comprehensive analytical form, many equations in mathematical physics can be written in the Airy's equation format by making appropriate transforms [23,24]. Given that the Airy's equation is linear, the analytical solution at the origin may be discovered using the power series solution approach. Their use in the approximate solution of differential equations with a simple turning point, the approximate solution of integrals with converging saddle points, and the mathematical modeling of physical processes is becoming more and more common [25-32]. However, this equation has not been examined in the multiplicative analysis.

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2. Preliminaries

This section will provide some fundamental definitions and characteristics of multiplicative calculus theories.

Definition 2.1. [33] Assume that $\varphi: A \subset \mathbb{R} \to \mathbb{R}$ is positive, differentiable in the usual case, and that $\varphi > 0$ for every t. $\varphi^*(t)$ is known as the multiplicative derivative of φ at t if the limit:

$$\varphi^*(t) = \lim_{h \to 0} \left[\frac{\varphi(t+h)}{\varphi(t)}\right]^{\frac{1}{h}}, \tag{2.1}$$

is positive and exists.

Definition 2.2. [33] Assume that $\varphi: A \to \mathbb{R}$ is positive, usual differentiable at t. The following is the relationship between classical and multiplicative derivatives:

$$\varphi^*(t) = e^{(\ln o\, \varphi)'(t)}.$$

Theorem 2.1. [33] Assume that φ, ψ are multiplicative differentiable and that κ is usual differentiable at t. The phrases listed below are offered for multiplicative derivative.

i) $(\vartheta\varphi)^*(t) = \varphi^*(t), \quad \vartheta \in \mathbb{R}^+,$

ii) $(\varphi\psi)^*(t) = \varphi^*(t)\psi^*(t),$

iii) $(\varphi/\psi)^*(t) = \varphi^*(t)/\psi^*(t),$

iv) $(\varphi^\kappa)^*(t) = \varphi^*(t)^{\kappa(t)}\, \varphi(t)^{\kappa'(t)},$

v) $(\varphi + \psi)^*(t) = \varphi^*(t)^{\frac{\varphi(t)}{\varphi(t)+\psi(t)}} \psi^*(t)^{\frac{\psi(t)}{\varphi(t)+\psi(t)}}.$

Definition 2.3. [33] Assume that φ is positive, bounded function on finite interval $[a, b]$. The symbol $\int_a^b \varphi(t)^{dt}$ is then known as multiplicative integral of φ on $[a, b]$. According to this definition, if φ is positive and Riemann integrable on $[a, b]$, then it is multiplicative integrable on $[a, b]$ and

$$\int_a^b \varphi(t)^{dt} = e^{\int_a^b (\ln o\, \varphi)(t) dt}.$$

On the other hand, if φ is Riemann integrable on [a, b], one may demonstrate that:

$$\int_a^b \varphi(t) dt = \ln \int_a^b (e^{\varphi(t)})^{dt}.$$

Theorem 2.2. [33] Assume that φ, ψ are positive, bounded function on $[a, b]$. The multiplicative derivative is provided with the following expressions.

i) $\int_a^b (\varphi(t)^\vartheta)^{dt} = (\int_a^b \varphi(t)^{dt})^\vartheta, \; \vartheta \in \mathbb{R},$

ii) $\int_a^b \varphi(t)\psi(t)^{dt} = \int_a^b \varphi(t)^{dt} \int_a^b \psi(t)^{dt},$

iii) $\int_a^b (\frac{\varphi(t)}{\psi(t)})^{dt} = \frac{\int_a^b \varphi(t)^{dt}}{\int_a^b \psi(t)^{dt}},$

iv) $\int_a^b \varphi(t)^{dt} = \int_a^c \varphi(t)^{dt} \int_c^b \varphi(t)^{dt}$, $c \in [a,b]$.

3. Some Methods for General Solution of Multiplicative Second-Order Linear Differential Equations with Variable Exponentials and Multiplicative Airy's Equation

In this section, firstly, multiplicative second-order differential equations will be considered and solved by using some transforms. Then, using the multiplicative power series method [20, 22] multiplicative Airy's equation will be solved.

The definition of multiplicative linear differential equations is given in [21, 34] by

$$(y^{*(n)}(\chi))^{a_n(\chi)}(y^{*(n-1)}(\chi))^{a_{n-1}(\chi)} \cdots (y^{**}(\chi))^{a_2(\chi)}(y^{*}(\chi))^{a_1(\chi)}(y(\chi))^{a_0(\chi)} = f(\chi), \tag{3.1}$$

where $f(\chi)$ is a positive function. In this section, we consider homogeneous multiplicative linear second-order (for $n = 2$ given in Eq. (3.1)) differential equations with variable exponentials:

$$y^{**}(\chi)(y^{*}(\chi))^{p(\chi)}y(\chi)^{q(\chi)} = 1, \tag{3.2}$$

where $p(\chi), q(\chi) \in C[a,b]$.

3.1. First Transform: Normalization

To eliminate the first-order term from the supplied second order equation, Lanczos [35] initially created this approach. This approach is used with Eq. (3.2). We start by taking into account the following transformation for Eq. (3.2).

$$y = u^{v}, \tag{3.3}$$

$$y^{*} = (u^{*})^{v}u^{v'},$$

$$y^{**} = (u^{**})^{v}(u^{*})^{2v'}u^{v''}.$$

(3.3) and its multiplicative derivatives are substituted into Eq. (3.2) and rearranged to produce:

$$u^{**}(u^{*})^{\frac{2v'}{v}+p}(u)^{\frac{v''}{v}+p\frac{v'}{v}+q} = 1. \tag{3.4}$$

If we take

$$\frac{2v'}{v} + p = 0, \tag{3.5}$$

in this case, we obtain:

$$v = e^{-\frac{1}{2}\int p d\chi}. \tag{3.6}$$

If the first and second order classical derivatives of Eq. (3.6)

$$v' = -\frac{p}{2}e^{-\frac{1}{2}\int p d\chi},$$

$$v'' = \left(-\frac{p'}{2} + \frac{p^2}{4}\right)e^{-\frac{1}{2}\int p d\chi},$$

are written in Eq. (3.4), we obtain:

$$u^{**}u^{r(\chi)} = 1, \tag{3.7}$$

where,

$$r(\chi) = -\frac{p'}{2} - \frac{p^2}{4} + q. \tag{3.8}$$

Eq. (3.7) is in a normal form of multiplicative equation, namely this equation does not contain the first-order term. Especially, for k (k is a constant), Eq. (3.7) transforms into multiplicative second order linear equation with constant exponentials:

$$u^{**}u^{k} = 1. \tag{3.9}$$

Consequently, this method says that if the solution of Eq. (3.7) is known, the solution of Eq. (3.2) is obtained by using the transform (3.3).

Example 3.1. Solve

$$y^{**}(y^{*})^{2cos\chi}y^{-sin\chi - sin^2\chi} = 1. \tag{3.10}$$

Solution Here $p = 2cos\chi,\ q = -sin\chi - sin^2\chi$ for Eq. (3.2) and therefore using (3.8),

$$r(\chi) = -1,$$

and by (3.7), we get

$$u^{**}u^{-1} = 1.$$

The solution of this equation is:

$u = {c_1}^{e^{-\chi}}{c_2}^{e^{\chi}}$, ($c_1, c_2$ are constants).

From Eq. (3.6)

$$v = e^{-sin\chi}.$$

Then the general solution of Eq. (3.10) is

$$y(\chi) = {c_1}^{e^{-\chi - sin\chi}}{c_2}^{e^{\chi - sin\chi}}.$$

Example 3.2. Find general solution of

$$y^{**}(y^{*})^{2\chi^2}y^{2\chi + \chi^4 - 4} = 1. \tag{3.11}$$

Solution $p = 2\chi^2,\ q = 2\chi + \chi^4 - 4$ and then from (3.8)

$$r(\chi) = -4,$$

and Eq. (3.7), we find:

$$u^{**}u^{-4} = 1.$$

This equation's solution is:

$u = {c_1}^{e^{-2\chi}}{c_2}^{e^{2\chi}}$, ($c_1, c_2$ are constants) and thus by (3.6)

$v = e^{-\frac{\chi^3}{3}}$.

Therefore, the general solution to the given equation (3.11) is:

$y(\chi) = c_1{}^{e^{-2\chi-\frac{\chi^3}{3}}} c_2{}^{e^{2\chi-\frac{\chi^3}{3}}}$.

3.2. Second Transform (Reduction of Order)

If a particular solution $y = y_1$ of Eq. (3.2) is known, then let take the second linear independence solution of the form [13]

$$y_2 = y_1{}^{\int u d\chi}. \tag{3.12}$$

If the first and second order multiplicative derivatives of (3.12)

$y_2^* = (y_1^*)^{\int u d\chi} y_1{}^{u}$,

$y_2^{**} = (y_1^{**})^{\int u d\chi} y_1^{*\,2u} y_1{}^{u'}$,

are considered in Eq. (3.2), we get

$$(y_1^{**}(y_1^*)^p y_1{}^q)^{\int u d\chi}((y_1^*)^2 y_1{}^p)^u y_1{}^{u'} = 1. \tag{3.13}$$

By remembering that $y = y_1$ is a particular solution of Eq. (3.2),

$$y_1^{**}(y_1^*)^p y_1{}^q = 1. \tag{3.14}$$

Therefore, if we set Eq. (3.14) in Eq. (3.13), we obtain homogeneous linear multiplicative equation:

$$\frac{u'}{u} = -\left(\frac{2 ln y_1^*}{ln y_1} + p\right). \tag{3.15}$$

The solution of Eq. (3.15) is

$$u = \frac{c}{(ln y_1)^2} e^{-\int p d\chi}, \tag{3.16}$$

where c is integral constant. Consequently, when $c = 1$, the second independent solution of Eq. (3.2) is of the form:

$$y_2 = y_1{}^{\int \frac{1}{(ln y_1)^2} e^{-\int p d\chi} d\chi}. \tag{3.17}$$

Then the general solution of Eq. (3.2) is

$$y(\chi) = y_1{}^{c_1} y_2{}^{c_2} = y_1{}^{\{c_1 + c_2 \int \frac{1}{(ln y_1)^2} e^{-\int p d\chi} d\chi\}}, \tag{3.18}$$

where c_1, c_2 are constants.

Example 3.3. For the given particular solution $y_1 = e^{\chi . sin\chi}$, $\chi > 0$ of

$$y^{**}(y^*)^{-\frac{2}{\chi}} y^{(1+\frac{2}{\chi^2})} = 1, \tag{3.19}$$

find the general solution.

Solution Here $p = -\frac{2}{\chi}$, and therefore using Eq. (3.17), the second independent solution of Eq. (3.19) is

$y_2 = e^{-\chi.cos\chi}$,

and so the general solution of Eq. (2.18) is

$y = e^{(c_1 sin\chi - c_2 cos\chi)\chi}$.

Example 3.4. Solve

$$y^{**}y^{*}y^{e^{-2\chi}} = 1, \quad (3.20)$$

with the knowledge of particular solution $y_1 = e^{\cos(e^{-\chi})}$, $\chi > 0$.

Solution The second solution of Eq. (3.20) regarding to Eq. (3.17)

$y_2 = e^{\sin(-e^{-\chi})}$,

where $p = 1$. So the general solution of Eq. (3.20)

$y = (e^{\cos(e^{-\chi})})^{c_1}(e^{\sin(-e^{-\chi})})^{c_2} = e^{c_1\cos(e^{-\chi}) + c_2\sin(-e^{-\chi})}$.

Example 3.5. Figure out the general solution to

$$y^{**}y^{\frac{-2}{cos^2\chi}} = 1, \quad (3.21)$$

giving the particular solution $y_1 = e^{tan\chi}$, $\chi > 0$.

Solution By (3.17) with $p = 0$

$y_2 = e^{-1-\chi\tan\chi}$,

and then the general solution of (3.21) is

$y = e^{c_1\tan\chi - c_2 - c_2\chi\tan\chi}$.

3.3. Third Transform

By reducting the order of Eq. (3.2), we will obtain the multiplicative Riccati equation. So, let take the transform:

$$y(\chi) = e^{\{\int z(\chi)^{d\chi}\}^{-1}}. \quad (3.22)$$

The multiplicative derivates of this function

$y^{*} = z^{\{\int z^{d\chi}\}^{-1}} = z^{-lny}$,

$y^{**} = \left(\frac{z^{lnz}}{z^{*}}\right)^{\{\int z^{d\chi}\}^{-1}} = \left(\frac{z^{lnz}}{z^{*}}\right)^{lny}$, are written in Eq. (3.2), we can get

$$\left(\frac{z^{lnz}}{z^*}\right)^{lny} z^{-p(\chi)lny} e^{q(\chi)lny} = 1,$$

or

$$z^* z^{p(\chi)} e^{-q(\chi)} = z^{lnz}. \tag{3.23}$$

Eq. (3.23) is a multiplicative Riccati equation. We can write this equation as

$$e^{\frac{z'}{z}} z^{p(\chi)} e^{-q(\chi)} = z^{lnz}. \tag{3.24}$$

If we apply the natural logarithmic on both sides, we get the result:

$$z' + p(\chi)zlnz - q(\chi)z = zln^2z. \tag{3.25}$$

From the transform

$$lnz = t, \tag{3.26}$$

we take classical Riccati equation:

$$t' + p(\chi)t - q(\chi) = t^2. \tag{3.27}$$

The general solution of problem is attained when solving this equation and reversing the transform (3.2). In contrast, taking note of

$$z(\chi) = (y^*)^{-\frac{1}{lny}},$$

it can be take Eq. (3.2).

Example 3.6. Figure out solution to

$$y^{**}(y^*)^{\frac{1}{\chi}} y^{-\frac{1}{\chi^2}} = 1. \tag{3.28}$$

Solution If we apply (3.22) to (3.28), we get multiplicative Riccati equation:

$$z^* z^{\frac{1}{\chi}} e^{-\frac{1}{\chi^2}} = z^{lnz},$$

with the particular solution $z_p = e^{-\frac{1}{\chi}}$. By applying the process for (3.23) - (3.27), we achieve:

$$t' + \frac{1}{\chi}t + \frac{1}{\chi^2} = t^2, \quad t_p = -\frac{1}{\chi}.$$

Its solution is

$$t = \frac{1-2c\chi^2}{\chi+2c\chi^3},$$

and from (3.26) and then (3.22), we obtain:

$$z = e^{\frac{1-2c\chi^2}{\chi+2c\chi^3}},$$

and the general solution as

$$y = e^{\frac{1+2c\chi^2}{\chi}}.$$

3.4. Multiplicative Airy's Equation

Here, multiplicative Airy's equation will be solved using the multiplicative power series method [14, 16].

With the particular choice $p(\chi) = 0, q(\chi) = -\chi$, we derive Airy's equation:

$$Ly = y^{**} y^{-\chi} = 1. \tag{3.29}$$

This equation is equivalent to the non-linear equation:

$$y''y - (y')^2 - \chi y^2 lny = 0,$$

or

$$(y'' - \chi y)y - [(y')^2 - \chi y^2(1 - lny)] = 0,$$

which can be obtained by using the properties of multiplicative calculus. Let's examine the solution of Eq. (3.29) as a multiplicative power series:

$$y(\chi) = \prod_{n=0}^{\infty} c_n{}^{\chi^n}, \tag{3.30}$$

where c_n are positive real constants. Taking second order multiplicative derivative of both sides of the Eq. (3.30),

$$y^{**}(\chi) = \prod_{n=2}^{\infty} c_n{}^{(n-1)n\chi^{n-2}}. \tag{3.31}$$

If Eq. (3.29) is constructed using the values discovered with (3.30) and (3.31), it is obtained:

$$c_2{}^2 \prod_{n=1}^{\infty} (c_{n+2}{}^{(n+1)(n+2)} . c_{n-1}{}^{-1})^{\chi^n} = 1 ,$$

$$c_2 = 1 ,$$

$$c_3{}^{2.3} . c_0{}^{-1} = 1 ,$$

$$c_4{}^{3.4} . c_1{}^{-1} = 1 ,$$

$$\vdots$$

$$c_{n+2}{}^{(n+1).(n+2)} . c_{n-1}{}^{-1} = 1 ,$$

and hence

$$c_{3n-1} = 1 ,$$

$$c_{3n} = c_0{}^{\frac{1}{2.3.5.6...(3n-1)3n}} ,$$

$$c_{3n+1} = c_1{}^{\frac{1}{3.4.6.7...3n(3n+1)}} .$$

Thus, the general solution of Eq. (3.29) is

$$y(\chi) = c_0{}^{\left(1+\sum_{n=1}^{\infty} \frac{\chi^{3n}}{2.3.5.6...(3n-1)3n}\right)} . c_1{}^{\left(\chi+\sum_{n=1}^{\infty} \frac{\chi^{3n+1}}{3.4.6.7...3n(3n+1)}\right)}.$$

4. Conclusion

In this paper, we have applied three transforms to multiplicative second-order linear differential equations in order to derive their general solutions. For each case, we present some examples for a better understanding of the methods. Further, multiplicative Airy's equation has been solved by the multiplicative power series method.

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