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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 153–164 (2024)
DOI:10.31801/cfsuasmas.1282587
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: March 15, 2023; Accepted: October 9, 2023

AFFINE MAPPINGS AND MULTIPLIERS FOR WEIGHTED ORLICZ SPACES OVER THE AFFINE GROUP $\mathbb{R}_+ \times \mathbb{R}$

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Abstract. Let $\mathbb{A} = \mathbb{R}_+ \times \mathbb{R}$ be the affine group with a right Haar measure μ, ω be a weight function on $\mathbb{A}$ and Φ be a Young function. We characterize the affine continuous mappings on the subsets of $L^{\Phi}(\mathbb{A}, \omega)$. Moreover we show that there exists an isometric isomorphism between the multiplier of the pair $(L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$ and the space of bounded measures $M(\mathbb{A})$.

1. Introduction

Orlicz spaces are an important concept in analysis and applications (see [19, 23, 24]). This concept extends the classical concept of L^p Lebesgue spaces for $p \geq 1$. A convex function $\Phi(x)$ is used in place of the function x^p appearing in the definition of L^p spaces. This function Φ is called a Young function. In addition to L^p spaces, several function spaces can be considered as Orlicz spaces; for example $L \log^+ L$ Zygmund spaces, which are Banach spaces related to Hardy-Littlewood maximal functions. Moreover, Sobolev spaces can be also considered as subspaces of Orlicz spaces (see [5]). Most of the features of Orlicz spaces have been investigated thoroughly (see [23], for example), especially, Orlicz spaces determined on measure spaces (see for example [12, 14, 17, 23]). In recent years, Orlicz spaces and their weighted cases are examined as Banach algebras over locally compact groups (lcg). Moreover their several properties are also studied (see [1, 20–22, 27, 28]).

On the other hand one of the basic problems in harmonic analysis is the description of multipliers. Multipliers have been considered in several contexts, for example Banach algebras and Banach modules theories, partial differential equations, the existence of invariant means, etc. Our aim in this paper is to investigate the affine continuous mappings for the weighted Orlicz space $L^{\Phi}(\mathbb{A}, \omega)$ over the affine group $\mathbb{A}$ and study the multiplier problem for $L^{\Phi}(\mathbb{A}) \cap L^1(\mathbb{A})$. The affine

2020 *Mathematics Subject Classification.* 46H25, 43A15, 46E30, 43A22, 43A20.

Keywords. Affine group, affine mapping, multiplier, weighted Orlicz space.

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group chosen is a prime example of a nonabelian group on which harmonic analysis and even more applied time-frequency analysis questions are studied (see [8, 9]).

For L^p spaces, in [16], Lau studied the affine mappings T between the subsets of Lebesgue spaces. In [27], Üster and Öztop studied continuous affine mappings on the subsets of Orlicz spaces. On the other hand the characterization of multipliers for weighted Lebesgue spaces has been given by Gaudry [10]. (See also [7].) In [10], Gaudry showed that the multiplier space of $L^1(G,\omega)$ can be characterized by $M(G,\omega)$. Moreover in [28], Üster characterized the compact mulipliers of $L^\Phi(G,\omega)$. Here G denotes a lcg. (See Section 2 for notation.)

The paper is organized as follows. In Section 2, we recall some basic definitions and notions on Orlicz and weighted Orlicz spaces. In Section 3, we study continuous affine mappings on subsets of weighted Orlicz space $L^\Phi(\mathbb{A},\omega)$ and we give a characterization for the multipliers of $L^\Phi(\mathbb{A})\cap L^1(\mathbb{A})$.

2. Preliminaries

We start this section by introducing some basic facts for an affine group and essential constructions on it.

Let $\mathbb{A} := (\mathbb{R}_+ \times \mathbb{R}, \cdot_{\mathbb{A}})$ be the affine group equipped with the multiplication

$$(s,t)\cdot_{\mathbb{A}}(x,y) = (sx, sy+t), \tag{1}$$

for $(s,t),\ (x,y) \in \mathbb{A}$. Note that $(1,0)\cdot_{\mathbb{A}}(s,t) = (s,t)\cdot_{\mathbb{A}}(1,0) = (s,t)$ and $(s,t)\cdot_{\mathbb{A}}(s^{-1},-s^{-1}t) = (s^{-1},-s^{-1}t)\cdot_{\mathbb{A}}(s,t) = (1,0)$. Thus $\mathbb{A}$, endowed with the multiplication (1), becomes a group and this group is called the affine group.

Since a mapping of the real line can be defined by $F_{s,t} : \mathbb{R} \to \mathbb{R}$ such that

$$F_{s,t}(x) = (s,t)\cdot x = sx+t,\ x \in \mathbb{R}$$

for any $(s,t) \in \mathbb{A}$, the affine group is also called the $sx+t$ group. $F_{s,t}$ is the affine mapping of the real line $\mathbb{R}$ and this operation is coherent with (1).

We can represent the affine group $\mathbb{A}$ in matrix form as

$$\mathbb{A} := \left\{ \begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix} : s>0,\, t\in\mathbb{R} \right\}.$$

The inverse and the identity elements are given by

$$\begin{pmatrix} s^{-1} & -s^{-1}t \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operations of the inversion and multiplication are continuous in the product topology. Thus the affine group $\mathbb{A}$ is a locally compact group and

$$d\nu(x,y) = \frac{dx}{x^2}\,dy$$

$$d\mu(x,y) = \frac{dx}{x}\,dy$$

are the left and right Haar measures, respectively (for more details see [13]). Now since

$$d\nu(x,y) = \frac{dx}{x^2}\,dy = \frac{1}{x}d\mu(x,y),$$

the affine group is not unimodular. The modular function on the affine group is $\Delta(x,y) = x^{-1}$.

Throughout this work we use the right Haar measure $d\mu$ on $\mathbb{A}$.

Let $f : \mathbb{A} \to \mathbb{C}$ and $(s,t) \in \mathbb{A}$. We use $L_{(s,t)}$ for the left translation and $R_{(s,t)}$ for the right translation given by

$$(L_{(s,t)}f)(x,y) := f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) \quad \text{and} \quad (R_{(s,t)}f)(x,y) := f((x,y) \cdot_{\mathbb{A}} (s,t)^{-1}).$$

Next we give some notions regarding Orlicz spaces, weighted Orlicz spaces and Young functions. Our main references are [12] and [23].

Definition 1. *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex,* $\Phi(0) = 0$ *and* $\lim\limits_{t\to\infty} \Phi(t) = +\infty$.

For a Young function Φ, its conjugate function Ψ is given by

$$\Psi(t) = \sup\{st - \Phi(s) : s \geq 0\} \quad (t \geq 0).$$

The pair (Φ,Ψ) of Young functions Φ,Ψ is said to be (Young) conjugate and we have

$$st \leq \Phi(s) + \Psi(t) \quad (\forall s,t \geq 0). \tag{2}$$

In this paper we only consider the real-valued Young functions. Clearly Φ is continuous and $\lim_{t\to\infty} \Phi(t) = \infty$. Note that the continuity of Φ may not imply the continuity of Ψ.

Let us recall the following facts about Orlicz spaces. Let (Φ,Ψ) be conjugate Young functions. Then the Orlicz space $L^{\Phi}(\mathbb{A})$ is defined to be

$$L^{\Phi}(\mathbb{A}) = \{f : \mathbb{A} \to \mathbb{C} : \int\limits_{\mathbb{A}} \Phi(\alpha|f(x,y)|)\,\frac{dx}{x}dy < \infty \text{ for some } \alpha > 0\}.$$

Here f and g in $L^{\Phi}(\mathbb{A})$ are equivalent if $f = g$ a.e. Recall that an Orlicz space is a Banach space with respect to (Orlicz) norm which is defined by

$$\|f\|_{\Phi} = \sup\left\{\int\limits_{\mathbb{A}} |f(x,y)\nu(x,y)|\,\frac{dx}{x}dy : \int\limits_{\mathbb{A}} \Psi(|\nu(x,y)|)\,\frac{dx}{x}dy \leq 1\right\}$$

for $f \in L^{\Phi}(\mathbb{A})$. Here (Φ,Ψ) are conjugate Young functions.

Another norm on an Orlicz space is the Luxemburg norm $N_{\Phi}(f)$ defined by

$$N_{\Phi}(f) = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{\lambda}\right)\frac{dx}{x}dy \leq 1\right\}.$$

Note that the Orlicz and Luxemburg norms are equivalent; that is,

$$N_{\Phi}(\cdot) \leq \|\cdot\|_{\Phi} \leq 2N_{\Phi}(\cdot).$$

We shall use the following definition in the last section. In [4] and [29], the main motivation to use this definition is to estimate the norm of the dilation operator. Here we use a result of Lemma 3.3 given in [29].

Given $\gamma > 0$ one can define

$$N_{\Phi,\gamma}(f) := \inf\{\lambda > 0 : \int_{\mathbb{A}} \Phi\left(\frac{|f(x,y)|}{\lambda}\right)\frac{dx}{x}dy \leq \ \gamma\}.$$

Here $N_{\Phi,1} = N_\Phi$ and these norms are equivalent on $L^\Phi(\mathbb{A})$:

$$\frac{\gamma_1}{\gamma_2}N_{\Phi,\gamma_1}(f) \leq N_{\Phi,\gamma_2}(f) \leq N_{\Phi,\gamma_1}(f)$$

for $0 < \gamma_1 \leq \gamma_2$.

For Orlicz spaces an important notion is the Δ_2-condition. Let us recall the following definition.

Definition 2. *Let $\Phi : [0,\infty) \to [0,\infty]$ be a Young function. Then Φ is said to satisfy Δ_2-condition (globally), if*

$$\Phi(2x) \leq M\Phi(x) \quad (x \geq 0)$$

for some absolute constant $M > 0$.

Note that if $\Phi \in \Delta_2$, then $L^\Phi(\mathbb{A})^* \cong L^\Psi(\mathbb{A})$, here * denotes the dual [23, Corollary 3.4.5]. Moreover if $\Psi \in \Delta_2$, then $L^\Phi(\mathbb{A})$ is a reflexive Banach space (see [14, 23] for more general cases.)

On the other hand, the weighted Orlicz space $L^\Phi(G,\omega)$ is defined by Osançlıol and Öztop in [20] over a lcg G and they consider the Banach algebra structure for $L^\Phi(G,\omega)$.

A weight function ω is a positive, locally integrable function on $\mathbb{A}$. In this paper we assume that ω is continuous (see [25, Section 3.7]). The space $L^\Phi(\mathbb{A},\omega)$ is defined by $\{f : f\omega \in L^\Phi(\mathbb{A})\}$. We also set

$$N_\Phi^\omega(f) = N_\Phi(f\omega) \tag{3}$$

for $f \in L^\Phi(\mathbb{A},\omega)$. Then $N_\Phi^\omega(\cdot)$ defines a norm on $L^\Phi(\mathbb{A},\omega)$ and $L^\Phi(\mathbb{A},\omega)$ is a Banach space with respect to this norm. Moreover, $L^\Psi(\mathbb{A},\omega^{-1})$ is the dual space of $(L^\Phi(\mathbb{A},\omega), N_\Phi^\omega(\cdot))$ if Φ fulfills the Δ_2-condition. Here the duality is given by

$$\langle f,h\rangle = \int_{\mathbb{A}} f(x,y)h(x,y)\frac{dx}{x}dy \quad (f \in L^\Phi(\mathbb{A},\omega),\ h \in L^\Psi(\mathbb{A},\omega^{-1})),$$

where (Φ,Ψ) are conjugate Young functions and the space $L^\Psi(\mathbb{A},\omega^{-1})$ is endowed with the norm $N_\Psi^{\omega^{-1}}(f) = N_\Psi(\frac{f}{\omega})$. So if Φ,Ψ fulfill the Δ_2-condition then $L^\Phi(\mathbb{A},\omega)$ is a reflexive Banach space (for the general case see [20]).

For $\Phi(x) = \frac{x^p}{p}$, $1 < p < \infty$, the conjugate Young function is $\Psi(y) = \frac{y^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^\Phi(\mathbb{A},\omega)$ and its norm are equal to the Lebesgue space $L^p(\mathbb{A},\omega)$

and its norm. For $p=1$ and $\Phi(x)=x$ the conjugate Young function is

$$\Psi(y)=\begin{cases} 0, & 0\le y\le 1\\ \infty, & \text{otherwise}\end{cases}$$

and we have $L^\Phi(\mathbb{A},\omega)=L^1(\mathbb{A},\omega)$. Note that for $p=1$, the Banach algebra $L^1(\mathbb{A},\omega)$ always has a bounded approximate identity.

As usual, $M(\mathbb{A},\omega)$ is the set of all complex bounded regular Borel measures λ on $\mathbb{A}$ with

$$\|\lambda\|_\omega=\int_{\mathbb{A}}\omega(s,t)d\lambda(s,t)<\infty.$$

We denote the space of all continuous functions f on $\mathbb{A}$ vanishing at infinity by $C^0(\mathbb{A},\omega^{-1})$ with the norm $\|f\|_{\infty,\omega^{-1}}=\|\frac{f}{\omega}\|_\infty$. Then $M(\mathbb{A},\omega)$ is realized as $(C^0(\mathbb{A},\omega^{-1}))^*$ by

$$\langle\lambda,f\rangle=\int_{\mathbb{A}}f(x,y)d\lambda(x,y)$$

(for the general case see [11]). If $\lambda\in M(\mathbb{A},\omega)$ and $f\in L^\Phi(\mathbb{A},\omega)$ the convolution of λ and f is defined by

$$(\lambda*f)(x,y)=\int_{\mathbb{A}}f((s,t)^{-1}\cdot_{\mathbb{A}}(x,y))d\lambda(s,t).$$

Moreover if f,g are measurable functions on $\mathbb{A}$ the convolution of f and g is defined by

$$(f*g)(x,y)=\int_{\mathbb{A}}f(s,t)g((s,t)^{-1}\cdot_{\mathbb{A}}(x,y))\frac{ds}{s}dt\quad((x,y)\in\mathbb{A}).$$

For each $(s,t)\in\mathbb{A}$, let $\delta_{(s,t)}(E)=1_E(s,t)$, where 1_E is the characteristic function of $E\subseteq\mathbb{A}$. Then

$$(\delta_{(s,t)}*f)(x,y)=f((s,t)^{-1}\cdot_{\mathbb{A}}(x,y))=L_{(s,t)}f(x,y)\quad((s,t)\in\mathbb{A})$$

where $L_{(s,t)^{-1}}$ is the left translation operator. For a function f on $\mathbb{A}$, we use $\widetilde{f}$ defined by $\widetilde{f}(x,y)=f((x,y)^{-1})$ for each $(x,y)\in\mathbb{A}$.

Throughout the paper we study $L^\Phi(\mathbb{A},\omega)$ with the weight ω and the Δ_2-condition on a Young function Φ.

3. Main Results

In this section we characterize the affine continuous mappings for $L^\Phi(\mathbb{A},\omega)$ over the affine gorup $\mathbb{A}$ and we study the multiplier problem for the space $L^\Phi(\mathbb{A},\omega)\cap L^1(\mathbb{A},\omega)$. Let us first give the following definitions.

Definition 3. *Let $C\subseteq L^\Phi(\mathbb{A},\omega)$. Then C is called left invariant if $L_{(x,y)}f\in C$ for each $f\in C$ and $(x,y)\in\mathbb{A}$.*

Notice that for $f \in L^{\Phi}(\mathbb{A},\omega)$ and $(x,y) \in \mathbb{A}$ we have $L_{(x,y)}f \in L^{\Phi}(\mathbb{A},\omega)$ and $N_{\Phi}^{\omega}(L_{(x,y)}f) \leq \omega(x,y)N_{\Phi}^{\omega}(f)$ (for the general lcgs see [20, Lemma 2.3]).

Definition 4. *Let X and Y be normed spaces and C, D be convex subsets of X and Y respectively. Then a mapping $f : C \to D$ is called affine if*

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$$

for each $x, y \in C$ and $\alpha \in [0,1]$.

For the subset K of $L^{\Phi}(\mathbb{A},\omega)$, we use $\operatorname{co} K$ for the convex hull of K. In addition to the norm topology on $L^{\Phi}(\mathbb{A},\omega)$, we will take the weak topology w and the weak* topology w^* for the pair $(L^{\Phi}(\mathbb{A},\omega), L^{\Phi}(\mathbb{A},\omega)^*)$ where (Φ,Ψ) is a conjugate pair.

Moreover, we make use of the following subsets of $M(\mathbb{A},\omega)$:

(i) $P(\mathbb{A},\omega) = \{\mu \in M(\mathbb{A},\omega) : \|\mu\|_{\omega} = 1 \text{ and } \mu \geq 0\}$,

(ii) $P_1(\mathbb{A},\omega) = \{h \in L^1(\mathbb{A},\omega) : \|h\|_{1,\omega} = 1 \text{ and } h \geq 0\}$,

(iii) $E(\mathbb{A},\omega) = \{\frac{\delta_{(x,y)}}{\omega(x,y)} : (x,y) \in \mathbb{A}\}$.

We omit the proof of the following Lemma which appears in [28] for general locally compact abelian groups. One can get the same result for nonabelian groups in a similar way.

Lemma 1. *We have $P(\mathbb{A},\omega) = \overline{P_1(\mathbb{A},\omega)}^{w^*} = \overline{\operatorname{co} E(\mathbb{A},\omega)}^{w^*}$. Here $\overline{\cdot}^{w^*}$ indicates weak* closure.*

Lemma 2. *The following are true.*

(i) *Let $f \in L^{\Phi}(\mathbb{A},\omega)$. Then the mapping $\mu \mapsto \mu * f$ is continuous from $(M(\mathbb{A},\omega), w^*)$ to $(L^{\Phi}(\mathbb{A},\omega), w)$.*

(ii) *Let $f \in L^1(\mathbb{A},\omega)$. Then the mapping $h \mapsto f * h$ is continuous from $(L^{\Phi}(\mathbb{A},\omega), w)$ to $(L^{\Phi}(\mathbb{A},\omega), w)$.*

Proof. (i) Let $\{\mu_{\alpha}\}_{\alpha} \subseteq M(\mathbb{A},\omega)$ be a net that is weak* convergent to μ, (Φ,Ψ) a conjugate Young pair and $f \in L^{\Phi}(\mathbb{A},\omega)$. Since $L^{\Phi}(\mathbb{A},\omega)$ is $M(\mathbb{A},\omega)$-module the mapping $\mu * f$ is well defined (see [20]). Let $T \in (L^{\Phi}(\mathbb{A},\omega))^*$, so there exists $g \in L^{\Psi}(\mathbb{A},\omega^{-1})$ such that

$$T(f) = \int_{\mathbb{A}} f(x,y)g(x,y)\frac{dx}{x}dy = \langle f, g\rangle.$$

Thus we obtain that

$$\begin{aligned} T(\mu_{\alpha} * f) &= \langle \mu_{\alpha} * f, g\rangle \\ &= \int_{\mathbb{A}} (\mu_{\alpha} * f)(x,y)g(x,y)\,\frac{dx}{x}dy \end{aligned}$$

$$= \int_{\mathbb{A}} \int_{\mathbb{A}} f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y))\, d\mu_\alpha(s,t) g(x,y)\, \frac{dx}{x} dy$$
$$= \int_{\mathbb{A}} \int_{\mathbb{A}} \tilde{f}((x,y)^{-1} \cdot_{\mathbb{A}} (s,t)) g(x,y)\, \frac{dx}{x} dy\, d\mu_\alpha(s,t)$$
$$= \int_{\mathbb{A}} (g * \tilde{f})(s,t)\, d\mu_\alpha(s,t)$$
$$= \langle g * \tilde{f}, \mu_\alpha \rangle.$$

Since $f \in L^\Phi(\mathbb{A}, \omega)$ and $g \in L^\Psi(\mathbb{A}, \omega^{-1})$, we have $g * \tilde{f} \in C_0(\mathbb{A}, \omega^{-1})$ (for the general case see [20].) This implies that $T(\mu_\alpha * f) = \langle g * \tilde{f}, \mu_\alpha \rangle \to \langle g * \tilde{f}, \mu \rangle = \langle \mu * f, g \rangle = T(\mu * f)$, i.e., $\mu_\alpha * f$ weakly converges to $\mu * f$ in $L^\Phi(\mathbb{A}, \omega)$.

(ii) Let $\{h_\alpha\}_\alpha \subseteq L^\Phi(\mathbb{A}, \omega)$ be a net that is weakly convergent to h and $f \in L^1(\mathbb{A}, \omega)$. We have $\lim_\alpha \langle h_\alpha, g \rangle = \langle h, g \rangle$ for all $g \in (L^\Phi(\mathbb{A}, \omega))^*$. Thus we obtain that

$$\langle h_\alpha * f, g \rangle = \int_{\mathbb{A}} (h_\alpha * f)(x,y) g(x,y) \frac{dx}{x} dy$$
$$= \int_{\mathbb{A}} \int_{\mathbb{A}} h_\alpha(s,t) f((s,t)^{-1} \cdot_{\mathbb{A}} (x,y)) g(x,y) \frac{ds}{s} dt \frac{dx}{x} dy$$
$$= \int_{\mathbb{A}} \int_{\mathbb{A}} h_\alpha(s,t) \tilde{f}((x,y)^{-1} \cdot_{\mathbb{A}} (s,t)) g(x,y) \frac{ds}{s} dt \frac{dx}{x} dy$$
$$= \int_{\mathbb{A}} h_\alpha(s,t) (g * \tilde{f})(s,t) \frac{ds}{s} dt$$
$$= \langle h_\alpha, g * \tilde{f} \rangle.$$

This gives that $\langle h_\alpha * f, g \rangle = \langle h_\alpha, g * \tilde{f} \rangle \to \langle h, g * \tilde{f} \rangle = \langle h * f, g \rangle$. $\square$

Theorem 1. *Let C, D be convex, closed, left invariant subsets of $L^{\Phi_2}(\mathbb{A}, \omega)$ and $L^{\Phi_1}(\mathbb{A}, \omega)$ respectively. If $T : C \to D$ is a continuous and affine mapping then the following are equivalent.*

(i) *$T(L_{(x,y)} f) = L_{(x,y)}(Tf)$ for each $(x,y) \in \mathbb{A}$ and $f \in C$.*
(ii) *$T(\nu * f) = \nu * T(f)$ for each $\nu \in P_1(\mathbb{A}, \omega)$ and $f \in C$.*

Proof. (i $\Rightarrow$ ii) Let $f \in C$ and assume that $T(L_{(x,y)} f) = L_{(x,y)}(Tf)$ for each $(x,y) \in \mathbb{A}$ and $\nu \in P_1(\mathbb{A}, \omega)$. Using Lemma 1, there exists a net $\{\nu_\alpha\}_\alpha$ in $\mathrm{co}\, E(\mathbb{A}, \omega)$, $\nu_\alpha = \sum_{i=1}^{n_\alpha} {\lambda_i}^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)}$ and ν_α weak* converges to ν. Then by Lemma 2, $\{\nu_\alpha * f\}_\alpha$

weakly converges to $\nu * f$ for each $f \in C$. Thus we have

$$\nu_\alpha * f = \left(\sum_{i=1}^{n_\alpha} \lambda_i{}^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)}\right) * f = \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} L_{(s_i^\alpha, t_i^\alpha)^{-1}} f.$$

As C is convex and left invariant, the net $\{\nu_\alpha * f\}_\alpha$ is contained in C. Now using Lemma 2 it follows that $\nu * f \in C$.

On the other hand since C and D are convex and closed they are weakly closed. Moreover since T is continuous and affine T is weakly continuous when C and D have their respective weak topologies (see [6, 26]). Then we get that

$$\begin{aligned}
T(\nu * f) &= \lim_\alpha T(\nu_\alpha * f) \\
&= \lim_\alpha T\left(\left(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)}\right) * f\right) \\
&= \lim_\alpha T\left(\sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} \left(\delta_{(s_i^\alpha, t_i^\alpha)} * f\right)\right) \\
&= \lim_\alpha T\left(\sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} (L_{(s_i^\alpha, t_i^\alpha)^{-1}} f)\right) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} T(L_{(s_i^\alpha, t_i^\alpha)^{-1}} f) \\
&= \lim_\alpha \sum_{i=1}^{n_\alpha} \frac{\lambda_i^\alpha}{\omega(s_i^\alpha, t_i^\alpha)} L_{(s_i^\alpha, t_i^\alpha)^{-1}} T(f) \\
&= \lim_\alpha \left(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \frac{\delta_{(s_i^\alpha, t_i^\alpha)}}{\omega(s_i^\alpha, t_i^\alpha)}\right) * T(f) \\
&= \lim_\alpha \nu_\alpha * T(f) \\
&= \nu * T(f).
\end{aligned}$$

(ii $\Rightarrow$ i) Conversely let $(x, y) \in \mathbb{A}$. Using Lemma 1 there exists a net $\{\nu_\alpha\}_\alpha \subseteq P_1(\mathbb{A}, \omega)$ such that $\{\nu_\alpha\}_\alpha$ converges to $\delta_{(x,y)^{-1}}$ in the weak* topology. If $T(\nu * f) = \nu * T(f)$ for each $\nu \in P_1(\mathbb{A}, \omega)$ and $f \in C$ we have that

$$\begin{aligned}
T(L_{(x,y)} f) &= T(\delta_{(x,y)^{-1}} * f) \\
&= \lim_\alpha T(\nu_\alpha * f) \\
&= \lim_\alpha \nu_\alpha * T(f) \\
&= \delta_{(x,y)^{-1}} * T(f) \\
&= L_{(x,y)} T(f).
\end{aligned}$$

This completes the proof. □

Theorem 2. *Let B be a weakly compact, bounded, left invariant, closed subset of $L^{\Phi}(\mathbb{A},\omega)$ and T be a continuous affine mapping from $P_1(\mathbb{A},\omega)$ to B. Then T commutes with all left translations if and only if there exists an $f \in B$ such that $T(g) = g * f$ for each $g \in P_1(\mathbb{A},\omega)$.*

Proof. Let $(x, y) \in \mathbb{A}$ and assume that $T(L_{(x,y)}g) = L_{(x,y)}(Tg)$ for each $g \in P_1(\mathbb{A},\omega)$. Using Theorem 1 we have $T(k * g) = k * T(g)$ for $k, g \in P_1(\mathbb{A},\omega)$. Let $\{u_\alpha\}_\alpha \subseteq P_1(\mathbb{A},\omega)$ be a bounded approximate identity for $L^1(\mathbb{A},\omega)$. Since B is weakly compact and $T(u_\alpha) \in B$ is bounded, there exists $f \in B$ such that $\{T(u_\alpha)\}_\alpha$ converges to f weakly. Thus

$$\begin{aligned} T(g) &= \lim_\alpha T(g * u_\alpha) \\ &= \lim_\alpha g * T(u_\alpha) \\ &= g * f \end{aligned}$$

and the result follows.

For the converse let $(x, y) \in \mathbb{A}$ and assume that $f \in B$ such that $T(g) = g * f$ for all $g \in P_1(\mathbb{A},\omega)$. Then

$$\begin{aligned} L_{(x,y)}T(g) &= L_{(x,y)}(g * f) \\ &= \delta_{(x,y)^{-1}} * (g * f) \\ &= (\delta_{(x,y)^{-1}} * g) * f \\ &= L_{(x,y)}g * f \\ &= T(L_{(x,y)}g) \end{aligned}$$

which gives the required result. □

Now our purpose is to obtain a characterization for the multipliers of $L^{\Phi}(\mathbb{A}) \cap L^1(\mathbb{A})$. We observe that the following result does not work for the weighted case and we give the result for the unweighted case.

We start with the definition of the left multiplier of $L^{\Phi}(\mathbb{A})$.

Definition 5. *Let T be a bounded linear operator from $L^{\Phi_1}(\mathbb{A})$ to $L^{\Phi_2}(\mathbb{A})$. Then T is said to be a left multiplier for $(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$ if $T(L_{(x,y)}f) = L_{(x,y)}(Tf)$ for all $f \in L^{\Phi_2}(\mathbb{A})$ and $(x, y) \in \mathbb{A}$. We write $\mathcal{M}(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$ for the set of left multipliers of $(L^{\Phi_2}(\mathbb{A}), L^{\Phi_1}(\mathbb{A}))$.*

Remark 1. *Observe that the normed space $L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ is a Banach space with the norm*

$$|||f||| = \|f\|_1 + N_\Phi(f)$$

and dense in $L^1(\mathbb{A})$.

The following lemma is important to us for our last result (for the proof see [29, Lemma 3.3].)

Lemma 3. *Let Φ be a Young function satisfying the Δ_2 condition. If $f \in L^{\Phi}(\mathbb{A})$ then* $\lim_{(a,b)\to(+\infty,+\infty)} N_{\Phi}(f + L_{(a,b)}f) = N_{\Phi,\frac{1}{2}}(f)$.

Now we have the tools to give a characterization of the multipliers of $L^{\Phi}(\mathbb{A}) \cap L^1(\mathbb{A})$.

Theorem 3. *Let $T : L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A}) \to L^1(\mathbb{A})$ be a linear mapping. Then the following are equivalent.*

(i) $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$.

(ii) *There exists a unique measure $\mu \in M(\mathbb{A})$ such that $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$.*
Furthermore the correspondence between T and μ defines an isometric isomorphism of $\mathcal{M}(L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$ onto $M(\mathbb{A})$.

Proof. Assume that $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A}), L^1(\mathbb{A}))$. Then for each $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ we obtain that

$$\|Tf\|_1 \leq \|T\|(\|f\|_1 + N_{\Phi}(f)). \tag{4}$$

By Lemma 3 we have $\lim_{(s,t)\to(\infty,\infty)} N_{\Phi}(f + L_{(s,t)}f) = N_{\Phi,\frac{1}{2}}(f)$. Using this fact together with (4) we have that

$$\begin{aligned} 2\|Tf\|_1 &= \lim_{(s,t)\to(\infty,\infty)} \|Tf + L_{(s,t)}Tf\|_1 \\ &= \lim_{(s,t)\to(\infty,\infty)} \|T(f + L_{(s,t)}f)\|_1 \\ &\leq \lim_{(s,t)\to(\infty,\infty)} \|T\|(\|f + L_{(s,t)}f\|_1 + N_{\Phi}(f + L_{(s,t)}f)) \\ &= \|T\|(2\|f\|_1 + N_{\Phi,\frac{1}{2}}(f)) \end{aligned}$$

for each $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$. Therefore we obtain

$$\|Tf\|_1 \leq \|T\|(\|f\|_1 + 2^{-1}N_{\Phi,\frac{1}{2}}(f)).$$

Applying this step n times we obtain

$$\|Tf\|_1 \leq \|T\|(\|f\|_1 + 2^{-n}N_{\Phi,\frac{1}{2}}(f))$$

for $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$. Since $\lim_{n\to\infty} 2^{-n} = 0$ we deduce that $\|Tf\|_1 \leq \|T\|\|f\|_1$.

Thus T defines a linear continuous mapping from $L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ to $L^1(\mathbb{A})$ commuting with left translations. Moreover since $L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ is dense in $L^1(\mathbb{A})$, T determines a unique map $S \in \mathcal{M}(L^1(\mathbb{A}))$ and $\|S\| \leq \|T\|$. Moreover there exists a unique $\mu \in M(\mathbb{A})$ such that $Sf = \mu * f$ for each $f \in L^1(\mathbb{A})$ and $\|\mu\| = \|S\|$ (see [30]). Therefore $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ and $\|\mu\| \leq \|T\|$.

Conversely, if $\mu \in M(\mathbb{A})$ and $Tf = \mu * f$ for each $f \in L^1(\mathbb{A}) \cap L^{\Phi}(\mathbb{A})$ we obtain

$$\|Tf\|_1 = \|\mu * f\|_1 \leq \|\mu\|\|f\|_1 \leq \|\mu\|\, |||f|||.$$

Therefore $T \in \mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ and $\|T\| \leq \|\mu\|$.

This gives to equivalence of (i) and (ii).

It is clear that the correspondence between T and μ defines an isometric isomorphism from $\mathcal{M}(L^1(\mathbb{A}) \cap L^\Phi(\mathbb{A}), L^1(\mathbb{A}))$ onto $M(\mathbb{A})$. $\square$

Declaration of Competing Interests The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgements The author is grateful to an anonymous referee for a careful reading of the manuscript and for helpful comments.

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