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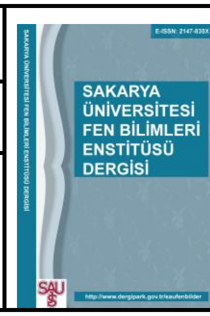

The residual power series method for solving fractional Klein-Gordon equation

Zeliha Körpınar*

ABSTRACT

In this article, the residual power series method (RPSM) for solving fractional Klein-Gordon equations is introduced. Residual power series algorithm gets Maclaurin expansion of the solution. The solutions of our equation are computed in the form of rapidly convergent series with easily calculable components by using mathematica software package. Reliability of the method is given with graphical consequences and series solutions. The found consequences show that the method is a power and efficient method in determination of solution the time fractional Klein-Gordon equations.

Keywords: Residual power series method, Fractional Klein-Gordon equation, Series solution.

Kesirli Klein-Gordon denklemi için residual power seri metodu

ÖZ

Bu makalede kesirli Klein-Gordon denklemlerinin çözümleri için Residual Power Seri metodu (RPSM) uygulanmıştır. Residual Power Seri algoritması çözümün Maclaurin açılımını verir. Bu denklemlerin çözümleri, Mathematica programı kullanılarak kolayca hesaplanan bileşenler ile hızlı yakınsak seriler formunda hesaplanmıştır. Metodun güvenilirliği, seri çözümler ve grafik sonuçlar yardımıyla verilmiştir. Bulunan sonuçlar, kullandığımız metodun kesirli Klein-Gordon denklemlerinin seri çözümlerinin belirlenmesinde güçlü ve etkili bir metot olduğunu göstermektedir.

Anahtar Kelimeler: Residual power seri metodu, Kesirli Klein-Gordon denklemleri, Seri çözüm.

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1. INTRODUCTION

In the last few years, considerable interest in fractional calculus used in many fields, such as regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, viscoelasticity, electrical circuits, electro-analytical chemistry, biology, control theory, etc. [1-4]. Besides there has been a significant theoretical development in fractional differential equations and its applications [5-10]. On the other hand, fractional derivatives supply an important implement for the definition of hereditary characteristics of different necessaries and treatment. This is the fundamental advantage of fractional differential equations in return classical integer-order problems.
In this paper, we apply the RPSM to find series solution for fractional Klein-Gordon equations. The RPSM was developed as an efficient method for fuzzy differential equations [11]. The RPSM is constituted with an repeated algorithm. It has been successfully put into practiced to handle the approximate solution of Lane-Emden equation [12,13], predicting and representing the multiplicity of solutions to boundary value problems of fractional order [14], constructing and predicting the solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations [15], the approximate solution of the nonlinear fractional KdV-Burgers equation [16], the approximate solutions of fractional population diffusion model [17], and the numerical solutions of linear non-homogeneous partial differential equations of fractional order [18].The proposed method is an alternative process for getting analytic Maclaurin series solution of problems.
In this paper, we consider the following the time-fractional Klein-Gordon equations of the form [19,20]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}+au(x,t)+bu^2(x,t)+cu^3(x,t), \tag{1}$$

$$t>0,\ 0<\alpha\le 1.$$

In the second section of this work, some preliminary results related to the Caputo derivative and the fractional power series are described. In Section 3, base opinion of the RPSM is constituted to construct the solution of the time fractional Klein-Gordon equations and some graphical consequens are included to demonstrate the reliability and efficiency of the method. Finally, consequences are introduced in Section 4.

2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS THEORY

We first illustrate the main descriptions and various features of the fractional calculus theory [2] in this section.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha(\alpha\ge 0)$ is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\quad \alpha>0,\ x>0,$$

$$J^0 f(x)=f(x). \tag{2}$$

Definition 2.2. The Caputo fractional derivatives of order α is defined as

$$D^\alpha f(x)=J^{m-\alpha}D^m f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\frac{d^m}{dt^m}f(t)dt, \tag{3}$$

$$m-1<\alpha\le m,\ x>0,$$

where D^m is the classical differential operator of order m.
For the Caputo derivative we have

$$D^\alpha x^\beta=0,\ \beta<\alpha,$$

$$D^\alpha x^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha},\ \beta\ge\alpha.$$

Definition 2.3. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order α of $u(x,t)$ is defined as [13,16],

$$D_t^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n u(x,\tau)}{\partial t^n}d\tau,$$

$$n-1<\alpha<n, \tag{4}$$

$$D_t^n u(x,t)=\frac{\partial^n u(x,t)}{\partial t^n},\quad n\in N,$$

Definition 2.4. A power series (PS) expansion of the form

$$\sum_{m=0}^{\infty}c_m(t-t_0)^{m\alpha}=c_0+c_1(t-t_0)^\alpha+c_2(t-t_0)^{2\alpha}+...,$$

$$0\le m-1<\alpha\le m,\ t\ge t_0,$$

is named fractional PS at $t=t_0$ [13].
Definition 2.5. A PS of the form

$$\sum_{m=0}^{\infty}f_m(x)(t-t_0)^{m\alpha}=f_0(x)+f_1(x)(t-t_0)^\alpha+f_2(x)(t-t_0)^{2\alpha}+...$$

$$0\le m-1<\alpha\le m, t\ge t_0, \tag{5}$$

is named fractional PS at $t=t_0$ [13].

Theorem 2.1.(see [16] for proof.) The fractional PS expansion of $u(x,t)$ at t_0 should be of the form

$$u(x,t)=\sum_{m=0}^{\infty}\frac{D_t^{m\alpha}u(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha},$$

$$0\le m-1<\alpha\le m, x\in I, t_0\le t<t_0+R, \tag{6}$$

which is a Generalized Taylor's series formula. If one set $\alpha = 1$ in Eq. (2.5), then the classical Taylor's series formula

$$u(x,t) = \sum_{m=0}^{\infty} \frac{\partial^m u(x,t_0)}{\partial t^m} \frac{(t-t_0)}{m!}, \; x \in I, t_0 \le t < t_0 + R,$$

is obtained [16].

3 APPLICATIONS FOR RPSM ALGORITHM AND GRAPHICAL RESULTS

Example 1.

Substituting $a = 1, b = 0$ and $c = 0$ into Eq.(1), consider fractional linear Klein-Gordon equation with initial condition:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u, \; t \ge 0, \; 0 < \alpha \le 1, \quad (7) \quad u(x,0) = 1 + \sin(x). \quad (8)$$

The exact solution for (7) for $\alpha = 1$ is [19]

$$u(x,t) = 1 + \sin(x) + \sum_{n=1}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (8)$$

We apply the RPSM to find out series solution for time fractional linear Klein-Gordon equation subject to given initial conditions by replacing its fractional power series expansion with its truncated residual function. From this equation a repetition formula for the calculation of coefficients is supplied, while coefficients in fractional PS expansion can be calculated repeatedly by repeated fractional differentiation of the truncated residual function [13,18].

The RPSM propose the solution for Eqs. (7) and (8) with a fractional power series at $t = 0$ [11]. Suppose that the solution takes the expansion form,

$$u = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \; 0 < \alpha \le 1, x \in I, 0 \le t < R. \quad (9)$$

Next, we let u_k to denote k. truncated series of u,

$$u_k = \sum_{n=0}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \; 0 < \alpha \le 1, x \in I, 0 \le t < R. \quad (10)$$

where $u_0 = f_0(x) = u(x,0) = f(x)$.

In this equations, the function $u(x,t)$ is assumed to be a function of time and space, which means that $u(x,t)$ is disappearing for $t < 0$ and $x < 0$ and this function is considered to be analytic on $t > 0$. Also, the function $f(x)$ is considered to be analytic on $x > 0$.

Also, Eq. (10) can be written as

$$u_k = f(x) + \sum_{n=1}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad (11)$$

$$0 < \alpha \le 1, \; 0 \le t < R, \; x \in I, \; k = \overline{1,\infty}.$$

At first, to find the value of coefficients $f_n(x)$, $n = 1,2,3,...,k$ in series expansion of Eq.(11), we define residual function Res; for Eq.(1) as

$$Res = \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - u$$

and the k-th residual function, Res_k as follows:

$$Res_k = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{\partial^2 u_k}{\partial x^2} - u_k, \; k = 1,2,3,... \quad (12)$$

As in [11-14], To give residual PS algorithm:

Firstly, we replace the k-th truncated series of u into Eq.(7).

Secondly, we find the fractional derivative formula $D_t^{(k-1)\alpha}$ of both $Res_{u,k}$, $k = \overline{1,\infty}$ and finally, we can solve found system $D_t^{(k-1)\alpha} Res_{u,k} = 0, 0 < \alpha \le 1, x \in I, t = 0, k = \overline{1,\infty}.$ (13) to get the required coefficients $f_n(x)$ for $n = \overline{1,k}.$ in Eq. (11).

Hence, to determine $f_1(x)$, we write $k = 1$ in Eq. (12),

$$Res_1 = \frac{\partial^\alpha u_1}{\partial t^\alpha} - \frac{\partial^2 u_1}{\partial x^2} - u_1, \quad (14)$$

where

$$u_1 = \frac{t^\alpha}{\Gamma(1+\alpha)} f_1(x) + f(x)$$

for

$$u_0 = f_0(x) = f(x) = u(x,0) = 1 + \sin(x).$$

Therefore,

$$Res_1 = f_1(x) - f''(x) - \frac{t^\alpha}{\Gamma(1+\alpha)} f_1''(x) - \left(\frac{t^\alpha}{\Gamma(1+\alpha)} f_1(x) + f(x) \right)$$

From Eq. (13) we deduce that $Res_1 = 0 \; (t = 0)$ and thus, $f_1(x) = 1.$ (15)

Therefore, the 1-st RPS approximate solutions are

$$u_1 = \frac{t^\alpha}{\Gamma(1+\alpha)} + 1 + \sin(x). \quad (16)$$

Similarly, to find out the form of the second unknown coefficient $f_2(x)$, we write $k = 2$ in Eq. (12)

$$Res_2 = \frac{\partial^\alpha u_2}{\partial t^\alpha} - \frac{\partial^2 u_2}{\partial x^2} - u_2,$$

where

$$u_2 = f(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2(x)$$

Therefore,

$$Res_2 = f_1(x) + \frac{t^\alpha}{\Gamma(1+\alpha)} f_2(x) - f''(x) - \frac{t^\alpha}{\Gamma(1+\alpha)} f_1''(x)$$

$$-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2^{''}(x)-(f(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_1(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2(x))$$

From Eq. (13) we deduce that $D_t^{\alpha} Res_2 = 0$ $(t=0)$ and thus,

$$f_2(x)=1 \tag{17}$$

Therefore, the 2 -st RPS approximate solutions are

$$u_2=\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+1+\sin(x).$$

Similarly to determine $f_3(x)$, we write $k=3$ in Eq. (12),

$$Res_3=\frac{\partial^{\alpha}u_3}{\partial t^{\alpha}}-\frac{\partial^2 u_3}{\partial x^2}-u_3,$$

where

$$u_3 = f(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_1(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2(x)+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} f_3(x)$$

Therefore,

$$\begin{aligned} Res_3(x,t) = {} & f_1(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_2(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_3(x) \\ & -(f^{''}(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_1^{''}(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2^{''}(x) \\ & +\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} f_3^{''}(x))-(f(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_1(x) \\ & +\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_2(x)+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} f_3(x)) \end{aligned} \tag{18}$$

From Eqs. (13) we deduce that $D_t^{2\alpha} Res_3 = 0$ $(t=0)$ and thus,

$$f_3(x)=\frac{1}{2} \tag{19}$$

Then,

$$u_3=\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{3\alpha}}{2\Gamma(1+3\alpha)}+1+\sin(x). \tag{20}$$

Similarly,

$$f_4(x)=\frac{1}{6},\ f_5(x)=\frac{1}{24}, \tag{21}$$

Therefore,

$$u_5=1+\sin(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} +\frac{t^{3\alpha}}{2\Gamma(1+3\alpha)}+\frac{t^{4\alpha}}{6\Gamma(1+4\alpha)}+\frac{t^{5\alpha}}{24\Gamma(1+5\alpha)}. \tag{22}$$

a)

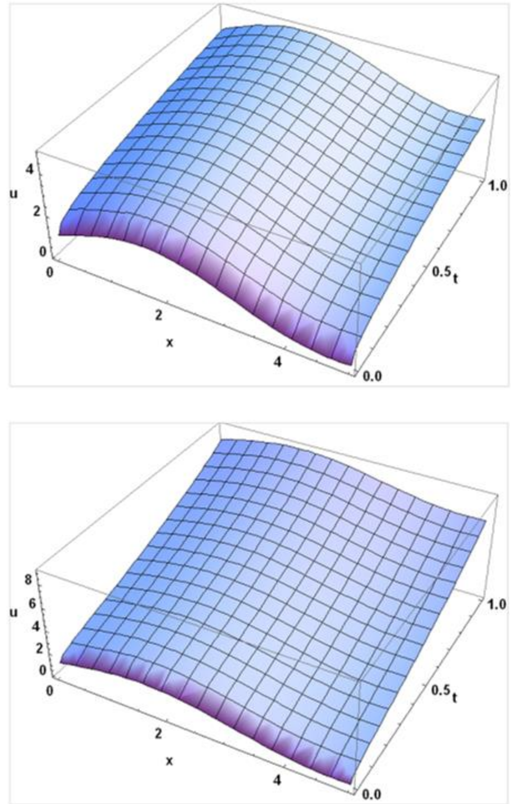


Figure 1.The surface graph of the exact solution $u(x,t)$ and the $u_5(x,t)$ approximate solution of the time fractional linear Klein-Gordon equation ($\alpha=0.3$)(a) $u_5(x,t)$, (b) $u(x,t)$.

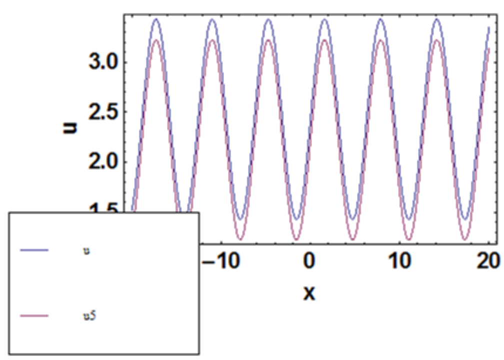


Figure 2. $u_5(x,t)$ and $u(x,t)$ solutions of the time fractional linear Klein-Gordon equation when $\alpha=0.5$, $t=0.4$.

These figure clear that $u_5(x,t)$ solution are closing the exact solution.

Example 2.

Substituting $a=0, b=-1$ and $c=0$ into Eq.(1),consider fractional nonlinear Klein-Gordon differential equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}-\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)^2=0, \tag{23}$$

$t \geq 0, 0 < \alpha \leq 1, x \in R,$

by the initial condition

$u(x,0) = 1 + \sin(x).$ (24)

For equation (23), the k-th residual function, Res_k as follows:

$$Res_k = \frac{\partial^{\alpha} u_k}{\partial t^{\alpha}} - \frac{\partial^2 u_k}{\partial x^2} + u_k^2, k = 1,2,3,... \qquad (25)$$

We apply repeating process as in the former application,

$$f_1(x,t) = -1 - 3\sin(x) - \sin^2(x),$$

$$f_2(x,t) = 2 - 2\cos(2x) + 11\sin(x) + 8\sin^2(x) + 2\sin^3(x),$$

$$f_3(x,t) = \frac{1}{8}(-153 + 244\cos(2x) - 3\cos(4x) - 306\sin(x) + 58\sin(3x)),$$

(26)

$$f_4(x,t) = \frac{1}{12}(657 - 1292\cos(2x) + 83\cos(4x) + 1322\sin(x) - 597\sin(3x) + 3\sin(5x)),$$

$$f_5(x,t) = \frac{1}{96}(-14442 + 31009\cos(2x) - 5990\cos(4x) + 15\cos(6x) - 29616\sin(x) + 22100\sin(3x) - 552\sin(5x)),$$

Therefore,

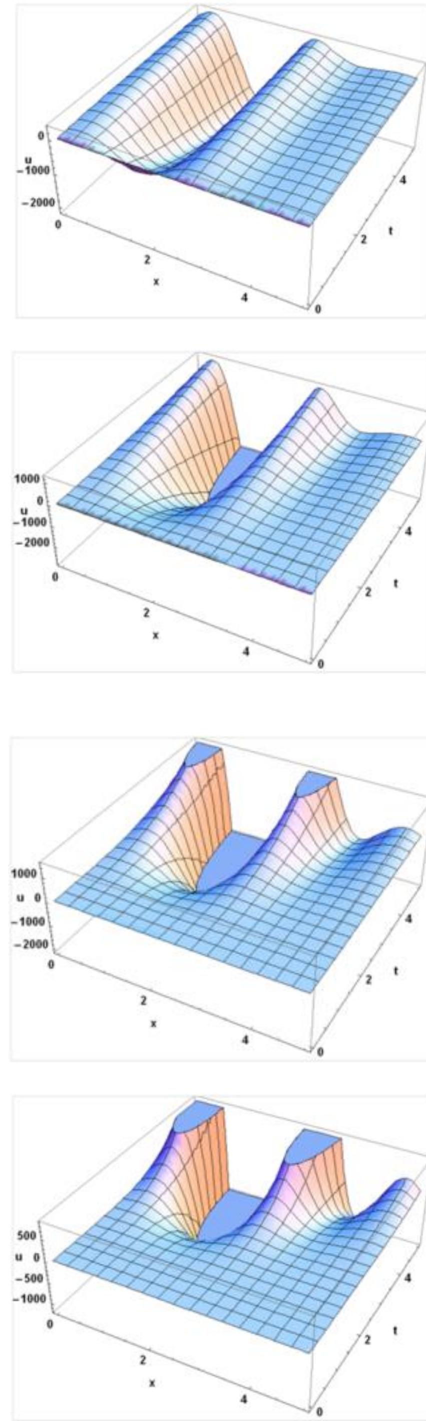
$$u_5 = 1 + \sin(x) + \left(-1 - 3\sin(x) - \sin^2(x)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} \qquad (26)$$

$$+ \left(2 - 2\cos(2x) + 11\sin(x) + 8\sin^2(x) + 2\sin^3(x)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$+ \left(\frac{1}{8}(-153 + 244\cos(2x) - 3\cos(4x) - 306\sin(x) + 58\sin(3x))\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \left(\frac{1}{12}(657 - 1292\cos(2x) + 83\cos(4x) + 1322\sin(x)\right.$$

$$\left. - 597\sin(3x) + 3\sin(5x)\right)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \left(\frac{1}{96}(-14442 + 31009\cos(2x)\right.$$

$$\left. - 5990\cos(4x) + 15\cos(6x) - 29616\sin(x) + 22100\sin(3x) - 552\sin(5x))\right)\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}.$$



a) b) c) d)

Figure 2.The surface graph of $u_5(x,t)$ approximate solution of the two dimensional time fractional nonlinear Klein-Gordon equation(a) $u_5(x,t)$ when $\alpha = 0.1$, (b) $u_5(x,t)$ when $\alpha = 0.3$, (c) $u_5(x,t)$ when $\alpha = 0.6$, (d) $u_5(x,t)$ when $\alpha = 0.9$.

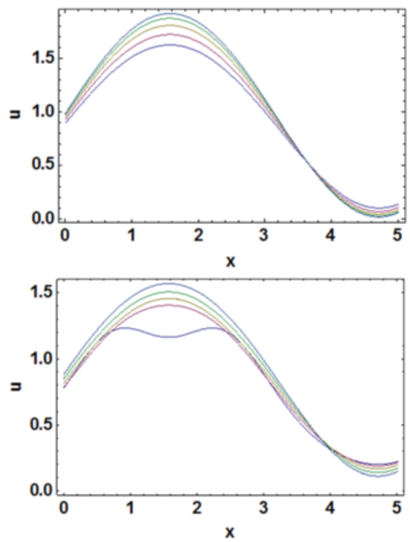


Figure 3. $u_5(x,t)$ solution of the two dimensional time fractional nonlinear Klein-Gordon equation when $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ ($t = 0.01$ and $t = 0.09$).

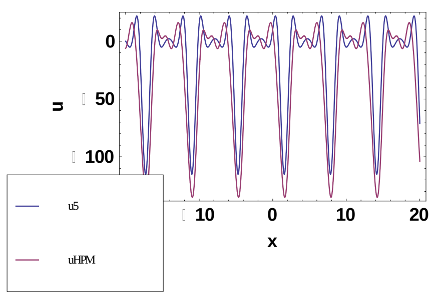


Figure 5. $u_5(x,t)$ and $u_{HPM}(x,t)$ solution of the two dimensional time fractional nonlinear Klein-Gordon equation when $\alpha = 0.5$, $t = 0.8$.

In figure 5, comparison among approximate solutions with known results is made.These results obtained by using residual power series method and homotopy perturbation method [19].

Example 3.

Substituting $a = -1, b = 0$ and $c = 1$ into Eq.(1),consider fractional nonlinear Klein-Gordon differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u(x,t)^3 = 0, \quad (27)$$

$t \geq 0, 0 < \alpha \leq 1, x \in R,$

by the initial condition

$$u(x,0) = -\sec h(x). \quad (28)$$

For equation (23), the k -th residual function, Res_k as follows:

$$Res_k = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{\partial^2 u_k}{\partial x^2} + u_k - u_k^3, k = 1,2,3,... \quad (29)$$

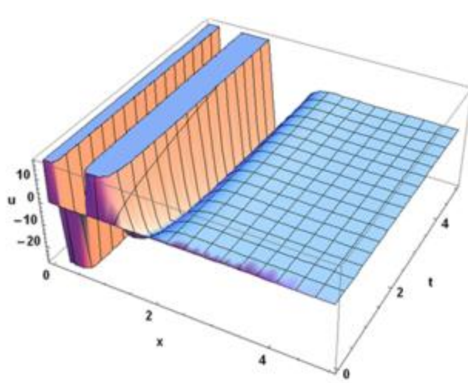
We apply repeating process as in the former application,

$$f_1(x,t) = \sec h(x) - \sec h(x)\tanh^2(x),$$
$$f_2(x,t) = (-5 + 4\cosh(2x))\sec h^5(x),$$
$$f_3(x,t) = \frac{1}{2}(117 - 112\cosh(2x) + 8\cosh(4x))\sec h^7(x),$$
$$f_4(x,t) = \frac{1}{6}(-5537 + 6000\cosh(2x)$$
$$+ 840\cosh(4x) + 16\cosh(6x))\sec h^9(x), \quad (30)$$
$$f_5(x,t) = \frac{1}{24}(436657 - 523208\cosh(2x) + 105320\cosh(4x)$$
$$- 5504\cosh(6x) + 32\cosh(8x))\sec h^{11}(x),$$

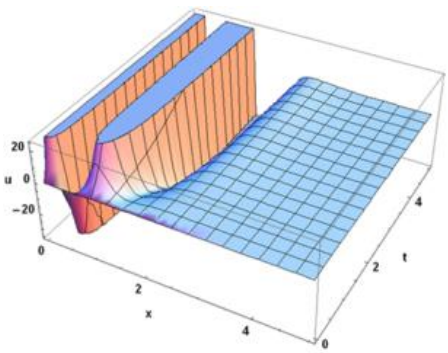
Therefore,

$$u_5 = -\sec h(x) + (\sec h(x) - \sec h(x)\tanh^2(x))\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ (-5 + 4\cosh(2x))\sec h^5(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$+ \frac{1}{2}(117 - 112\cosh(2x) + 8\cosh(4x))\sec h^7(x)$$
$$\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{1}{6}(-5537 + 6000\cosh(2x)$$
$$+ 840\cosh(4x) + 16\cosh(6x)) \quad (31)$$
$$\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{1}{24}(436657 - 523208\cosh(2x)$$
$$+ 105320\cosh(4x) - 5504\cosh(6x)$$
$$+ 32\cosh(8x))\sec h^{11}(x)\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}.$$

a)



b)



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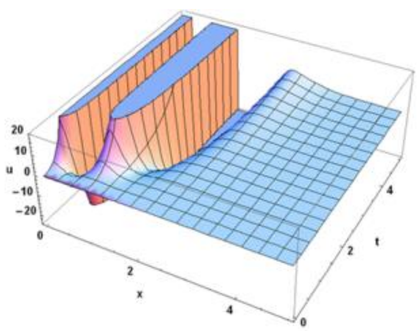

d)

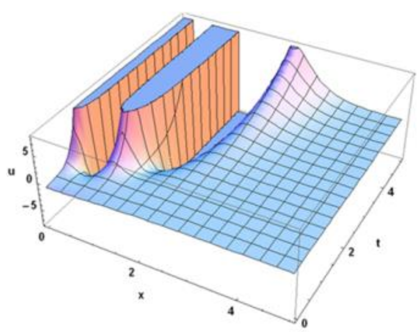

Figure 6. The surface graph of $u_5(x,t)$ approximate solution of the time fractional nonlinear Klein-Gordon equation(a) $u_5(x,t)$ when $\alpha = 0.1$, (b) $u_5(x,t)$ when $\alpha = 0.3$, (c) $u_5(x,t)$ when $\alpha = 0.6$, (d) $u_5(x,t)$ when $\alpha = 0.9$.

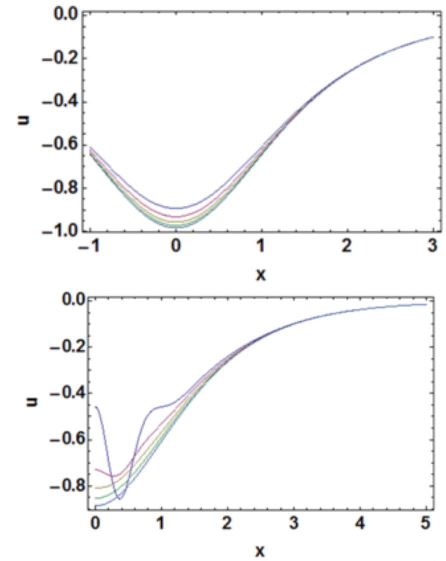

Figure 7. $u_5(x,t)$ solution of the time fractional nonlinear Klein-Gordon equation when $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ ($t = 0.01$ and $t = 0.09$).

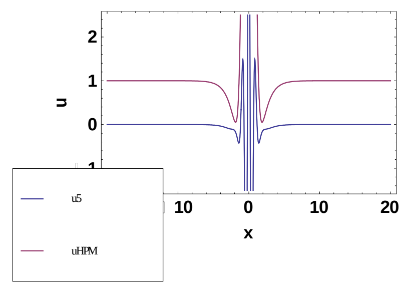

Figure 8. $u_5(x,t)$ and $u_{HPM}(x,t)$ solution of the two dimensional time fractional nonlinear Klein-Gordon equation when $\alpha = 0.8$, $t = 0.8$.

In figure 8, comparison among approximate solutions with known results is made.These results obtained by using residual power series method and homotopy perturbation method [19].

For example 3, we give a part of mathematica.

```
f0[x] = -Sech[x];
u1[x] = f0[x] + f1[x] * t^α/Γ[1 + α]
u1[x] = -Sech[x] + t^α/Γ[1 + α] * f1[x];
Simplify[D[u, t^α/Γ[1 + α]] - D[D[u, x], x] + u - u^3]
-Sech[x] - Sech[x]^3 + Sech[x] Tanh[x]^2 + f1[x] + t^α/Γ[1 + α] f1[x] +
 (Sech[x] - ·t^α/Γ[1 + α] f1[x])^3 - t^α/Γ[1 + α] f1''[x]
t^α/Γ[1 + α] = 0;
Solve[
 -Sech[x] + Sech[x] Tanh[x]^2 + f1[x] + 3 (t^α/Γ[1 + α])^2 Sech[x] f1[x]^2 -
   (t^α/Γ[1 + α])^3 f1[x]^3 + t^α/Γ[1 + α] (f1[x] - 3 Sech[x]^2 f1[x] - f1''[x]) = 0, f1
f1[x] → Sech[x] - Sech[x] Tanh[x]^2
...........
```

4 CONCLUSIONS

In this study the RPSM with new strategies has employed to obtain approximate analytical solution of Klein-Gordon equations.The fundamental objective of this paper to introduce in an algorithmic form and implement a new analytical repeated algorithm derived from on the RPS. This algorithm provides accurate numerical solutions without discretization for nonlinear differential equations.

For example 1, $u_{RPSM}(x,t)$ and $u_{Exact}(x,t)$ solutions of the time fractional linear Klein-Gordon equation compared in figure 2. For example 2, $u_{RPSM}(x,t)$ and $u_{HAM}(x,t)$ solutions of the two dimensional time fractional nonlinear Klein-Gordon equation compared in figure 5 and For example 3, $u_{RPSM}(x,t)$ and $u_{HAM}(x,t)$ solutions of the two dimensional time fractional nonlinear Klein-Gordon equation compared in figure 8. Graphical and numerical consequences are introduced to illustrate the solutions. From the results, it is clear that the RPSM yields very accurate and convergent approximate solutions using only a few iterates in fractional problems. The work emphasized our belief that the present method can be applied as an alternative to get analytic solutions of different kinds of fractional linear and nonlinear partial differential equations applied in mathematics, physics and engineering.

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