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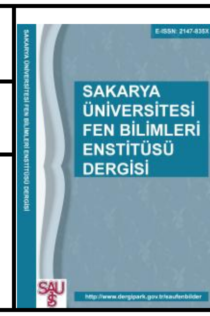

Rotational Hypersurfaces in $\mathbb{S}^3(r)\times\mathbb{R}$ Product Space

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ABSTRACT

We consider rotational hypersurfaces in $\mathbb{S}^3(r)\times\mathbb{R}$ product space of five dimensional Euclidean space $\mathbb{E}^5$. We calculate the mean curvature and the Gaussian curvature, and give some results

Keywords: 5-space, rotational hypersurface, shape operator, Gaussian curvature, mean curvature

$\mathbb{S}^3(r)\times\mathbb{R}$ Çarpım Uzayındaki Dönel Hiperyüzeyler

ÖZ

Beş boyutlu Öklid uzayı $\mathbb{E}^5$ içindeki $\mathbb{S}^3(r)\times\mathbb{R}$ çarpım uzayının dönel hiperyüzeylerini ele aldık. Hiperyüzeylerin ortalama eğriliği ve Gauss eğriliğini hesapladık ve bunların bazı sonuçlarını verdik

Anahtar Kelimeler: 5-boyut, dönel hiperyüzey, şekil operatörü, Gauss eğriliği, ortalama eğrilik

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1. INTRODUCTION (GİRİŞ)

When we focus on the ruled (helicoid) and rotational characters in literature, we see Bour's theorem in [2]. About helicoidal surfaces in Euclidean 3-space, do Carmo and Dajczer [4] prove that, by using a result of Bour [2], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface.

Magid, Scharlach and Vrancken [6] introduce the affine umbilical surfaces in 4-space. Vlachos [12] consider hypersurfaces in E^4 with harmonic mean curvature vector field. Scharlach [11] studies the affine geometry of surfaces and hypersurfaces in 4-space. Cheng and Wan [3] consider complete hypersurfaces of 4-space with constant mean curvature.

General rotational surfaces as a source of examples of surfaces in the four dimensional Euclidean space were introduced by Moore [7, 8]. Ganchev and Milousheva [5] consider the analogue of these surfaces in the Minkowski 4-space. They classify completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points. Arslan et al [1] study on generalized rotation surfaces in E^4. Moruz and Munteanu [9] consider hypersurfaces in the Euclidean space E^4 defined as the sum of a curve and a surface whose mean curvature vanishes. They call them minimal translation hypersurfaces in E^4 and give a classification of these hypersurfaces.

We consider the rotational hypersurfaces in $S^3(r)\times R$ of Euclidean 5-space E^5 in this paper. We give some basic notions of the five dimensional Euclidean geometry in Section 2. In Section 3, we give the definition of a rotational hypersurface in $S^3(r)\times R$ of E^5. Then we calculate the mean curvature and the Gaussian curvature of the rotational hypersurface.

2. PRELIMINARIES (ÖN HAZIRLIK)

In the next representations and definitions we inspire the three dimensional Euclidean space and the book of O'Neill [10], and then extend it to the dimension five.

In this section, we will introduce the first and second fundamental forms, matrix of the shape operator $\mathbf{S}$, Gaussian curvature K and the mean curvature H of hypersurface $\mathbf{M}=\mathbf{M}(r,\theta_1,\theta_2,\theta_3)$ in Euclidean 5-space E^5. In the rest of this work, we shall identify a vector $\vec{\alpha}$ with its transpose.

Definition 1. Let $\mathbf{M}=\mathbf{M}(r,\theta_1,\theta_2,\theta_3)$ be an isometric immersion of a hypersurface M^4 in the E^5. The *vector product* of

$$\vec{x}=(x_1,x_2,x_3,x_4,x_5),\ \vec{y}=(y_1,y_2,y_3,y_4,y_5),$$
$$\vec{z}=(z_1,z_2,z_3,z_4,z_5),\ \vec{w}=(w_1,w_2,w_3,w_4,w_5)$$

on E^5 is defined as follows:

$$\vec{x}\times\vec{y}\times\vec{z}\times\vec{w}=\det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5\\ x_1 & x_2 & x_3 & x_4 & x_5\\ y_1 & y_2 & y_3 & y_4 & y_5\\ z_1 & z_2 & z_3 & z_4 & z_5\\ w_1 & w_2 & w_3 & w_4 & w_5\end{pmatrix}. \quad (1)$$

Definition 2. For a hypersurface $\mathbf{M}(r,\theta_1,\theta_2,\theta_3)$ in 5-space, the first fundamental form matrix (g_{ij}) of $\mathbf{M}$ is as follows

$$\mathrm{I}=\begin{pmatrix} E & F & A & D\\ F & G & B & J\\ A & B & C & Q\\ D & J & Q & S\end{pmatrix}, \quad (2)$$

and then

$$\begin{aligned}\det \mathrm{I} = &\left(EG-F^2\right)CS+B^2D^2+A^2J^2-A^2GS+F^2Q^2\\ &-CGD^2-CJ^2E-B^2SE-GQ^2E-2ABJD+2CFJD\\ &-2BFQD+2AGQD+2BJQE+2ABFS-2AFJQ,\end{aligned} \quad (3)$$

and the second fundamental form matrix (h_{ij}) of $\mathbf{M}$ is as follows

$$\mathrm{II}=\begin{pmatrix} L & M & P & X\\ M & N & T & Y\\ P & T & V & Z\\ X & Y & Z & I\end{pmatrix} \quad (4)$$

and then

$$\begin{aligned}\det \mathrm{II} = &\left(LN-M^2\right)VI+2IMPT-2MPYZ-2PTXY+2NPXZ\\ &-2MTXZ+2VMXY+2LTYZ-NIP^2+T^2X^2-LIT^2\\ &-NVX^2-LVY^2-LNZ^2-M^2Z^2+P^2Y^2,\end{aligned} \quad (5)$$

where

$$A=\mathbf{M}_r\cdot\mathbf{M}_{\theta_2},\ B=\mathbf{M}_{\theta_1}\cdot\mathbf{M}_{\theta_2},\ C=\mathbf{M}_{\theta_2}\cdot\mathbf{M}_{\theta_2},$$
$$D=\mathbf{M}_r\cdot\mathbf{M}_{\theta_3},\ J=\mathbf{M}_{\theta_1}\cdot\mathbf{M}_{\theta_3},\ Q=\mathbf{M}_{\theta_2}\cdot\mathbf{M}_{\theta_3},$$
$$S=\mathbf{M}_{\theta_3}\cdot\mathbf{M}_{\theta_3},\ P=\mathbf{M}_{r\theta_2}\cdot e,\ T=\mathbf{M}_{\theta_1\theta_2}\cdot e,$$
$$V=\mathbf{M}_{\theta_2\theta_2}\cdot e,\ Z=\mathbf{M}_{\theta_2\theta_3}\cdot e,\ X=\mathbf{M}_{r\theta_3}\cdot e,$$
$$Y=\mathbf{M}_{\theta_1\theta_3}\cdot e,\ Z=\mathbf{M}_{\theta_2\theta_3}\cdot e,\ I=\mathbf{M}_{\theta_3\theta_3}\cdot e,$$

$$e=\frac{\mathbf{M}_r\times\mathbf{M}_{\theta_1}\times\mathbf{M}_{\theta_2}\times\mathbf{M}_{\theta_3}}{\left\|\mathbf{M}_r\times\mathbf{M}_{\theta_1}\times\mathbf{M}_{\theta_2}\times\mathbf{M}_{\theta_3}\right\|} \tag{6}$$

is the *Gauss map* (i.e. the unit normal vector), "$\cdot$" means dot product, and some partial differentials that we represent are $\mathbf{M}_r=\frac{\partial\mathbf{M}}{\partial r}$, $\mathbf{M}_{\theta_1\theta_3}=\frac{\partial\mathbf{M}}{\partial\theta_1\partial\theta_3}$.

Definition 3. Following product matrices:

$$\begin{pmatrix} E & F & A & D\\ F & G & B & J\\ A & B & C & Q\\ D & J & Q & S\end{pmatrix}^{-1}\begin{pmatrix} L & M & P & X\\ M & N & T & Y\\ P & T & V & Z\\ X & Y & Z & I\end{pmatrix},$$

gives *the matrix of the shape operator* $\mathbf{S}$ as follows:

$$\mathbf{S}=\frac{1}{\det \mathrm{I}}\begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14}\\ s_{21} & s_{22} & s_{23} & s_{24}\\ s_{31} & s_{32} & s_{33} & s_{34}\\ s_{41} & s_{42} & s_{43} & s_{44}\end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} s_{11}= {} & AJ^2P-CJ^2L-B^2LS+B^2XD+CJMD-BJPD-BMQD\\ & -CGXD+GPQD+ABMS-ABJX-AJMQ+BJLQ\\ & +BJLQ-CFMS+CGLS-AGPS+BFPS+CFJX\\ & +AGQX-BFQX-FJPQ+FMQ^2-GLQ^2, \end{aligned}$$

$$\begin{aligned} s_{12}= {} & AJ^2T-CJ^2M-B^2MS+B^2YD+CJND-BJTD-BNQD\\ & -CGYD+GQTD+ABNS-ABJY-AJNQ+BJMQ\\ & +BJMQ-CFNS+CGMS+CFJY-AGST+BFST\\ & +AGQY-BFQY+FNQ^2-GMQ^2-FJQT, \end{aligned}$$

$$\begin{aligned} s_{13}= {} & AJ^2V-CJ^2P-B^2PS+B^2ZD+CJTD-BJVD-CGZD\\ & -BQTD+GQVD-ABJZ+ABST+BJQP+BJPQ\\ & +CFJZ+CGPS-AJQT-CFST+AGQZ-AGSV\\ & -BFQZ+BFSV-FJQV-GPQ^2+FQ^2T, \end{aligned}$$

$$\begin{aligned} s_{14}= {} & AJ^2Z-CJ^2X-B^2SX+B^2DI-BJZD+CJYD-BQYD\\ & +GQZD-ABJI+CFJI+AGQI-BFQI-CGDI\\ & +ABSY-AJQY+BJQX-AGSZ+BFSZ+BJQX\\ & -CFSY+CGSX-FJQZ+FOQY-GQ^2X, \end{aligned}$$

$$\begin{aligned} s_{21}= {} & -A^2MS+A^2JX-CMD^2+BPD^2+CJLD-ABXD\\ & -AJPD+AMQD-BLQD+AMQD+CFXD+CMSE\\ & -BPSE-CJXE-FPQD+BQXE+JPQE-MQ^2E\\ & +ABLS-AJLQ-CFLS+AFPS-AFQX+FLQ^2, \end{aligned}$$

$$\begin{aligned} s_{22}= {} & A^2JY-A^2NS-CND^2+BTD^2+CJMD-ABYD\\ & +ANQD-BMQD-AJTD+ANQD+CFYD+CNSE\\ & -CJYE-BSTE+BQYE-FQTD-NQ^2E+JQTE\\ & +ABMS-AJMQ-CFMS+AFST-AFQY+FMQ^2, \end{aligned}$$

$$\begin{aligned} s_{23}= {} & A^2JZ-A^2ST-CTD^2+BVD^2-ABZD+CJPD\\ & -AJVD-BQPD+CFZD+AQTD+AQTD-CJZE\\ & +CSTE+BQZE-BSVE-FQVD+JQVE-OQTE\\ & +ABPS-AJPQ-CFPS-AFQZ+AFSV+FPQ^2, \end{aligned}$$

$$\begin{aligned} s_{24}= {} & -A^2SY+A^2JI-CYD^2+BZD^2-AJZD+CJXD\\ & +AQYD-BQXD+AQYD-BSZE+CSYE-FQZD\\ & +JQZE-Q^2YE-AFQI-ABDI+CFDI-CJEI\\ & +BQEI+ABSX+AFSZ-AJQX-CFSX+FQ^2X, \end{aligned}$$

$$\begin{aligned} s_{31}= {} & AJ^2L-F^2PS+F^2QX+BMD^2-GPD^2-J^2PE-AJMD\\ & -BJLD+AGXD-BFXD+2FJPD-FMQD+GLQD\\ & -BMSE+BJXE+JMQE+GPSE-GQXE+AFMS\\ & -AGLS+BFLS-AFJX-FJLQ, \end{aligned}$$

$$\begin{aligned} s_{32}= {} & AJ^2M-F^2ST+F^2QY+BND^2-GTD^2-J^2TE-AJND\\ & -BJMD+AGYD-BFYD-FNQD+GMQD-BNSE\\ & +2FJTD+BJYE+JNQE+GSTE-GQYE+AFNS\\ & -AGMS+BFMS-AFJY-FJMQ, \end{aligned}$$

$$\begin{aligned} s_{33}= {} & AJ^2P+F^2OZ-F^2SV+BTD^2-GVD^2-J^2VE-BJPD\\ & -AJTD+AGZD-BFZD+2FJVD+GQPD+BJZE\\ & -FQTD-BSTE+JQTE-GQZE+GSVE-AFJZ\\ & -AGPS+BFPS+AFST-FJQP, \end{aligned}$$

$$\begin{aligned} s_{34}= {} & AJ^2X-F^2SZ+BYD^2-GZD^2-J^2ZE+F^2QI-AJYD\\ & -BJXD+2FJZD-FQYD+GQXD-BSYE+JQYE\\ & +GSZE-AFJI+AGDI-BFDI+BJEI-GQEI\\ & +AFSY-AGSX+BFSX-FJQX, \end{aligned}$$

$$\begin{aligned} s_{41}= {} & A^2JM-A^2GX-CF^2X+F^2PQ+B^2LD-B^2XE-ABMD\\ & +CFMD-CGLD+AGPD-BFPD-CJME+BJPE\\ & +BMQE+CGXE-GPQE-ABJL+CFJL+2ABFX\\ & -AFJP-AFMQ+AGLQ-BFLQ, \end{aligned}$$

$$
\begin{aligned}
s_{42} = {} & A^2JN - A^2GY - CF^2Y + F^2QT + B^2MD - B^2YE - ABND \\
& +CFND - CGMD - CJNE + AGTD - BFTD + BJTE \\
& +BNQE + CGYE - GQTE - ABJM + CFJM + 2ABFY \\
& -AFJT - AFNQ + AGMQ - BFMQ, \\
s_{43} = {} & A^2JT - A^2GZ - CF^2Z + F^2QV + B^2PD - B^2ZE - ABTD \\
& -CGPD + CFTD + AGVD - BFVD - CJTE + BJVE \\
& +CGZE + BQTE - GQVE - ABJP + 2ABFZ + CFJP \\
& -AFJV + AGPQ - BFPQ - AFQT, \\
s_{44} = {} & A^2JY + F^2QZ + B^2XD - A^2GI - CF^2I - B^2EI - ABYD \\
& +AGZD - BFZD + CFYD - CGXD + BJZE - CJYE \\
& +BQYE - GQZE + 2ABFI + CGEI - ABJX - AFJZ \\
& +CFJX - AFQY + AGQX - BFQX.
\end{aligned}
$$

Definition 4. The formulas of the Gaussian and the mean curvatures are, respectively, as follow:

$$K = \det(\mathbf{S}) = \frac{\det \mathrm{II}}{\det \mathrm{I}}, \tag{8}$$

and

$$H = \frac{1}{4} tr(\mathbf{S}), \tag{9}$$

where

$$tr(\mathbf{S}) = \frac{\Omega}{\det \mathrm{I}},$$

$$
\begin{aligned}
\Omega = {} & 2AJ^2P - CJ^2L - B^2LS - A^2NS + 2A^2JY + F^2OZ - F^2SV + F^2QZ \\
& +2BTD^2 + 2B^2XD - GVD^2 - J^2VE - A^2GI - CF^2I - B^2EI + 2CJMD \\
& -2ABYD - 2BJPD + ANQD - BMQD - 2AJTD + ANQD - BMQD \\
& +2AGZD - 2BFZD + 2CFYD - 2CGXD + CNSE + 2FJVD + GQPD \\
& +2BJZE - 2CJYE + GPQD - FQTD - 2BSTE + BQYE - FQTD - CND^2 \\
& +BQYE + JQTE - NQ^2E + JQTE - GQZE + GSVE - GQZE + 2ABFI \\
& +CGEI + 2ABMS - 2ABJX - AJMQ + BJLQ - AJMQ + BJLQ - 2CFMS \\
& +CGLS - 2AFJZ - 2AGPS + 2BFPS + 2CFJX + 2AFST - AFQY - GLQ^2 \\
& +AGQX - BFQX - FJQP - AFQY + AGQX - BFQX - FJPQ + 2FMQ^2.
\end{aligned}
$$

A hypersurface $\mathbf{M}$ is minimal if $H = 0$ identically on $\mathbf{M}$.

3. ROTATIONAL HYPERSURFACES (DÖNEL HİPERYÜZEYLER)

We define the rotational hypersurface in $\mathbb{S}^3(r)\times\mathbb{R}$ product space of $\mathbb{E}^5$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \to \Pi$ be a curve in a plane Π in $\mathbb{E}^5$, and let ℓ be a straight line in Π.

Definition 5. A *rotational hypersurface* in $\mathbb{S}^3(r)\times\mathbb{R}$ of $\mathbb{E}^5$ is hypersurface created by rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively).

We may suppose that ℓ is the line spanned by the vector $(0,0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$
\mathbf{Z}(\theta_1,\theta_2,\theta_3) =
\begin{pmatrix}
\cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1 & -\cos\theta_1\sin\theta_2 & -\cos\theta_1\cos\theta_2\sin\theta_3 & 0 \\
\sin\theta_1\cos\theta_2\cos\theta_3 & \cos\theta_1 & -\sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2\sin\theta_3 & 0 \\
\sin\theta_2\cos\theta_3 & 0 & \cos\theta_2 & -\sin\theta_2\sin\theta_3 & 0 \\
\sin\theta_3 & 0 & 0 & \cos\theta_3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

where $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$. The matrix $\mathbf{Z}$ can be found by solving the following equations simultaneously;

$$\mathbf{Z}\ell = \ell, \quad \mathbf{Z}^t\mathbf{Z} = \mathbf{Z}\mathbf{Z}^t = I_5, \quad \det\mathbf{Z} = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_5-axis of $\mathbb{E}^5$. Parametrization of the profile curve is given by

$$\gamma(r) = (r, 0, 0, 0, \varphi(r)),$$

where $\varphi(r) : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $r \in I$. So, the rotational hypersurface which is spanned by the vector $(0,0,0,0,1)$, is as follows:

$$\mathbf{R}(r,\theta_1,\theta_2,\theta_3) = \mathbf{Z}(\theta_1,\theta_2,\theta_3)\gamma(r)^t$$

in $\mathbb{E}^5$, where $r \in I$, $\theta_1, \theta_2, \theta_3 \in [0, 2\pi]$. Then we see the rotational hypersurface as follows:

$$
\mathbf{R}(r,\theta_1,\theta_2,\theta_3) =
\begin{pmatrix}
r\cos\theta_1\cos\theta_2\cos\theta_3 \\
r\sin\theta_1\cos\theta_2\cos\theta_3 \\
r\sin\theta_2\cos\theta_3 \\
r\sin\theta_3 \\
\varphi(r)
\end{pmatrix}
=
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5
\end{pmatrix}. \tag{10}
$$

Here, we see the sphere with radius $r : \sum_{i=1}^{4} x_i^2 = r^2$ in $\mathbb{S}^3(r)\times\mathbb{R}$ product space of $\mathbb{E}^5$. When $\theta_2 = \theta_3 = 0$, we have rotational surface of $\mathbb{E}^3$.

Next, we obtain the mean curvature and the Gaussian curvature of the rotational hypersurface in (10).

Theorem 1. *The Gaussian curvature and the mean curvature of the rotational hypersurface in* (10) *are as follow, respectively,*

$$K=\frac{\varphi'^3\varphi''}{r^3(1+\varphi'^2)^3},$$

and

$$H=\frac{r\varphi''+3\varphi'^3+3\varphi'}{4r\left(1+\varphi'^2\right)^{3/2}}.$$

where $r\in\mathbb{R}-\{0\}$, $0\le\theta_1,\theta_2,\theta_3\le 2\pi$ *and* $\varphi(r):I\subset\mathbb{R}\to\mathbb{R}$ *is a differentiable function for all* $r\in I$.

Proof. Using the first differentials of (10) with respect to $r,\theta_1,\theta_2,\theta_3,$ we get the first quantities in (2) as follow:

$$\mathrm{I}=\begin{pmatrix}1+\varphi'^2 & 0 & 0 & 0\\ 0 & r^2\cos^2\theta_2\cos^2\theta_3 & 0 & 0\\ 0 & 0 & r^2\cos^2\theta_3 & 0\\ 0 & 0 & 0 & r^2\end{pmatrix}.$$

We have

$$\det\mathrm{I}=r^6(1+\varphi'^2)\cos^2\theta_2\cos^4\theta_3,$$

where $\varphi=\varphi(r),$ $\varphi'=\dfrac{d\varphi}{dr}.$ Using the second differentials with respect to $r,\theta_1,\theta_2,\theta_3,$ we have the second quantities in (4) as follow:

$$\mathrm{II}=\frac{1}{\sqrt{\det I}}\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&b&0\\0&0&0&c\end{pmatrix},$$

where

$$a=r^3\varphi''\cos\theta_2\cos^2\theta_3,$$
$$b=r^4\varphi'\cos\theta_2\cos^4\theta_3,$$
$$c=r^4\varphi'\cos\theta_2\cos^2\theta_3$$

and

$$\det\mathrm{II}=\frac{r^3\varphi'^3\varphi''\cos^2\theta_2\cos^4\theta_3}{(1+\varphi'^2)^2}.$$

The Gauss map of the rotational hypersurface (10), using (6), is

$$e_{\mathbf{R}}=\frac{1}{\sqrt{\mathbf{W}}}\begin{pmatrix}-\cos\theta_1\cos\theta_2\cos\theta_3\\ -\sin\theta_1\cos\theta_2\cos\theta_3\\ -\sin\theta_2\cos\theta_3\\ -\sin\theta_3\\ 1\end{pmatrix},$$

where $\mathbf{W}=\sqrt{1+\varphi'^2}.$

Using (7), we get the matrix of the shape operator of the rotational hypersurface (10) as follows:

$$\mathbf{S}=\begin{pmatrix}\frac{\varphi''}{(1+\varphi'^2)^{3/2}}&0&0&0\\0&\frac{\varphi'}{r(1+\varphi'^2)^{1/2}}&0&0\\0&0&\frac{\varphi'}{r(1+\varphi'^2)^{1/2}}&0\\0&0&0&\frac{\varphi'}{r(1+\varphi'^2)^{1/2}}\end{pmatrix}.$$

Finally, using (8) and (9), respectively, we calculate the Gaussian curvature and the mean curvature of the rotational hypersurface (10) as follow:

$$K=\det(\mathbf{S})=\frac{\det\mathrm{II}}{\det\mathrm{I}}=\frac{\varphi'^3\varphi''}{r^3(1+\varphi'^2)^3},$$

and

$$H=\frac{1}{4}tr(\mathbf{S})=\frac{r\varphi''+3\varphi'^3+3\varphi'}{4r\left(1+\varphi'^2\right)^{3/2}}.$$

Corollary 1. *Let* $\mathbf{R}:M^4\longrightarrow\mathbb{E}^5$ *be an isometric immersion given by* (10). *Then* M^4 *has constant Gaussian curvature if and only if*

$$\varphi'^3\varphi''-Cr^3(1+\varphi'^2)^3=0.$$

Corollary 2. *Let* $\mathbf{R}:M^4\longrightarrow\mathbb{E}^5$ *be an isometric immersion given by* (10) *. Then* M^4 *has constant mean curvature (CMC) if and only if*

$$\left(r\varphi''+3\varphi'^3+3\varphi'\right)^2-16Cr^2\left(1+\varphi'^2\right)^3=0.$$

Corollary 3. *Let* $\mathbf{R}: M^4 \longrightarrow \mathbb{E}^5$ *be an isometric immersion given by* (10)*. Then* M^4 *has zero Gaussian curvature if and only if*

$$\varphi(r)=c_1, \text{ or}$$
$$\varphi(r)=\pm\left(c_1r+c_2\right), \text{ or}$$
$$\varphi(r)=\pm\sqrt{-1}\left(c_1r+c_2\right).$$

Proof. Solving the 2nd order differential eq. $K=0,$ i.e. $\varphi'^3\varphi''=0,$ we get the solutions.

Corollary 4. *Let* $\mathbf{R}: M^4 \longrightarrow \mathbb{E}^5$ *be an isometric immersion given by* (10)*. Then* M^4 *has zero mean curvature if and only if*

$$\varphi(r)=\pm\int\frac{dr}{\sqrt{c_1r^6-1}}+c_2.$$

Proof. When we solve the 2nd order differential eq. $H=0,$ i.e.

$$r\varphi''+3\varphi'^3+3\varphi'=0,$$

we get the solution.

4. SONUÇLAR (CONCLUSION)

In the present paper, we define a new kind rotational hypersurface with 4-parameters $\mathbb{S}^3(r)\times\mathbb{R}$ product space of five dimensional Euclidean space $\mathbb{E}^5$. It can be extended higher dimensions, for example $\mathbb{S}^7(r)\times\mathbb{R}$. Moreover, the topic can also be transformed into the Minkowski geometry.

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