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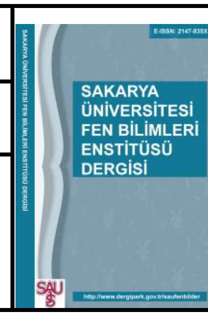

Monotonluğu Koruyan Matrisler

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ÖZ

Negatif olmayan üçgensel matris dönüşümlerinin çok geniş bir ailesi için, monoton azalan dizilerin monoton azalan dizilere dönüşebilmesi koşullarını elde ettik.

Anahtar Kelimeler: monoton azalan diziler, genelleştirilmiş Hausdorff matrisleri, Üst üçgen matrisler, toplanabilme matrisleri

Monotonicity Preserving Matrices

ABSTRACT

We obtain the conditions for a large class of nonnegative triangular matrix transformations to map positive monotone decreasing sequences into positive monotone decreasing sequences.

Keywords: monotone decreasing sequences, generalized Hausdorff matrices, Upper traingular matrices, summability matrices.

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1. INTRODUCTION

A positive sequence $x = \{x_n\}$ is a monotone decreasing sequence, or simply decreasing, if $x_n \geq x_{n+1}$ for each $n \geq 0$.A matrix A is called triangular if $a_{nk} = 0$ for each k>n, and a triangle if it is triangular and $a_{nn} \neq 0$ for each n. We shall say that a matrix preserves monotonicity if it maps every monotone decreasing sequence into a monotone decreasing sequence.

Bennett [1] prove the following theorem.

Theorem 1. Let $A = (a_{nk})$ be a matrix with nonnegative entries, and consider the associated transform $x \to y$, given by

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k.$$

Then the following conditions are equivalent.

(i) $y_0 \geq y_1 \geq \cdots \geq 0$ whenever $x_0 \geq x_1 \geq \cdots \geq x_n$

(ii) $\sum_{k=0}^{r} a_{nk} \geq \sum_{k=0}^{r} a_{n+1,k}, \qquad n, r = 0,1,2, \ldots$

This theorem yields a number of corollaries for some well-known summability matrices. Bennett has noted that the positive Hausdorff matrices map decreasing sequences into decreasing sequences. As our first result we shall show that the same is true for generalized Hausdorff matrices.

2. PRELIMINARIES

An ordinary Hausdorff matrix H is a triangular matrix with nonzero entries

$$\binom{n}{k} \Delta^{n-k} \mu_k \qquad (1)$$

where μ_n is a real or complex sequence and Δ is the forward difference operator defined by

$\Delta \mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{n+1} \mu_k = \Delta^n \mu_k - \Delta^n \mu_{k+1}$.

There are several generalizations of Hausdorff matrices. One of this is called the H-J matrices. A sequence (λ_n) is called acceptable sequence if it satisfies the following properties:

$$0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \ldots,$$

with $\lambda_n \to \infty$, but slowly enough so that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

The nonzero entries of an H-J matrix $H(\mu; \lambda)$ are defined by

$$h(\mu; \lambda)_{nk} = \lambda_{k+1} \ldots \lambda_n [\mu_k, \ldots, \mu_n], \; 0 \leq k \leq n$$

where μ_n is a real or complex sequence, [.] is the divided difference operator defined by

$$[\mu_k, \mu_{k+1}] = \frac{1}{\lambda_{k+1} - \lambda_k} [\mu_k - \mu_{k+1}]$$

and

$$[\mu_k, \ldots, \mu_n] = \frac{1}{\lambda_n - \lambda_k} ([\mu_k, \ldots, \mu_{n-1}] - [\mu_{k+1}, \ldots, \mu_n])$$

and where it is understood that $\lambda_{k+1} \lambda_{k+2} \ldots \lambda_n = 1$ when $k = n$.

Hausdorff [5] defined this generalization for $\lambda_0 = 0$, and, Jakimovski [6] extended this class for $\lambda_0 > 0$.

3. MAIN THEOREMS

Theorem 2. A positive H-J matrix, with $\lambda_0 = 0$, maps decreasing sequences into decreasing sequences.

Proof: Using (ii) of Bennett's theorem

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k} = \sum_{k=0}^{r} \lambda_{k+1} \ldots \lambda_n [\mu_k, \ldots, \mu_n] - \sum_{k=0}^{r} \lambda_{k+1} \ldots \lambda_{n+1} [\mu_k, \ldots, \mu_{n+1}]. \qquad (2)$$

From the definition of divided differences,

$$(\lambda_{n+1} - \lambda_k)[\mu_k, \ldots \mu_{n+1}] = [\mu_k, \ldots \mu_n] - [\mu_{k+1}, \ldots \mu_{n+1}]$$

Substituting into (2), we have

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k} = \sum_{k=0}^{r} \lambda_{k+1} \ldots \lambda_n (\lambda_{n+1} - \lambda_k - \lambda_{n+1}) [\mu_k, \ldots, \mu_{n+1}] + \sum_{k=0}^{r} \lambda_{k+1} \ldots \lambda_n [\mu_{k+1}, \ldots, \mu_{n+1}]$$

$$= -\sum_{k=0}^{r} \lambda_k \ldots \lambda_n [\mu_k, \ldots, \mu_{n+1}] + \sum_{k=1}^{r+1} \lambda_k \ldots \lambda_n [\mu_k, \ldots, \mu_{n+1}]$$

$$= \lambda_{r+1} \ldots \lambda_n [\mu_{kr+1}, \ldots, \mu_{n+1}] - \lambda_0 \ldots \lambda_n [\mu_0, \ldots, \mu_{n+1}].$$

Since $\lambda_0 = 0$ and the matrix is positive,

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k}$$

$$= \lambda_{r+1} \dots \lambda_n [\mu_{r+1}, \dots, \mu_{n+1}] \geq 0. \qquad (3)$$

□

The E-J generalized Hausdorff matrices, denoted by $H_\mu^\alpha = (h_{nk}^{(\alpha)})$, were defined independently by Endl [2] and Jakimovski [6], with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k^{(\alpha)}, \;\; 0 \leq k \leq n,$$

for any $\alpha \geq 0$. For $\alpha = 0$, the E-J matrices reduce to the ordinary Hausdorff matrices.

Corollary 1. For $0 \leq \alpha < 1$, a positive E-J generalized Hasudorff matrix maps decreasing seuences into decreasing sequences.

Proof.

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k}$$
$$= \sum_{k=0}^{r} \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k$$
$$- \sum_{k=0}^{r} \binom{n+1+\alpha}{n+1-k} \Delta^{n+1-k} \mu_k. \qquad (4)$$

Since

$$\Delta^{n+1-k} \mu_k = \Delta^{n-k} \mu_k - \Delta^{n-k} \mu_{k+1},$$

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k}$$

$$= \sum_{k=0}^{r} \binom{n+\alpha}{n-k} \{\Delta^{n+1-k} \mu_k + \Delta^{n-k} \mu_{k+1}\}$$
$$- \sum_{k=0}^{r} \binom{n+1+\alpha}{n+1-k} \Delta^{n+1-k} \mu_k$$

$$= \sum_{k=0}^{r} \binom{n+\alpha}{n-k} \left(1 - \frac{n+1+\alpha}{n+1-k}\right) \Delta^{n+1-k} \mu_k$$
$$+ \sum_{k=0}^{r} \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_{k+1}$$

$$= \sum_{k=0}^{r} \binom{n+\alpha}{n+1-k} \Delta^{n+1-k} \mu_k$$
$$+ \sum_{k=1}^{r+1} \binom{n+\alpha}{n+1-k} \Delta^{n+1-k} \mu_k.$$

For $0 \leq \alpha < 1$, the binomial form $\binom{n+\alpha}{n+1}$, will vanish, so the above equality will be

$$\sum_{k=0}^{r} h_{nk} - \sum_{k=0}^{r} h_{n+1,k} = \binom{n+\alpha}{n-r} \Delta^{n-r} \mu_{r+1} \geq 0.$$

□

Corollary 2. Every positive Hausdorff matrix maps decreasing sequences into decreasing sequences.

Proof: Ordinary Hausdorff matrices are the special case of H-J matrices obtained by setting $\lambda_n = n$, or by using an E-J matrices with $\alpha = 0$.

□

Let $\{p_n\}$ be a nonnegative sequence with $p_0 > 0$. A Nörlund matrix is a triangular matrix B with entries

$$b_{nk} = \frac{p_{n-k}}{P_n},$$

where

$$P_n = \sum_{k=0}^{n} p_{n-k.}$$

Corollary 3. Every Nörlund matrix, with decreasing sequence $\{p_n\}$, preserves decreasing sequences.
Proof: Using (ii) of Bennett's theorem

$$\sum_{k=0}^{r} b_{nk} - \sum_{k=0}^{r} b_{n+1,k} = \sum_{k=0}^{r} \frac{p_{n-k}}{P_n} - \sum_{k=0}^{r} \frac{p_{n+1-k}}{P_{n+1}}$$

Since $\{p_n\}$ is decreasing, $p_{n-k} \geq p_{n+1-k}$, we can write the above equation as

$$\sum_{k=0}^{r} \frac{p_{n-k}}{P_n} - \sum_{k=0}^{r} \frac{p_{n+1-k}}{P_{n+1}} \geq \sum_{k=0}^{r} \frac{p_{n-k}}{P_n} - \sum_{k=0}^{r} \frac{p_{n-k}}{P_{n+1}}$$

$$= \left(\frac{1}{P_n} - \frac{1}{P_{n+1}}\right) \sum_{k=0}^{r} p_{n-k}$$

$$= \frac{p_{n+1}}{P_n P_{n+1}} \sum_{k=0}^{r} p_{n-k}.$$

Since (p_k) is nonnegative sequence,

$$\sum_{k=0}^{r} b_{nk} - \sum_{k=0}^{r} b_{n+1,k} \geq \frac{p_{n+1}}{P_n P_{n+1}} \sum_{k=0}^{r} p_{n-k} \geq 0.$$

□

There are also some non-summability matrices that preserve decreasing sequences.

A factorable matrix is a lower triangular matrix with entries $a_{nk} = a_n b_k$, where a_n depends only upon n and b_k depends only upon k.

Theorem 3. Let A be a positive factorable matrix. If a_n is non-increasing the A preserves decreasing sequences.

Proof:

$$\sum_{k=0}^{r} a_{nk} - \sum_{k=0}^{r} a_{n+1,k} = \sum_{k=0}^{r} a_n b_k - \sum_{k=0}^{r} a_{n+1} b_k$$

$$= (a_n - a_{n+1}) \sum_{k=0}^{r} b_k \geq 0.$$

□

Let (p_k) be a nonnegative sequence with $p_0 > 0$, and define $P_n = \sum_{k=0}^{n} p_k$. A weighted mean matrix $(\widetilde{N}, p)$ is a triangular matrix with entries p_k/P_n. A weighted mean matrix is the special case of the factorable matrix.

Corollary 4. Every nonnegative weighted mean matrix preserves decreasing sequences.

Proof: The proof is easy to verify since P_n is non-increasing.

□

It is clear that every upper triangular matrices are monotonicity preserving matrices if they have decreasing in columns; i.e., $a_{nk} \geq a_{n+1,k}$. The following theorem gives another approach for every upper triangular matrices.

Theorem 4. Let A be positive upper triangular matrix with entries $c_{nk} \geq c_{n+1,k+1}$, for $k = n, n+1, \ldots$ then A is a monotonicity preserving matrix.

Proof:

$$y_n - y_{n+1} = \sum_{k=n}^{\infty} c_{nk} x_k - \sum_{k=n+1}^{\infty} c_{n+1,k}\, x_k$$

$$= \sum_{k=n}^{\infty} c_{nk} x_k - \sum_{k=n}^{\infty} c_{n+1,k+1}\, x_{k+1}.$$

Since $\{x_n\}$ is a positive decreasing sequence and and A is a positive matrix with $c_{nk} \geq c_{n+1,k+1}$, then

$$y_n - y_{n+1} \geq 0$$

where $x_n - x_{n+1} \geq 0$.

□

An infinite matrix will be called a band matrix if it has only a finite number of nonzero diagonals. The width of a band matrix refers to the number of nonzero diagonals. An upper band matrix A has finite number of the diagonals on or above the main diagonal.

Corollary 5. Let A be an upper banded matrix with entries $a_{jk} \geq a_{j+1,k+1} \geq 0$. Then A is a monotonicity preserving matrix.

In [1] Bennett remarked that matrices of the form

$$\begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot \\ 0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot \\ 0 & 0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_n & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{8}$$

with $a_1, a_2, \ldots, a_n \geq 0$, are monotonicity-preserving.

Corollary 6. Let A be an upper banded matrix with entries $a_{ij} = a_{j-i+1}$, where $j = i, i+1, \ldots, n+i-1$, and $a_1, a_2, \ldots, a_n \geq 0$, is monotonicity-preserving.

Corollary 7. Bidiagonal matrices are monotonicity decreasing if each $a_{nk} \geq a_{n+1,k+1}$.

Proof: Let $y = Ax$ where A is a bidiagonal matrix with two nonzero diagonals a_{nj} and a_{nk}. Then,

$$y_n - y_{n+1} = \sum_{i=j}^{k} a_{ni} x_i - \sum_{i=j+1}^{k+1} a_{n+1,i}\, x_i$$

$$= \sum_{i=j}^{k} a_{ni} x_i - \sum_{i=j}^{k} a_{n+1,i+1}\, x_{i+1}$$

$$\geq \sum_{i=j}^{k} a_{ni} x_i - \sum_{i=j}^{k} a_{n+1,i+1}\, x_i$$

$$= \sum_{i=j}^{k} (a_{ni} - a_{n+1,i+1}) x_i \geq 0.$$

□

We shall call a matrix D an $r-fold$ weighted shift if, for some positive integer r, it has entries

$$d_{nk} = \begin{cases} a_k & k = n,\ n+1,\ \ldots, n+r \\ 0 & otherwise \end{cases},$$

where a_k is a positive sequence.

Corollary 8. A $r - fold$ weighted shift D matrix preserves decreasing sequences if $\{a_k\}$ is a decreasing sequences.

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