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Research Article

Hermite-Hadamard type inequalities for (p, h)-convex functions on $\mathbb{R}^n$

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Abstract

In this paper, the concept of the (p, h)-convex function is introduced, which generalizes the p-convex function and the h-convex function, and Hermite-Hadamard type inequalities for (p, h)-convex functions on $\mathbb{R}^n$ are established. Furthermore, some mappings related to the above inequalities are studied and some known results are generalized.

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1. Introduction

Let $p \in (0, 1]$. We say that a set $\mathcal{D} \subset \mathbb{R}^n$ is *p-convex* if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ and $\lambda, \mu \in (0, 1)$ with $\lambda^p + \mu^p = 1$,

$$\lambda\mathbf{x} + \mu\mathbf{y} \in \mathcal{D}. \tag{1.1}$$

One can see that $p-$convex sets are just convex with $p = 1$ and are star-shaped convex sets as $p \to 0$. The proceeding definition shows that there is a big gap between $p-$convex sets and convex sets for $0 < p < 1$. It is easy to see that $\mathbb{R}$, $\mathbb{R}_+ := [0, \infty)$ and $[0, a) \subset \mathbb{R}(a > 0)$ are all p-convex sets, and we can check that if $\mathcal{D}_i \subset \mathbb{R}$ are p-convex sets, $i = 1, 2, \ldots, n$, then the Cartesian product of $\mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ is also a p-convex set. More applications of p-convex sets can be found in [16, 23] and their references.

Definition 1.1. Let $0 < p \leq 1$, $\mathcal{D}$ be a p-convex set and $h : [0, 1] \to \mathbb{R}_+$ be a given function. A function $f : \mathcal{D} \to \mathbb{R}$ is called (p, h)-convex on $\mathcal{D}$ if

$$f(\lambda\mathbf{x} + \mu\mathbf{y}) \leq h(\lambda)f(\mathbf{x}) + h(\mu)f(\mathbf{y}) \tag{1.2}$$

holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ and $\lambda, \mu \in (0, 1)$ with $\lambda^p + \mu^p = 1$.

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The notation of the (p,h)-convex function generalizes some known classes of the usual p-convex function and the h-convex function, which are obtained by putting in (1.2) $h(t)=t$ [24] and $p=1$ [10], respectively. The h-convex function was introduced by Varošanec [24] and unifies the convex function, the s-convex function (in the second sense) [3], the P-function [21] and the Godunova-Levin function [12]. Lots of properties and applications of them can be found in literatures, see e.g. [4, 8, 11, 13, 15, 19, 20, 25]. In particular, if f satisfies (1.2) with $h(t)=t,\ h(t)=t^s(s\in(0,1)), h(t)\equiv 1$ and $h(t)=1/t\ (0<t<1)$, then f is said to be a *p-convex function, (p,s)-convex function (in the second sense), (p,P)-function* and a *p-Godunova-Levin function*, respectively. Moreover, it is not difficult to see that $h\left(2^{-1/p}\right)>0$ if f is a nonnegative and nontrivial (p,h)-convex function. Throughout the paper, we assume that the function h in Definition 1.1 is always (Lebesgue) integrable on the interval $[0,1]$.

Convex type functions are important in both pure and applied mathematics. The famous Hermite-Hadamard inequality for the convex function is stated as follows:

Theorem 1.2. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an integrable convex function. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}.$$

Dragomier, Pečarić and Persson [9] extended it for Godunova-Levin functions and P-functions in 1995. Dragomir and Fitzpatrick[7] obtained an analogue inequalities for s-convex functions (in the second sense) in 1999. Sarikaya, Saglam and Yildririm extended it to $h-$convex functions in 2008.

Theorem 1.3 ([22])**.** *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an integral $h-$convex function on $[a,b]$ with $h(1/2)>0$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le[f(a)+f(b)]\int_0^1 h(x)dx.$$

For any $\alpha,\beta>0$, we define the Beta function by

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

It is well known that

$$B\left(\alpha,\beta\right)=B(\beta,\alpha),\quad B\left(\alpha+1,\beta\right)=\frac{\alpha}{\alpha+\beta}B\left(\alpha,\beta\right).$$

In 2021, Ekem, Kemali, Tnaztepe and Adilov [10] established the Hermite-Hadamard inequality for p-convex functions as follows.

Theorem 1.4. *If $f:\mathbb{R}_+\to\mathbb{R}_+$ is an integrable p-convex function, then, for any $[a,b]\subset\mathbb{R}_+$, we have*

$$\begin{aligned}2^{\frac{1}{p}-1}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)&\le\frac{1}{b-a}\int_a^b f(x)dx\\&\le\frac{1}{2(b-a)}\left\{\left[a+\frac{b}{p}B\left(\frac{1}{p},\frac{1}{p}\right)\right]f(a)+\left[b+\frac{a}{p}B\left(\frac{1}{p},\frac{1}{p}\right)\right]f(b)\right\}.\end{aligned}$$

Comparing Theorem 1.2 and Theorem 1.4, the former is better than the latter at the end point $p=1$ (i.e. the function is convex). Therefore, we are more interested in the case of $0<p<1$ in the next.

Meanwhile, there are many literatures dedicated to develop the Hermite-Hadamard type inequalities to multidimensions. In the sequel, unless otherwise specified, we denote that $\mathbb{R}^1=\mathbb{R}$, $\mathbb{R}^1_+=\mathbb{R}_+$, $\mathbb{R}^n$ is the Euclidean space of dimension n and $\mathbb{R}^n_+$ is the usual n times Cartesian product of $\mathbb{R}_+$. $[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n$ denotes by $[\mathbf{a},\mathbf{b}]=[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$

and the Lebesgue measure of it by $|[\mathbf{a},\mathbf{b}]| = \prod_{i=1}^{n}(b_i - a_i)$. Denote by $L(E)$ the set of (Lebesgue) integrable functions on the measurable set $E \subset \mathbb{R}^n$. For any points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define the product of vectors by

$$\mathbf{x} \circ \mathbf{y} = (x_1 y_1, x_2 y_2, \ldots, x_n y_n),$$

the quotient of vectors by

$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}\right), y_i \neq 0, i = 1, 2, \ldots, n,$$

the linear combination of vectors by

$$a\mathbf{x} + b\mathbf{y} = (ax_1 + by_1, ax_2 + by_2, \ldots, ax_n + by_n), \ a, b \in \mathbb{R},$$

and the power of vectors by

$$c^{\mathbf{x}} = (c^{x_1}, c^{x_2}, \ldots, c^{x_n}), \ 1 < c < \infty.$$

Definition 1.5. Let $0 < p \leq 1$, $h : [0,1] \to [0,\infty)$ be a given function and $\mathcal{D}_i \subset \mathbb{R}$ be p-convex sets, $i = 1, 2, \ldots, n$. A function $f : \mathcal{D} := \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \subset \mathbb{R}^n \to \mathbb{R}$ is called coordinated (p,h)-convex if for every $i \in \{1, 2, ..., n\}$ the partial mapping $f_i : \mathcal{D}_i \to \mathbb{R}$, $f_i(u) = f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$ is (p,h)-convex for all given $x_j \in \mathcal{D}_j$, $j \neq i$.

In particular, at the point $p = 1$, the corresponding function f is said to be coordinated h-convex on $\mathcal{D}$.

If $h(t) = t$, $h(t) = t^s (s \in (0,1)), h(t) \equiv 1$ and $h(t) = 1/t$ $(0 < t < 1)$ in Definition 1.5, then the function f is called a *coordinated p-convex function, coordinated (p,s)-convex function (in the second sense), coordinated (p,P)-function* and a *coordinated p-Godunova-Levin function*, respectively, and, furthermore, if $p = 1$, then f is called a *coordinated convex function, coordinated s-convex function (in the second sense), coordinated P-function* and a *coordinated Godunova-Levin function*, respectively.

In 2001, Dragomir [6] proved the following Hermite-Hadamard type inequality for co-ordinated convex functions on the plane.

Theorem 1.6. *Let $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^2 \to \mathbb{R}$ be coordinated convex on $[\mathbf{a},\mathbf{b}]$ and $f \in L([\mathbf{a},\mathbf{b}])$. Then*

$$\begin{aligned} f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) &\leq \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2)\, dx_1 dx_2 \\ &\leq \frac{f(a_1,a_2) + f(a_1,b_2) + f(b_1,a_2) + f(b_1,b_2)}{4}. \end{aligned}$$

Thereafter, Alomari and Darus [1] extended similar results for coordinated s-convex functions. Latif and Alomari [17] considered the case of coordinated h-convex functions.

Theorem 1.7 ([17])**.** *Let $h : [0,1] \to \mathbb{R}_+$ with $h(1/2) > 0$ and $h \in L([0,1])$. Suppose that $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^2 \to \mathbb{R}$ be a coordinated h-convex function on $[\mathbf{a},\mathbf{b}]$ and $f \in L([\mathbf{a},\mathbf{b}])$. Then*

$$\begin{aligned} &\frac{1}{4h^2(1/2)} f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \\ \leq\ &\frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2)\, dx_1 dx_2 \\ \leq\ &[f(a_1,a_2) + f(a_1,b_2) + f(b_1,a_2) + f(b_1,b_2)] \left(\int_0^1 h(t) dt\right)^2. \end{aligned}$$

Another interesting topic is to give some applications of the Hermite-Hadamard inequalities for convex type functions. Let $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^n \to \mathbb{R}$. Define the mappings $\mathfrak{H} : [\mathbf{0},\mathbf{1}] = [0,1]^n \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathfrak{I} : [\mathbf{0},\mathbf{1}] = [0,1]^n \subset \mathbb{R}^n \to \mathbb{R}$ by

$$\mathfrak{H}(\mathbf{t}) = \frac{1}{|[\mathbf{a},\mathbf{b}]|} \int_{[\mathbf{a},\mathbf{b}]} f\left(\mathbf{t} \circ \mathbf{x} + (\mathbf{1} - \mathbf{t}) \circ \frac{\mathbf{a}+\mathbf{b}}{2}\right) d\mathbf{x}, \tag{1.3}$$

$$\mathfrak{I}(\mathbf{t}) = \frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\mathbf{t}\circ\mathbf{x} + (\mathbf{1}-\mathbf{t})\circ\mathbf{y}\right) d\mathbf{x}d\mathbf{y}, \tag{1.4}$$

respectively. Clearly,

$$\mathfrak{H}(\mathbf{0}) := \mathfrak{H}(0,\dots,0) = f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right),$$

$$\mathfrak{H}(\mathbf{1}) := \mathfrak{H}(1,\dots,1) = \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f\left(\mathbf{x}\right) d\mathbf{x},$$

$$\mathfrak{I}(\mathbf{0}) := \mathfrak{I}(0,\dots,0) = \mathfrak{I}(\mathbf{1}) := \mathfrak{I}(1,\dots,1) = \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x},$$

$$\mathfrak{I}(\mathbf{1/2}) := \mathfrak{I}(1/2,\dots,1/2) = \frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) d\mathbf{x}d\mathbf{y}.$$

In 1992, Dragomir [5] studied some properties of $\mathfrak{H}$ and $\mathfrak{I}$ for $n=1$. In 2021, the authors [10] obtained the following conclusions for p-convex functions.

Theorem 1.8 ([10]). *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an integrable p-convex function. For any $[a,b]\subset\mathbb{R}_+$, define the mappings $\mathfrak{H}$ and $\mathfrak{I}$ by* (1.3) *and* (1.4) *for $n=1$, respectively.*

(i) *If f is a decreasing function, then $\mathfrak{H}(t)$ and $\mathfrak{I}(t)$ are both p-convex functions on the interval* $[0,1]$.

(ii) *For any $t\in(0,1]$, we have*

$$\mathfrak{H}(t) \geq 2^{\frac{1}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right),$$

and

$$2^{1-\frac{1}{p}}\mathfrak{I}(t) \geq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) dxdy.$$

In 2001, Dragomir [6] extended his previous results [5] to coordinated convex functions on $\mathbb{R}^2$.

Theorem 1.9 ([6]). *Let $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^2\to\mathbb{R}$ be integrable coordinated convex on $[\mathbf{a},\mathbf{b}]$. The mappings $\mathfrak{H}$ and $\mathfrak{I}$ are as in* (1.3) *and* (1.4) *with $n=2$, respectively. Then*

(i) *the mappings $\mathfrak{H}$ and $\mathfrak{I}$ are both coordinated convex on* $[0,1]^2$.

(ii)

$$\sup_{\mathbf{t}\in[0,1]^2}\mathfrak{H}(\mathbf{t}) = \mathfrak{H}(\mathbf{1}), \quad \inf_{\mathbf{t}\in[0,1]^2}\mathfrak{H}(\mathbf{t}) = \mathfrak{H}(\mathbf{0}).$$

(iii)

$$\sup_{\mathbf{t}\in[0,1]^2}\mathfrak{I}(\mathbf{t}) = \mathfrak{I}(\mathbf{0}) = \mathfrak{I}(\mathbf{1}), \quad \inf_{\mathbf{t}\in[0,1]^2}\mathfrak{I}(\mathbf{t}) = \mathfrak{I}(\mathbf{1/2}).$$

It is notable that Theorem 1.9 reduces to Theorem 1 and Theorem 2 in [5] with $n=1$. In 2008, Alomari and Darus [2] extended the properties of the mapping $\mathfrak{H}$ in Theorem 1.9 to coordinated s-convex functions on $\mathbb{R}^2$. In 2013, Matłok [18] obtained similar results for coordinated $h-$convex functions.

Theorem 1.10 ([18]). *Let $h_1,\ h_2:(0,1)\to[0,\infty)$ with $h_1,\ h_2\in L((0,1))$ and $h_1(1/2)h_2(1/2) > 0$. Suppose that the function $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^2\to\mathbb{R}$, $f\in L([\mathbf{a},\mathbf{b}])$ and its partial mappings $f(\cdot,x_2)$ and $f(x_1,\cdot)$ are h_1-convex on $[a_1,b_1]$ and h_2-convex on $[a_2,b_2]$ respectively. $\mathfrak{H}:[\mathbf{0},\mathbf{1}]=[0,1]^2\to\mathbb{R}$ is defined by* (1.3) *with $n=2$. Then*

(i) *the partial mappings $\mathfrak{H}(\cdot,t_2)$ and $\mathfrak{H}(t_1,\cdot)$ are h_1-convex and h_2-convex on $[0,1]$ respectively.*

(ii)

$$\frac{1}{4h_1(1/2)h_2(1/2)}\mathfrak{H}(\mathbf{0}) \le \mathfrak{H}(\mathbf{t})$$

holds for all $\mathbf{t} \in [\mathbf{0}, \mathbf{1}]$.

Recently, the authors [14] extended the above inequalities and mappings to more general case—-coordinated strongly h-convex functions on $\mathbb{R}^n$.

With these motivations, the main purpose of the present paper is to establish Hermite-Hadamard type inequalities for the new convex type functions–(p, h)-convex functions. Furthermore, some mappings related to the above inequalities are studied.

2. Hermite-Hadamard's inequalities for (p, h)-convex functions

In this section, we fist establish the Hermite-Hadamard type inequalities to (p, h)-convex functions on the real line.

Theorem 2.1. *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be an integrable* (p, h)*-convex function. Then, for all* $[a, b] \subset \mathbb{R}_+$*, we have*

$$\begin{aligned}
\frac{1}{2h\left(2^{-\frac{1}{p}}\right)} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) &\le \frac{1}{b-a}\int_a^b f(x)dx \\
&\le \frac{1}{b-a}\min\left\{b\left[f(b)\int_0^1 h(t)dt + f(a)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1}t^{p-1}dt\right],\right. \\
&\quad \left. a\left[f(a)\int_0^1 h(t)dt + f(b)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1}t^{p-1}dt\right]\right\} \\
&\le \frac{1}{2(b-a)}\left\{\left[a\int_0^1 h(t)dt + b\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1}t^{p-1}dt\right]f(a)\right. \\
&\quad \left. + \left[b\int_0^1 h(t)dt + a\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1}t^{p-1}dt\right]f(b)\right\}.
\end{aligned}$$

Proof. First, we prove the later two inequalities. Without loss of generality, we may suppose that $0 < a < b < \infty$ and $0 < p < 1$. Let $M = \left(\frac{b}{a}\right)^{\frac{p}{1-p}}$. By changing the variable $x = t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a$, there is some $t_0 \in \left[\frac{1}{M+1,1}\right)$ such that

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{1}{p}\int_{t_0}^1 f\left(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a\right)\left[bt^{\frac{1}{p}-1} - a(1-t)^{\frac{1}{p}-1}\right]dt \\
&\le \frac{1}{p}\int_{t_0}^1 f\left(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a\right)\cdot bt^{\frac{1}{p}-1}dt \\
&\le \frac{1}{p}\int_0^1 f\left(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a\right)\cdot bt^{\frac{1}{p}-1}dt,
\end{aligned}$$

which, combing the (p, h)-convexity and nonnegativity of f, means that

$$\int_a^b f(x)dx \le \frac{b}{p}\left[f(b)\int_0^1 h\left(t^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt + f(a)\int_0^1 h\left((1-t)^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt\right]. \tag{2.1}$$

And, with some simple changing of variables, we have

$$\int_0^1 h\left(t^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt = p\int_0^1 h(t)dt, \tag{2.2}$$

and

$$\int_0^1 h\left((1-t)^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt = p\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1}t^{p-1}dt. \tag{2.3}$$

Then it follows from (2.1), (2.2) and (2.3) that

$$\int_a^b f(x)dx \leq b\left[f(b)\int_0^1 h(t)dt + f(a)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt\right]. \tag{2.4}$$

By a similar argument, changing the variable $x = (1-t)^{\frac{1}{p}}b + t^{\frac{1}{p}}a$ yields that

$$\begin{aligned}\int_a^b f(x)dx &\leq \frac{a}{p}\left[f(a)\int_0^1 h\left(t^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt + f(b)\int_0^1 h\left((1-t)^{\frac{1}{p}}\right)t^{\frac{1}{p}-1}dt\right] \\ &= a\left[f(a)\int_0^1 h(t)dt + f(b)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt\right].\end{aligned} \tag{2.5}$$

Then it follows from (2.4) and (2.5) that

$$\begin{aligned}\int_a^b f(x)dx \leq \min\Bigg\{ & b\left[f(b)\int_0^1 h(t)dt + f(a)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt\right], \\ & a\left[f(a)\int_0^1 h(t)dt + f(b)\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt\right]\Bigg\},\end{aligned}$$

and the third inequality is easily obtained by it.

Next we prove the first inequality. The (p,h)-convexity of f shows that, for all $x, w > 0$,

$$f\left(\frac{x+w}{2^{\frac{1}{p}}}\right) \leq h\left(2^{-\frac{1}{p}}\right)[f(x)+f(w)].$$

Let $x = ta + (1-t)b$, $w = (1-t)a + tb$ and $t \in (0,1)$. Then,

$$f\left(\frac{x+w}{2^{\frac{1}{p}}}\right) \leq h\left(2^{-\frac{1}{p}}\right)[f(ta+(1-t)b) + f((1-t)a+tb)].$$

Integrating both side on $[0,1]$, we achieve that

$$\frac{1}{2h\left(2^{-\frac{1}{p}}\right)} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which completes the proof of Theorem 2.1. □

If taking $h(t) = t^s (0 < s \leq 1)$ in Theorem 2.1, then

$$\int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt = \int_0^1 \left(1-t^p\right)^{\frac{1}{p}-1} t^{\frac{1}{p}+s-1}dt = \frac{s}{p}B\left(\frac{s}{p}, \frac{1}{p}\right), \tag{2.6}$$

which, combing Theorem 2.1, implies the following result.

Corollary 2.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable (p,s)-convex function. Then, for all $[a,b] \subset \mathbb{R}_+$, we have*

$$\begin{aligned}& 2^{\frac{s}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{(s+1)(b-a)}\min\left\{b\left[f(b) + \frac{s}{p}B\left(\frac{s}{p},\frac{1}{p}\right)f(a)\right], a\left[f(a) + \frac{s}{p}B\left(\frac{s}{p},\frac{1}{p}\right)f(b)\right]\right\} \\ &\leq \frac{1}{2(s+1)(b-a)}\left\{\left[a + \frac{s}{p}B\left(\frac{s}{p},\frac{1}{p}\right)b\right]f(a) + \left[\frac{s}{p}B\left(\frac{s}{p},\frac{1}{p}\right)a + b\right]f(b)\right\}.\end{aligned}$$

Remark 2.3. In particular, Corollary 2.2 improves Theorem 1.4 directly with $s = 1$.

Especially, if $h \equiv 1$ on $[0,1]$, then $\int_0^1 h(t)dt = \int_0^1 h(t)\left(1-t^p\right)^{\frac{1}{p}-1} t^{p-1}dt = 1$ $(0 < p \leq 1)$ and we have:

Corollary 2.4. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an integrable (p,P)- convex function. Then, for all $[a,b]\subset\mathbb{R}_+$,*

$$\frac{1}{2}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{a}{b-a}[f(a)+f(b)].$$

Next, we will extend the above inequalities to high-dimensions.

Theorem 2.5. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ and $f\in L\left(\mathbb{R}_+^n\right)$. If the partial mappings f_i are (p_i,h_i)-convex functions on $\mathbb{R}_+$ for $i=1,2,\cdots,n$, respectively, then, for all $[\mathbf{a},\ \mathbf{b}]\subset\mathbb{R}_+^n$, we have*

$$\begin{aligned}
&\frac{1}{2^n\prod_{i=1}^n h_i\left(2^{-\frac{1}{p_i}}\right)}f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{\mathbf{p}}}}\right)\\
\le\ &\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})d\mathbf{x}\\
\le\ &\frac{1}{2^n|[\mathbf{a},\mathbf{b}]|}\sum_{\substack{c_i,\tilde{c}_i=a_i\ \text{or}\ b_i,\\ c_i\neq\tilde{c}_i,\ i=1,\dots,n}}\prod_{i=1}^n\left[c_i\int_0^1 h_i(t)dt+\tilde{c}_i\int_0^1 h_i(t)\left(1-t^{p_i}\right)^{\frac{1}{p_i}-1}t^{p_i-1}dt\right]f\left(\mathbf{c}\right).
\end{aligned}$$

Proof. By Fubini's theorem and the third inequality of Theorem 2.1, an induction argument shows that

$$\begin{aligned}
&\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})d\mathbf{x}\\
=\ &\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}f\left(x_1,\dots,x_n\right)dx_1\dots dx_n\\
\le\ &\frac{1}{2}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\left\{\left[a_1\int_0^1 h_1(t)dt+b_1\int_0^1 h_1(t)\left(1-t_1^p\right)^{\frac{1}{p_1}-1}t^{p_1-1}dt\right]f(a_1,x_2,\dots,x_n)\right.\\
&\left.+\left[b_1\int_0^1 h_1(t)dt+a_1\int_0^1 h_1(t)\left(1-t_1^p\right)^{\frac{1}{p_1}-1}t^{p_1-1}dt\right]f(b_1,x_2,\dots,x_n)\right\}dx_2\dots dx_n\\
=\ &\frac{1}{2}\sum_{\substack{c_1,\tilde{c}_1=a_1\ \text{or}\ b_1,\\ c_1\neq\tilde{c}_1}}\left[c_1\int_0^1 h_1(t)dt+\tilde{c}_1\int_0^1 h_1(t)\left(1-t_1^p\right)^{\frac{1}{p_1}-1}t^{p_1-1}dt\right]\times\\
&\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}f(c_1,x_2,\dots,x_n)dx_2\dots dx_n\\
\le\ &\frac{1}{2^2}\sum_{\substack{c_1,\tilde{c}_1=a_1\ \text{or}\ b_1,\\ c_1\neq\tilde{c}_1}}\left[c_1\int_0^1 h_1(t)dt+\tilde{c}_1\int_0^1 h_1(t)\left(1-t^{p_1}\right)^{\frac{1}{p_1}-1}t^{p_1-1}dt\right]\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3}\\
&\left\{\sum_{\substack{c_2,\tilde{c}_2=a_2\ \text{or}\ b_2,\\ c_2\neq\tilde{c}_2}}\left[c_2\int_0^1 h_2(t)dt+\tilde{c}_2\int_0^1 h_2(t)\left(1-t^{p_2}\right)^{\frac{1}{p_2}-1}t^{p_2-1}dt\right]f\left(c_1,c_2,x_3,\dots,x_n\right)\right\}\\
&dx_3\dots dx_n\\
=\ &\frac{1}{2^2}\sum_{\substack{c_i,\tilde{c}_i=a_i\ \text{or}\ b_i,\\ c_i\neq\tilde{c}_i,\ i=1,2}}\prod_{i=1}^2\left[c_i\int_0^1 h_i(t)dt+\tilde{c}_i\int_0^1 h_i(t)\left(1-t^{p_i}\right)^{\frac{1}{p_i}-1}t^{p_i-1}dt\right]\\
&\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3}f\left(c_1,c_2,x_3,\dots,x_n\right)dx_3\dots dx_n\\
\le\ &\cdots
\end{aligned}$$

$$= \frac{1}{2^n} \sum_{\substack{c_i,\tilde{c}_i=a_i \text{ or } b_i,\\ c_i\neq\tilde{c}_i,\ i=1,\ldots,n}} \prod_{i=1}^{n} \left[c_i \int_0^1 h_i(t)dt + \tilde{c}_i \int_0^1 h_i(t)\left(1-t^{p_i}\right)^{\frac{1}{p_i}-1} t^{p_i-1}dt \right] f\left(c_1,c_2,\ldots,c_n\right).$$

This finishes the second inequality in Theorem 2.5.

On the other hand, by induction again, the Fubini's theorem and the first inequality in Theorem 2.1 tell us that

$$\begin{aligned}
&\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \\
\geq\ & \frac{1}{2h_1\left(2^{-\frac{1}{p_1}}\right)} \int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2^{\frac{1}{p_1}}}, x_2,\ldots,x_n\right) dx_2\ldots dx_n \\
\geq\ & \frac{1}{2^2 h_1\left(2^{-\frac{1}{p_1}}\right) h_2\left(2^{-\frac{1}{p_2}}\right)} \int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3} f\left(\frac{a_1+b_1}{2^{\frac{1}{p_1}}}, \frac{a_2+b_2}{2^{\frac{1}{p_2}}}, x_2,\ldots,x_n\right) dx_3\ldots dx_n \\
\geq\ & \cdots \\
\geq\ & \frac{1}{2^n \prod_{i=1}^n h_i\left(2^{-\frac{1}{p_i}}\right)} f\left(\frac{a_1+b_1}{2^{\frac{1}{p_1}}},\ldots,\frac{a_n+b_n}{2^{\frac{1}{p_n}}}\right),
\end{aligned}$$

which completes the proof of the theorem. □

Taking $p_1 = p_2 = \cdots = p_n = p$ and $h_1 = \cdots = h_n = h$ in Theorem 2.5 implies that

Corollary 2.6. *Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be an integrable coordinated (p,h)-convex function. Then , for all $[\mathbf{a},\ \mathbf{b}] \subset \mathbb{R}_+^n$,*

$$\begin{aligned}
&\frac{1}{2^n h^n\left(2^{-\frac{1}{p}}\right)} f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{p}}}\right) \\
\leq\ & \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \\
\leq\ & \frac{1}{2^n|[\mathbf{a},\mathbf{b}]|} \sum_{\substack{c_i,\tilde{c}_i=a_i \text{ or } b_i,\\ c_i\neq\tilde{c}_i,\ i=1,\ldots,n}} \prod_{i=1}^{n} \left[c_i \int_0^1 h(t)dt + \tilde{c}_i \int_0^1 h(t)\left(1-t^{p}\right)^{\frac{1}{p}-1} t^{p-1}dt \right] f\left(\mathbf{c}\right).
\end{aligned}$$

And if $h_i = t^{s_i}$ on $(0,1)$, $i = 1,\ldots,n$ in Theorem 2.5, we have:

Corollary 2.7. *Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ and $f \in L\left(\mathbb{R}_+^n\right)$. If the partial mappings f_i are (p_i,s_i)-convex functions on $\mathbb{R}_+$ for $i = 1,2,\cdots,n$, respectively, then, for any $[\mathbf{a},\ \mathbf{b}] \subset \mathbb{R}_+^n$, we have*

$$\begin{aligned}
&2^{\sum_{i=1}^n \frac{s_i}{p_i}-n} f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{\mathbf{p}}}}\right) \\
\leq\ & \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \\
\leq\ & \frac{1}{2^n\prod_{i=1}^n(1+s_i)|[\mathbf{a},\mathbf{b}]|} \sum_{\substack{c_i,\tilde{c}_i=a_i \text{ or } b_i,\\ c_i\neq\tilde{c}_i, i=1,2,\ldots,n}} \prod_{i=1}^{n} \left[c_i + \tilde{c}_i \frac{s_i}{p_i} B\left(\frac{s_i}{p_i},\frac{1}{p_i}\right)\right] f(\mathbf{c}).
\end{aligned}$$

If taking $p_1 = p_2 = \cdots = p_n = p$ and $s_1 = s_2 = \cdots = s_n = s$, we obtain

Corollary 2.8. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be an integrable coordinated (p,s)-convex functions. Then, for all $[\mathbf{a},\ \mathbf{b}]\subset\mathbb{R}_+^n$, we have*

$$\begin{aligned} 2^{n\left(\frac{s}{p}-1\right)} f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{p}}}\right) &\le \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \\ &\le \frac{1}{[2(s+1)]^n\,|[\mathbf{a},\mathbf{b}]|}\sum_{\substack{c_i,\tilde{c}_i=a_i \text{ or } b_i,\\ c_i\neq\tilde{c}_i, i=1,2,\dots,n}}\prod_{i=1}^{n}\left[c_i+\tilde{c}_i\frac{s}{p}B\left(\frac{s}{p},\frac{1}{p}\right)\right]f(\mathbf{c}). \end{aligned}$$

Especially, if $s=1$, we have

Corollary 2.9. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be an integrable coordinated p-convex functions. Then, for $[\mathbf{a},\ \mathbf{b}]\subset\mathbb{R}_+^n$, we have*

$$\begin{aligned} 2^{n\left(\frac{1}{p}-1\right)} f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{p}}}\right) &\le \frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \\ &\le \frac{1}{4^n\,|[\mathbf{a},\mathbf{b}]|}\sum_{\substack{c_i,\tilde{c}_i=a_i \text{ or } b_i,\\ c_i\neq\tilde{c}_i, i=1,2,\dots,n}}\prod_{i=1}^{n}\left[c_i+\tilde{c}_i\frac{1}{p}B\left(\frac{1}{p},\frac{1}{p}\right)\right]f(\mathbf{c}). \end{aligned}$$

3. Some properties of mappings related to Hermite-Hadamard's inequalities

In this section, we mainly study some properties of the mappings defined in the first section.

Theorem 3.1. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an integrable (p,h)-convex function and the mapping $\mathfrak{H}:[0,1]\subset\mathbb{R}_+\to\mathbb{R}_+$ be defined by (1.3) for $n=1$.*

(i) *If f is a decreasing function, then $\mathfrak{H}(t)$ is a (p,h)-convex function in $[0,1]$.*

(ii) *For any $t\in(0,1]$, the following inequalities holds:*

$$\mathfrak{H}(t)\ge\frac{1}{2h\left(2^{-\frac{1}{p}}\right)}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right). \tag{3.1}$$

Furthermore, if f is a decreasing function on $\mathbb{R}_+$, then, for any $t\in(0,1]$,

$$\frac{1}{2h\left(2^{-\frac{1}{p}}\right)}\mathfrak{H}(0)\le\mathfrak{H}(t)\le\left[h\left(t^{\frac{1}{p}}\right)+2h\left(2^{-\frac{1}{p}}\right)h\left((1-t)^{\frac{1}{p}}\right)\right]\mathfrak{H}(1). \tag{3.2}$$

Proof. (i) Let $x,y\in[0,1]$ and $\lambda,\mu\ge0$ with $\lambda^p+\mu^p=1$. Using $\lambda+\mu\le1$, monotonicity and (p,h)-convexity of f, for all$t_1,t_2\in[0,1]$, we have

$$\begin{aligned} &\mathfrak{H}(\lambda t_1+\mu t_2)\\ =\;& \frac{1}{b-a}\int_a^b f\Big((\lambda t_1+\mu t_2)x+[1-(\lambda t_1+\mu t_2)]\frac{a+b}{2}\Big)dx\\ =\;& \frac{1}{b-a}\int_a^b f\Big(\lambda t_1x-\lambda t_1\frac{a+b}{2}+\mu t_2x-\mu t_2\frac{a+b}{2}+\frac{a+b}{2}\Big)dx\\ \le\;& \frac{1}{b-a}\int_a^b f\Big(\lambda t_1x-\lambda t_1\frac{a+b}{2}+\mu t_2x-\mu t_2\frac{a+b}{2}+(\lambda+\mu)\frac{a+b}{2}\Big)dx\\ =\;& \frac{1}{b-a}\int_a^b f\left(\lambda\left[t_1x+(1-t_1)\frac{a+b}{2}\right]+\mu\left[t_2x+(1-t_2)\frac{a+b}{2}\right]\right)dx\\ \le\;& \frac{1}{b-a}\int_a^b \left\{h(\lambda)\left[f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)\right]+h(\mu)\left[f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right]\right\}dx\\ =\;& h(\lambda)\mathfrak{H}(t_1)+h(\mu)\mathfrak{H}(t_2). \end{aligned}$$

(ii) Changing variable $u = tx + (1-t)\frac{a+b}{2}$, then

$$\mathfrak{H}(t) = \frac{1}{t(b-a)} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(u)du = \frac{1}{m-n} \int_{m}^{n} f(u)du,$$

where $m = tb + ((1-t)\frac{a+b}{2})$,$n = ta + (1-t)\frac{a+b}{2}$. By virtue of Theorem 1, we obtain

$$\frac{1}{m-n} \int_{m}^{n} f(u)du \geq \frac{1}{2h\left(2^{-\frac{1}{p}}\right)} f\left(\frac{m+n}{2^{\frac{1}{p}}}\right) = \frac{1}{2h\left(2^{-\frac{1}{p}}\right)} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right).$$

This finishes the proof of (3.1).

On the other hand, the monotonicity and nonnegativity of f show that

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \geq f\left(\frac{a+b}{2}\right) = \mathfrak{H}(0),$$

and

$$\begin{aligned}
\mathfrak{H}(t) &\leq \frac{1}{b-a} \int_a^b f\left(t^{\frac{1}{p}}x + (1-t)^{\frac{1}{p}} \frac{a+b}{2}\right) dx \\
&\leq h\left(t^{\frac{1}{p}}\right) \frac{1}{b-a} \int_a^b f(x)dx + h\left((1-t)^{\frac{1}{p}}\right) f\left(\frac{a+b}{2}\right) \\
&\leq h\left(t^{\frac{1}{p}}\right) \mathfrak{H}(1) + h\left((1-t)^{\frac{1}{p}}\right) f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \\
&\leq h\left(t^{\frac{1}{p}}\right) \mathfrak{H}(1) + 2h\left(2^{-\frac{1}{p}}\right) h\left((1-t)^{\frac{1}{p}}\right) \mathfrak{H}(1) \\
&= \left[h\left(t^{\frac{1}{p}}\right) + 2h\left(2^{-\frac{1}{p}}\right) h\left((1-t)^{\frac{1}{p}}\right)\right] \mathfrak{H}(1),
\end{aligned}$$

the last inequality is obtained by Theorem 2.1. Thus we finish the proof of the theorem. □

Theorem 3.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable (p,h)-convex function and the mapping $\mathfrak{I}(t) : [0,1] \subset \mathbb{R}_+ \to \mathbb{R}_+$ be defined by (1.4) for $n = 1$.*

(i) *If f is a decreasing function, then $\mathfrak{I}(t)$ is a (p,h)-convex function in $[0,1]$.*

(ii) *For any $t \in (0,1]$, we have*

$$\mathfrak{I}(t) \geq \frac{1}{2h\left(2^{-\frac{1}{p}}\right)(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) dxdy. \tag{3.3}$$

Furthermore, if f is a decreasing function, then

$$\frac{1}{2h\left(2^{-\frac{1}{p}}\right)} \mathfrak{I}\left(\frac{1}{2}\right) \leq \mathfrak{I}(t) \leq \left[h\left(t^{\frac{1}{p}}\right) + h\left((1-t)^{\frac{1}{p}}\right)\right] \mathfrak{I}(1). \tag{3.4}$$

Proof. (i) Let $t_1, t_2, \lambda, \mu \in [0,1]$ and $\lambda^p + \mu^p = 1$. For any $x, y \in [a,b]$, similar to the proof of Theorem 3.1, the monotonicity, $(p,h)-$convexity of f and the basic fact of $\lambda + \mu \leq 1$ show that

$$\begin{aligned}
\mathfrak{I}(\lambda t_1 + \mu t_2) &= \frac{1}{(b-a)^2} \int_a^b \int_a^b f((\lambda t_1 + \mu t_2)x + [1 - (\lambda t_1 + \mu t_2)]y) dxdy \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\lambda[t_1 x + (1-t_1)y] + \mu[t_2 x + (1-t_2)y]\right) dxdy \\
&\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left[h(\lambda) f\left(t_1 x + (1-t_1)y\right) + h(\mu) f\left(t_2 x + (1-t_2)y\right)\right] dxdy \\
&= h(\lambda)\mathfrak{H}(t_1) + h(\lambda)\mathfrak{h}(t_2).
\end{aligned}$$

(ii) It is not difficult to see that

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \le h\left(2^{-\frac{1}{p}}\right)[f(tx+(1-t)y)+f(ty+(1-t)x)]$$

holds for all $t\in[0,1]$ and $x,y\in[a,b]$. Integrating the both sides of the above inequality on $[a,b]^2$, the Fubini theorem means that

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right)dxdy \le 2h\left(2^{-\frac{1}{p}}\right)\mathfrak{I}(t)$$

holds for all $t\in(0,1]$, which gives the desire result.

Furthermore, the monotone decreasing and nonnegativity of f yield that

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right)dxdy \ge \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy = \mathfrak{I}\left(\frac{1}{2}\right)$$

and, for any $t\in(0,1]$,

$$\begin{aligned}\mathfrak{I}(t) &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t^{\frac{1}{p}}x+(1-t)^{\frac{1}{p}}y\right)dxdy\\ &\le h\left(t^{\frac{1}{p}}\right)\frac{1}{(b-a)^2}\int_a^b\int_a^b f(x)dxdy + h\left((1-t)^{\frac{1}{p}}\right)\frac{1}{(b-a)^2}\int_a^b\int_a^b f(y)dxdy\\ &= \left[h\left(t^{\frac{1}{p}}\right)+h\left((1-t)^{\frac{1}{p}}\right)\right]\mathfrak{I}(0) = \left[h\left(t^{\frac{1}{p}}\right)+h\left((1-t)^{\frac{1}{p}}\right)\right]\mathfrak{I}(1).\end{aligned}$$

Thus we complete the proof. □

As a consequence of Theorem 3.1 and Theorem 3.2, we have:

Corollary 3.3. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an integrable (p,s)-convex function and the mappings $\mathfrak{H}$ and $\mathfrak{I}$ are defined by (1.3) and (1.4) with $n=1$, respectively.*

(i) *If f is a decreasing function, then $\mathfrak{H}(t)$ and $\mathfrak{I}(t)$ are both (p,s)-convex functions on $[0,1]$.*

(ii) *For any $t\in(0,1]$, the following inequalities holds:*

$$\begin{aligned}\mathfrak{H}(t) &\ge 2^{\frac{s}{p}-1}f\left(\frac{a+b}{2^{\frac{1}{p}}}\right),\\ \mathfrak{I}(t) &\ge \frac{2^{\frac{s}{p}-1}}{(b-a)^2}\int_a^b\int_a^b f\Big(\frac{x+y}{2^{\frac{1}{p}}}\Big)dxdy.\end{aligned}$$

Furthermore, if f is a decreasing function on $\mathbb{R}_+$, then, for any $t\in(0,1]$,

$$\begin{aligned}2^{\frac{s}{p}-1}\mathfrak{H}(0) &\le \mathfrak{H}(t)\le\left[t^{\frac{s}{p}}+2^{1-\frac{s}{p}}(1-t)^{\frac{s}{p}}\right]\mathfrak{H}(1),\\ 2^{\frac{s}{p}-1}\mathfrak{I}\left(\frac{1}{2}\right) &\le \mathfrak{I}(t)\le\left[t^{\frac{s}{p}}+(1-t)^{\frac{s}{p}}\right]\mathfrak{I}(1).\end{aligned}$$

In particular, taking $s=1$ in Corollary 3.3, it extends Theorem 1.8.

Next we will generalize the proceeding results to multi-dimensions.

Theorem 3.4. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be integrable and f_i be $(p_i,h_i)-$convex for $i=1,\dots,n$, respectively. The mapping $\mathfrak{H}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}$ is defined by (1.3), then*

(i) *If f_i is monotone decreasing on $\mathbb{R}_+$, $i=1,\dots,n$, then the partial function $\mathfrak{H}_i$ is a (p_i,h_i)-convex function on $[0,1]$, $i=1,\dots,n$.*

(ii) *For any $\mathbf{t}\in(\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n$, the following inequalities holds:*

$$\mathfrak{H}(\mathbf{t}) \ge \frac{1}{2^n\prod_{i=1}^n h_i\left(2^{-\frac{1}{p_i}}\right)}f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{\mathbf{p}}}}\right). \tag{3.5}$$

Furthermore, if f_i is monotone decreasing on $\mathbb{R}_+$, $i = 1, \ldots, n$, respectively, we have

$$\frac{1}{2^n \prod_{i=1}^n h_i\left(2^{-\frac{1}{p_i}}\right)} \mathfrak{H}(\mathbf{0}) \le \mathfrak{H}(\mathbf{t}) \le \prod_{i=1}^n \left[h_i\left(t_i^{\frac{1}{p_i}}\right) + 2h_i\left(2^{-\frac{1}{p_i}}\right) h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right] \mathfrak{H}(\mathbf{1}). \quad (3.6)$$

Proof. (i) Without loss of generality, we just prove that $\mathfrak{H}_1(\cdot)$ is a (p_1, h_1)-convex function on $[0,1]$. For any $\xi, \eta, u, v \in [0,1]$ and $\xi^{p_1} + \eta^{p_1} = 1$, the Fubini theorem and the monotonicity of f_1 imply that that

$$\begin{aligned}
& \mathfrak{H}_1(\xi u + \eta v) \\
= & \frac{1}{\prod_{j=1}^n (b_j - a_j)} \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f\left((\xi u + \eta v) x_1 + [1 - (\xi u + \eta v)] \frac{a_1 + b_1}{2},\right. \\
& \left. t_2 x_2 + (1 - t_2) \frac{a_2 + b_2}{2}, \ldots, t_n x_n + (1 - t_n) \frac{a_n + b_n}{2}\right) dx_1 dx_2 \ldots dx_n \\
\le & \frac{1}{\prod_{j=1}^n (b_j - a_j)} \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f\left(\xi\left[u x_1 + (1-u)\frac{a_1 + b_1}{2}\right] + \eta\left[v x_1 + (1-v)\frac{a_1 + b_1}{2}\right],\right. \\
& \left. t_2 x_2 + (1 - t_2) \frac{a_2 + b_2}{2}, \ldots, t_n x_n + (1 - t_n) \frac{a_n + b_n}{2}\right) dx_1 dx_2 \ldots dx_n \\
\le & h_1(\xi) \mathfrak{H}_1(u) + h_1(\eta) \mathfrak{H}_1(v).
\end{aligned}$$

(ii) It follows from (3.1) and Fubini's theorem that

$$\begin{aligned}
& \mathfrak{H}(\mathbf{t}) \\
& \ge \frac{1}{2h_1\left(2^{-\frac{1}{p}}\right) \prod_{i=2}^n (b_i - a_i)} \\
& \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} f\left(\frac{a_1 + b_1}{2^{\frac{1}{p_1}}}, t_2 x_2 + (1 - t_2)\frac{a_2 + b_2}{2}, \ldots, t_n x_n + (1 - t_n)\frac{a_n + b_n}{2}\right) dx_2 \ldots dx_n \\
& \ge \frac{1}{2^2 h_1\left(2^{-\frac{1}{p_1}}\right) h_2\left(2^{-\frac{1}{p_2}}\right) \prod_{i=3}^n (b_i - a_i)} \int_{a_n}^{b_n} \cdots \int_{a_3}^{b_3} \\
& f\left(\frac{a_1 + b_1}{2^{\frac{1}{p_1}}}, \frac{a_2 + b_2}{2^{\frac{1}{p_2}}}, t_3 x_3 + (1 - t_3)\frac{a_3 + b_3}{2}, \ldots, t_n x_n + (1 - t_n)\frac{a_n + b_n}{2}\right) dx_3 \ldots dx_n \\
& \ge \cdots \\
& \ge \frac{1}{2^n \prod_{i=1}^n h_i\left(2^{-\frac{1}{p_i}}\right)} f\left(\frac{a_1 + b_1}{2^{\frac{1}{p_1}}}, \ldots, \frac{a_n + b_n}{2^{\frac{1}{p_n}}}\right).
\end{aligned}$$

On the other hand, the monotonicity of f_i, $i = 1, \ldots, n$, and (3.2) show that

$$\begin{aligned}
f\left(\frac{a_1 + b_1}{2^{\frac{1}{p_1}}}, \ldots, \frac{a_n + b_n}{2^{\frac{1}{p_n}}}\right) \ge f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2^{\frac{1}{p_2}}}, \ldots, \frac{a_n + b_n}{2^{\frac{1}{p_n}}}\right) \ge \cdots \ge f\left(\frac{\mathbf{a} + \mathbf{b}}{2}\right) \\
= H(0),
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{H}(\mathbf{t}) \le & \frac{\left[h_1\left(t_1^{\frac{1}{p_1}}\right) + 2h_1\left(2^{-\frac{1}{p_1}}\right) h_1\left((1-t_1)^{\frac{1}{p_1}}\right)\right]}{\prod_{i=i}^n (b_i - a_i)} \int_{a_1}^{b_1} \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \\
& f\left(x_1, t_2 x_2 + (1 - t_2)\frac{a_2 + b_2}{2}, \ldots, t_n x_n + (1 - t_n)\frac{a_n + b_n}{2}\right) dx_2 \ldots dx_n dx_1
\end{aligned}$$

$$\leq \frac{\prod_{i=1}^{2}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+2h_i\left(2^{-\frac{1}{p_i}}\right)h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]}{\prod_{i=i}^{n}(b_i-a_i)}\times$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3} f\left(x_1,x_2,x_3,\ldots,t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_3\ldots dx_ndx_1dx_2$$

$$\leq\cdots$$

$$\leq \frac{\prod_{i=1}^{n}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+2h_i\left(2^{-\frac{1}{p_i}}\right)h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]}{\prod_{i=i}^{n}(b_i-a_i)}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f\left(x_1,\ldots,x_n\right)dx_1\ldots dx_n$$

$$=\prod_{i=1}^{n}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+2h_i\left(2^{-\frac{1}{p_i}}\right)h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]\mathfrak{H}(\mathbf{1}).$$

Therefore, we obtain the desired results. □

Theorem 3.5. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be integrable and f_i be $(p_i,h_i)-$convex ,$i=1,2,\cdots,n$, respectively. Define the mapping $\mathfrak{I}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}_+$ by (1.4), then*

(i) *If f_i is monotone decreasing, $i=1,2,\ldots,n$, then $\mathfrak{I}_i$ is a (p_i,h_i)-convex function on $[0,1]$, $i=1,2,\cdots,n$.*

(ii) *For any $\mathbf{t}\in(\mathbf{0},\mathbf{1}]$, the following inequality holds:*

$$\mathfrak{I}(\mathbf{t})\geq\frac{1}{2^n\prod_{i=1}^{n}h_i\left(2^{-\frac{1}{p_i}}\right)}\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{\mathbf{x}+\mathbf{y}}{2^{\frac{1}{\mathbf{p}}}}\right)d\mathbf{x}d\mathbf{y}. \tag{3.7}$$

Furthermore, if f_i is monotone decreasing, $i=1,2,\ldots,n$, we have

$$\frac{1}{2^n\prod_{i=1}^{n}h_i\left(2^{-\frac{1}{p_i}}\right)}\mathfrak{I}\left(\frac{\mathbf{1}}{2}\right)\leq\mathfrak{I}(\mathbf{t})\leq\prod_{i=1}^{n}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]\mathfrak{I}(\mathbf{1}). \tag{3.8}$$

Proof. Similar to the proof of Theorem 3.4 (i), Theorem 3.5 (i) is easily achieved by Theorem 3.2 (i). Now we turn to prove the second part of Theorem 3.5. It follows from (3.3) and the Fubini theorem that

$$\mathfrak{I}(\mathbf{t})$$

$$\geq \frac{1}{2h_1\left(2^{-\frac{1}{p_1}}\right)\prod_{i=1}^{n}(b_i-a_i)^2}\int_{a_n}^{b_n}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_2}^{b_2}$$

$$f\left(\frac{x_1+y_1}{2^{\frac{1}{p_1}}},t_2x_2+(1-t_2)\,y_2,\ldots,t_nx_n+(1-t_n)\,y_n\right)dx_1dy_1\ldots\ dx_ndy_n$$

$$\geq \frac{1}{2^2h_1\left(2^{-\frac{1}{p_1}}\right)h_2\left(2^{-\frac{1}{p_2}}\right)\prod_{i=1}^{n}(b_i-a_i)^2}\int_{a_n}^{b_n}\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3}\int_{a_3}^{b_3}$$

$$f\left(\frac{x_1+y_1}{2^{\frac{1}{p_1}}},\frac{x_2+y_2}{2^{\frac{1}{p_2}}},t_3x_3+(1-t_3)\,y_3,\ldots,t_nx_n+(1-t_n)\,y_n\right)dx_1dy_1\ldots\ dx_ndy_n$$

$$\geq\ \cdots$$

$$\geq \frac{1}{2^n\prod_{i=1}^{n}h_i\left(2^{-\frac{1}{p_i}}\right)}\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{x_1+y_1}{2^{\frac{1}{p_1}}},\ldots,\frac{x_n+y_n}{2^{\frac{1}{p_n}}}\right)d\mathbf{x}d\mathbf{y}.$$

Furthermore, similar to the proof of the first part of (3.6), the monotonicity of f_i tell us that

$$\begin{aligned} &\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{x_1+y_1}{2^{\frac{1}{p_1}}},\ldots,\frac{x_n+y_n}{2^{\frac{1}{p_n}}}\right)d\mathbf{x}d\mathbf{y} \\ \geq\; &\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)d\mathbf{x}d\mathbf{y} = \mathfrak{I}\left(\frac{\mathbf{1}}{\mathbf{2}}\right). \end{aligned} \tag{3.9}$$

On the other hand, the right-hand side of (3.6) and the Fubini theorem imply that

$$\begin{aligned} &\mathfrak{I}(\mathbf{t}) \\ &\leq \left[h_1\left(t_1^{\frac{1}{p_1}}\right)+h_1\left((1-t_1)^{\frac{1}{p_1}}\right)\right]\frac{1}{(b_1-a_1)\prod_{i=2}^n(b_i-a_i)^2}\times \\ &\int_{a_1}^{b_1}\int_{a_n}^{b_n}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_2}^{b_2} f\left(x_1,\frac{x_2+y_2}{2^{\frac{1}{p_2}}},\ldots,\frac{x_n+y_n}{2^{\frac{1}{p_n}}}\right)dx_2dy_2\ldots\; dx_ndy_ndx_1 \\ &\leq \prod_{i=1}^{2}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]\frac{1}{(b_1-a_1)(b_2-a_2)\prod_{i=3}^n(b_i-a_i)^2}\times \\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_n}^{b_n}\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3}\int_{a_3}^{b_3} f\left(x_1,x_2,\frac{x_3+y_3}{2^{\frac{1}{p_3}}},\ldots,\frac{x_n+y_n}{2^{\frac{1}{p_n}}}\right)dx_3dy_3\ldots\; dx_ndy_ndx_1dx_2 \\ &\leq\cdots \\ &\leq \prod_{i=1}^{n}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})\,d\mathbf{x} \\ &= \prod_{i=1}^{n}\left[h_i\left(t_i^{\frac{1}{p_i}}\right)+h_i\left((1-t_i)^{\frac{1}{p_i}}\right)\right]\mathfrak{I}(\mathbf{1}), \end{aligned}$$

which, combing with (3.9), finishes the proof of Theorem 3.5. □

Corollary 3.6. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be integrable and coordinated $(p,h)-$convex. Suppose that the mappings $\mathfrak{H}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}_+$ and $\mathfrak{I}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}_+$ are defined by* (1.3) *and* (1.4)*, respectively.*

(i) *If f_i are monotone decreasing on $\mathbb{R}_+$, $i=1,\ldots,n$, then the functions $\mathfrak{H}$ and $\mathfrak{I}$ are both coordinated $(p,h)-$convex on $[\mathbf{0},\mathbf{1}]$.*

(ii) *For any $\mathbf{t}\in(\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n$, the following inequalities holds:*

$$\begin{aligned} \mathfrak{H}(\mathbf{t}) &\geq \frac{1}{2^nh^n\left(2^{-\frac{1}{p}}\right)}f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{p}}}\right), \\ \mathfrak{I}(\mathbf{t}) &\geq \frac{1}{2^nh^n\left(2^{-\frac{1}{p}}\right)}\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{\mathbf{x}+\mathbf{y}}{2^{\frac{1}{p}}}\right)d\mathbf{x}d\mathbf{y}. \end{aligned}$$

Furthermore, if f_i is monotone decreasing on $\mathbb{R}_+$, $i=1,\ldots,n$, respectively, we have

$$\begin{aligned} \frac{1}{2^nh^n\left(2^{-\frac{1}{p}}\right)}\mathfrak{H}(\mathbf{0}) &\leq \mathfrak{H}(\mathbf{t}) \leq \prod_{i=1}^{n}\left[h\left(t_i^{\frac{1}{p}}\right)+2h\left(2^{-\frac{1}{p}}\right)h\left((1-t_i)^{\frac{1}{p}}\right)\right]\mathfrak{H}(\mathbf{1}), \\ \frac{1}{2^nh^n\left(2^{-\frac{1}{p}}\right)}\mathfrak{I}\left(\frac{\mathbf{1}}{2}\right) &\leq \mathfrak{I}(\mathbf{t}) \leq \prod_{i=1}^{n}\left[h\left(t_i^{\frac{1}{p}}\right)+h\left((1-t_i)^{\frac{1}{p}}\right)\right]\mathfrak{I}(\mathbf{1}). \end{aligned}$$

Corollary 3.7. *Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ be integrable and f_i be $(p_i,s_i)-$convex for $i=1,\ldots,n$. Suppose that the mappings $\mathfrak{H}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}_+$ and $\mathfrak{I}(\mathbf{t}):[\mathbf{0},\mathbf{1}]\subset\mathbb{R}_+^n\to\mathbb{R}_+$ are defined by* (1.3) *and* (1.4)*, respectively.*

(i) *If f_i is monotone decreasing on $\mathbb{R}_+$, $i = 1, \ldots, n$, then the partial functions $\mathfrak{H}_i$ and $\mathfrak{I}_i$ are both (p_i, s_i)-convex on $[0, 1]$, $i = 1, \ldots, n$.*

(ii) *For any $\mathbf{t} \in (\mathbf{0}, \mathbf{1}] \subset \mathbb{R}^n_+$, the following inequalities holds:*

$$\begin{aligned} \mathfrak{H}(\mathbf{t}) &\geq 2^{\sum_{i=1}^n \frac{s_i}{p_i} - n} f\left(\frac{\mathbf{a}+\mathbf{b}}{2^{\frac{1}{\mathbf{p}}}}\right), \\ \mathfrak{I}(\mathbf{t}) &\geq 2^{\sum_{i=1}^n \frac{s_i}{p_i} - n} \frac{1}{|[\mathbf{a},\mathbf{b}]|^2} \int_{[\mathbf{a},\mathbf{b}]} \int_{[\mathbf{a},\mathbf{b}]} f\left(\frac{\mathbf{x}+\mathbf{y}}{2^{\frac{1}{\mathbf{p}}}}\right) d\mathbf{x} d\mathbf{y}. \end{aligned}$$

Furthermore, if f_i is monotone decreasing on $\mathbb{R}_+$, $i = 1, \ldots, n$, we have

$$\begin{aligned} 2^{\sum_{i=1}^n \frac{s_i}{p_i} - n} \mathfrak{H}(\mathbf{0}) &\leq \mathfrak{H}(\mathbf{t}) \leq \prod_{i=1}^n \left[t_i^{\frac{s_i}{p_i}} + 2^{1-\frac{s_i}{p_i}} (1-t_i)^{\frac{s_i}{p_i}} \right] \mathfrak{H}(\mathbf{1}), \\ 2^{\sum_{i=1}^n \frac{s_i}{p_i} - n} \mathfrak{I}\left(\frac{\mathbf{1}}{2}\right) &\leq \mathfrak{I}(\mathbf{t}) \leq \prod_{i=1}^n \left[t_i^{\frac{s_i}{p_i}} + (1-t_i)^{\frac{s_i}{p_i}} \right] \mathfrak{I}(\mathbf{1}). \end{aligned}$$

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