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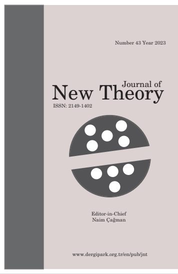

On the Orbit Problem of Free Lie Algebras

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Abstract — By operationalizing F_n as a free Lie Algebra of finite rank n, this work considers the orbit problem for F_n. The orbit problem is the following: given an element $u \in F_n$ and a finitely generated subalgebra H of F_n, does H meet the orbit of u under the automorphism group $AutF_n$ of F_n? It is proven that the orbit problem is decidable for finite rank n, $n \geqslant 2$. Furthermore, we solve a particular instance of the problem – i.e., whether H contains a primitive element of F_n. In addition, some applications are provided. Finally, the paper inquires the need for further research.

Keywords *Orbit, automorphism, free Lie algebras*

Mathematics Subject Classification (2020) 17B01, 17B40

1. Introduction

The orbit problem is one of the most studied algorithmic problems in algebra. The problem generally concerns a subalgebra H of an algebra F, the orbit of an element u of F under the action of a subgroup G of Aut$F$, and it is checked whether or not the subalgebra contains the orbit of a given element. Indeed, the orbit problem has been extensively studied in various algebraic structures, including groups, Lie algebras, and associative algebras. Computational group theory, in particular, has been a prominent field where the orbit problem has been investigated. Whitehead's [1] work in computational group theory proved the decidability of the orbit problem for free groups. This means that there exists an algorithm that can effectively determine whether the orbit of a given element under the action of a subgroup of the automorphism group lies within a subgroup of a free group. In [2, 3], the authors established similar results regarding the orbit problem of finitely generated subgroups. The problem was also studied for a cyclic subgroup of the automorphism group of a free group, e.g., [4, 5]. Furthermore, in [6], Kozen focused on the decidability of the orbit problem for infinite algebras. In 2011, Bahturin and Olshanskii [7] investigated if the subalgebra membership problem is decidable for free Lie algebras. The membership problem for free Lie algebras asks whether a given element belongs to a subalgebra of a free Lie algebra. This problem's decidability would imply a systematic and algorithmic approach to determine whether a given element belongs to a subalgebra. The results of this study determined that the subalgebra membership problem for free Lie algebras is, in fact, undecidable. This means that there is no general algorithm that can solve this problem for all cases. Consequently, the subalgebra membership problem for free Lie algebras remains an open and challenging research topic in algebraic computation. It is worth noting that even though the

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subalgebra membership problem is undecidable for free Lie algebras, specific cases may exist where the problem can be solved. In the context of the present paper, the orbit problem is a particular case of the membership problem. In the case of free Lie algebras, the orbit problem considers whether an automorphic image of a given Lie element is contained in a finitely generated subalgebra, while the membership problem asks whether a given element belongs to a free Lie algebra or a given finitely generated subalgebra.

This paper considers the orbit problem for finitely generated free Lie algebras. The technique used to solve the problem is inspired by the results of a similar problem in groups [3]. We give algorithms if an automorphic image of a given Lie element u is contained by a given finitely generated subalgebra H of a free Lie algebra F_n with finite rank n such that $n \geq 2$. Moreover, we prove that it is decidable whether or not a primitive element is contained by a given finitely generated subalgebra H.

2. Preliminaries

Let F_n be a free Lie algebra generated by $X = \{x_1, x_2, \cdots, x_n\}$ over a field K of characteristic 0. Denote by $U(F_n)$, the universal enveloping algebra of F_n, i.e., the free associative algebra with the same generating set X over the field K. There is the augmentation homomorphism $\varepsilon : U(F_n) \to K$ defined by $\varepsilon(x_i) = 0,\ i \in \{1, 2, \cdots, n\}$. Fox derivations [8,9]

$$\frac{\partial}{\partial x_i} : U(F_n) \to U(F_n), \quad i \in \{1, 2, \cdots, n\}$$

satisfy the following conditions for each $a, b \in K$ and $u, v \in U(F_n)$,

i. $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, (Kronecker delta)

ii. $\frac{\partial}{\partial x_i}(au + bv) = a\frac{\partial}{\partial x_i}(u) + b\frac{\partial}{\partial x_i}(v)$

iii. $\frac{\partial}{\partial x_i}(uv) = u\frac{\partial}{\partial x_i}(v) + \varepsilon(v)\frac{\partial}{\partial x_i}(u)$

such that $\frac{\partial}{\partial x_i}(a) = 0$, for any $a \in K$. A primitive element in F_n is an element belonging to a free generating set of F_n. Given an arbitrary element u in F_n, the rank of u, denoted by rank(u), is defined as the least number of free generators from X on which the image of u under any automorphism of F_n can depend. This definition is in line with the work of [9]. We introduce the left $U(F_n)$-module M_u generated by the elements $\frac{\partial u}{\partial x_i}$, for $i \in \{1, \cdots, n\}$. The algebra $U(F_n)$ as a left $U(F_n)$-module is a free cyclic module. It is known that any left ideal of a free associative algebra is a free module of unique rank [10]. We denote the rank of the module M_u as rank(M_u).

Lemma 2.1. [11] Let $u \in F_n$ and $\varphi \in \mathrm{Aut}F_n$. Then, rank$(M_{\varphi(u)})$ =rank(M_u) =rank(u).

Lemma 2.2. [11] Let H be a subalgebra generated by $\{x_1, x_2, \cdots, x_r\}$, $1 \leq r < n$ and $u \in F_n$. If rank$(M_u) \leq r$, then there is an automorphism φ of F_n such that $\varphi(u) \in H$.

For an element u of F_n, we write $u = u(x_1, \cdots, x_k)$ if u depends on the generators $x_1, \cdots, x_k$. We use bracket notation $[u, v]$ to denote the Lie product of elements u and v of F_n. Lie monomials of F_n are defined in the usual way as non-zero Lie products of elements of X. The degree of a monomial is the length of this product. We call an element u of F_n is homogeneous if it is a linear combination of the monomials with the same degree. A subalgebra of F_n generated by a set Y is denoted by $\langle Y \rangle$.

Definition 2.3. [12] We define elementary transformations of F_n by one of the following transformations applied to X

i. A non-singular linear transformation is applied to X

ii. An element x of X is replaced by $x + u(x_1 \cdots, x_k)$ where u is an expression in the elements

$x_1, \cdots, x_k$ of $X \backslash \{x\}$

In [12], Cohn proved that every automorphism of a finitely generated free Lie algebra is a composition of elementary transformations.

Proposition 2.4. [12] Every automorphism φ of F_2 belongs to the general linear group $GL_2(K)$ and is defined by

$$\begin{aligned} \varphi \quad : \quad & x_1 \rightarrow \alpha x_1 + \beta x_2 \\ & x_2 \rightarrow \gamma x_1 + \delta x_2 \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in K$ and $\alpha\delta - \beta\gamma \neq 0$.

Proposition 2.5. [13] An endomorphism $\varphi : F_2 \rightarrow F_2$ defined as

$$\begin{aligned} \varphi \quad : \quad & x_1 \rightarrow \alpha x_1 + \beta x_2 \\ & x_2 \rightarrow \gamma x_1 + \delta x_2 \end{aligned}$$

is an automorphism if and only if

$$[\varphi(x_1), \varphi(x_2)] = k[x_1, x_2]$$

where $\alpha, \beta, \gamma, \delta \in K$, $k = \alpha\delta - \beta\gamma \neq 0$.

Thus, we can decide whether a given pair of elements of F_2 generates this algebra with this criterion.

3. The Orbit Problem

In this section, we discuss the decidability of the orbit problem and the existence of primitive elements in a finitely generated subalgebra H of a free Lie algebra F_n with finite rank $n \geq 2$. Decidability of the orbit problem means that there exists an algorithm or a systematic procedure that can determine whether the orbit of a given element under the action of a given subgroup of automorphisms belongs to a subalgebra. Firstly, we prove that for the case of rank 2, it is possible to decide whether the orbit of a given element $u \in F_2$ under the action of $\mathrm{Aut}F_2$ is in H. This result is significant because it establishes a decision algorithm for a specific case of the orbit problem. In addition, we show that for rank 2, it is also possible to decide whether or not H contains a primitive element. Furthermore, we extend the results to larger ranks and provide algorithms to solve the orbit problem and determine the existence of primitive elements in H, for $n > 2$.

3.1. Case of Rank $n = 2$

Theorem 3.1. Given $u \in F_2$ and a finitely generated subalgebra H of F_2, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_2$.

Proof.
Let F_2 be a free Lie algebra generated by $\{x_1, x_2\}$ and H be a finitely generated subalgebra of F_2. Given $\varphi \in \mathrm{Aut}F_2$ defined by $\varphi(x_1) = a$ and $\varphi(x_2) = b$. Since φ is an automorphism, the set $\{a, b\}$ freely generates F_2. For any element $u = u(x_1, x_2) \in F_2$,

$$\varphi(u(x_1, x_2)) = u(\varphi(x_1), \varphi(x_2)) = u(a, b)$$

Thus, if $u \in F_2$, then $\varphi(u) \in H$ if and only if $u(a, b) \in H$ for some free generating set $\{a, b\}$ of F_2. By Proposition 2.5,

$$[a, b] = [\varphi(x_1), \varphi(x_2)] = \lambda[x_1, x_2]$$

where $\lambda \in K \backslash \{0\}$. Hence, we obtain that there exists an automorphism φ such that $\varphi(u) \in H$ if and only if the following system admits a solution

$$[a, b] = \lambda[x_1, x_2]$$

and

$$u(a, b) = h$$

where $h \in H$, $\lambda \in K\backslash\{0\}$, and a and b are free generators of F_2. This completes the proof. □

Example 3.2. Given a subalgebra H generated by the subset $\{[x_1, x_2], x_2\}$ of F_2. Consider the element $u(x_1, x_2) = x_1 + [[x_1, x_2], x_1]$ of F_2. We find a solution to the system

$$[a, b] = \lambda[x_1, x_2]$$

and

$$u(a, b) = h$$

where $h \in H$, $\lambda \in K\backslash\{0\}$, and a and b are free generators of F_2. It implies

$$u(a, b) = a + [[a, b], a] = \alpha x_2 + \beta[[x_1, x_2], x_2] \tag{1}$$

where $\alpha, \beta \in K\backslash\{0\}$. By grading, $a = \alpha x_2$, and replacing a in Equation 1, $b = -\dfrac{\beta}{\alpha^2}x_1$ can be obtained. Then, by the equation

$$[a, b] = -\frac{\beta}{\alpha}[x_1, x_2]$$

a and b are free generators. Hence, by Theorem 3.1, there exists an automorphism φ such that $\varphi(u) \in H$.

Corollary 3.3. Let H be a subalgebra of F_2. It is decidable whether or not H contains a primitive element.

We consider a tuple element of F_2 rather than a single element in the following theorem.

Theorem 3.4. Let $u_1, u_2, \cdots, u_k \in F_2$ and $H_1, H_2, \cdots, H_k, H$, and K be subalgebras of F_2. The following problems are decidable

i. whether $\varphi(u_1) \in H_1, \cdots, \varphi(u_k) \in H_k$, for some $\varphi \in \mathrm{Aut}F_2$

ii. whether $\varphi(K) \subseteq H$, for some $\varphi \in \mathrm{Aut}F_2$

Proof.
Let $u_1, u_2, \cdots, u_k \in F_2$ and $H_1, H_2, \cdots, H_k, H$, and K be subalgebras of F_2.

i. We prove this statement as Theorem 3.1, by reduction to a system of equations. Let F_2 be a free Lie algebra generated by $\{x_1, x_2\}$. We consider the system

$$[a, b] = \alpha[x_1, x_2]$$

and

$$u_i(a, b) = h_i, \quad i \in \{1, 2, \cdots, k\}$$

where $h_1, \cdots, h_k \in H$, $\alpha \in K\backslash\{0\}$, and a and b are free generators of F_2. Clearly, if this system admits a solution, then $\varphi(u_i) \in H_i$, $i \in \{1, 2, \cdots, k\}$.

ii. This statement is a particular case of *i*, when $\{u_1, u_2, \cdots, u_k\}$ is a generating set of K and $H_1 = H_2 = \cdots = H_k = H$. □

3.2. Case of Rank $n > 2$

Theorem 3.5. Let u be a homogeneous element of F_n and H be a subalgebra of F_n. If H is a free factor of F_n or rank$H = 1$, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$.

PROOF.
Assume that H is a free factor of F_n, i.e., $F_n = H * G$ where rank$H = r$, $1 < r < n$, and G is a subalgebra of F_n. Let $u \in F_n$ and M_u be the left $U(F_n)-$module generated by $\frac{\partial u}{\partial x_i}$, $i \in \{1, \cdots, n\}$. By Lemma 2.1,

$$\mathrm{rank}u = \mathrm{rank}M_u = \mathrm{rank}kM_{\varphi(u)}$$

for some $\varphi \in \mathrm{Aut}F_n$. By [9], we can compute a minimum rank element v in the automorphic orbit

$$\mathrm{Orb}(u) = \{\psi(u) : \psi \in \mathrm{Aut}F_n\}$$

of u. If rank$v = r$, it is easily verified that $\phi(v) \in H$ for some automorphism ϕ of F_n by Lemma 2.2. Thus, $\phi(v) = \varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$. Assume that $H = \langle y \rangle$, for an element y of F_n. Given $u = \alpha u_1$ and $y = \beta y_1$ where $\alpha, \beta \in K\backslash\{0\}$ and $u_1, y_1 \in F_n$. If $\varphi(u) \in H$, then

$$\varphi(u) = \alpha\varphi(u_1) = \gamma y = \gamma\beta y_1$$

where $\gamma \in K$. It implies $\alpha = \gamma\beta$ and $\varphi(u_1) = y_1$. Therefore, we obtain $\varphi(u) \in H$ if and only if $\varphi(u_1) = y_1$, i.e., u_1 and y_1 are in each other's automorphic orbit if and only if $\varphi(u) \in H$. $\square$

We require the following technical result.

Theorem 3.6. Let $u \in F_n$. $A = \{x_1, x_2, \cdots, x_{n-1}, u\}$ is a free generating set of F_n if and only if $u = \alpha x_n + f(x_1, \cdots, x_{n-1})$ where $\alpha \in K\backslash\{0\}$ and $f(x_1, \cdots, x_{n-1})$ is an element of F_n depends on the free generators $x_1, \cdots, x_{n-1}$.

PROOF.
If A is a free generating set then the Jacobian matrix $J(A)$ is invertible over $U(F_n)$ by [14]. The Jacobian matrix

$$J(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \end{pmatrix}$$

can be reduced to

$$J(A)^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial u}{\partial x_n} \end{pmatrix}$$

by applying elementary transformations to its rows. Clearly, $J(A)$ is invertible if and only if $J(A)^*$ is invertible. Therefore, $\frac{\partial u}{\partial x_n}$ is an invertible element of $U(F_n)$. Since the only invertible elements of $U(F_n)$ are the elements of the field K, $\frac{\partial u}{\partial x_n}$ belongs to K. Thus, for a nonzero element $\alpha \in K$, $\frac{\partial u}{\partial x_n} = \alpha$ and the element u is of the form

$$\alpha x_n + f(x_1, \cdots, x_{n-1})$$

Conversely, if

$$u \in Kx_n + \langle x_1, \cdots, x_{n-1}\rangle$$

then $J(A)$ is invertible. Hence, A is a free generating set. $\square$

Proposition 3.7. Let $\{v_1, \cdots, v_{n-1}\}$ be a primitive subset of F_n. Then, there exists a set

$$A = \{w \in F_n \mid \{v_1, \cdots, v_{n-1}, w\} \text{ is a free generating set of } F_n\}$$

PROOF.
Let $\{v_1, \cdots, v_{n-1}\}$ be a primitive subset in F_n and φ be an automorphism of F_n defined by

$$\begin{aligned} \varphi : x_i &\to v_i \\ x_n &\to z \end{aligned}$$

where $1 \leq i \leq n-1$ and $z \in F_n$. Then, $\{x_1, \cdots, x_{n-1},\ \varphi^{-1}(z)\}$ is a free generating set of F_n. This shows that

$$\varphi^{-1}(z) \in Kx_n + \langle x_1, \cdots, x_{n-1}\rangle$$

by Theorem 3.6. Thus, $z \in K\varphi(x_n)+\langle v_1, \cdots, v_{n-1}\rangle$, and we obtain a free generating set $\{v_1, \cdots, v_{n-1}, z\}$. Hence, we obtain a set A such that

$$A = K\varphi(x_n) + \langle v_1, \cdots, v_{n-1}\rangle = Kz + \langle v_1, \cdots, v_{n-1}\rangle, \quad z \in F_n$$

$\square$

Theorem 3.8. Given $u \in F_n$ and a subalgebra H of F_n. If $\text{rank}H = n-1$, then it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$.

PROOF.
Let H be a subalgebra of F_n generated by the set $\{v_1, \cdots, v_{n-1}\}$ freely and φ be an automorphism of F_n. Assume that $\varphi(x_i) = v_i$, for $1 \leq i \leq n-1$. Consider the set

$$A = \{w \in F_n \mid \{v_1, \cdots, v_{n-1}, w\} \text{ is a free generating set of } F_n\}$$

It implies $w = \varphi(x_n)$. By [15],

$$F_n/\langle x_n\rangle \cong \langle x_1, \cdots, x_{n-1}\rangle$$

and $\langle x_1, \cdots, x_{n-1}\rangle$ is a free Lie algebra. For $u \in F_n$,

$$u + \langle x_n\rangle \in \langle x_1, \cdots, x_{n-1}\rangle$$

Thus, it is obtained

$$u = \sum \alpha_s[\cdots[x_n, x_{i_1}], \cdots], x_{i_s}] + f(x_1, \cdots, x_{n-1})$$

where $x_{i_1}, x_{i_2}, \cdots, x_{i_s} \in \{x_1, \cdots, x_{n-1}\}$. We compute

$$\begin{aligned} \varphi(u) &= \sum \alpha_s[\cdots[\varphi(x_n), \varphi(x_{i_1})]\cdots], \varphi(x_{i_s})] + \varphi(f(x_1, \cdots, x_{n-1})) \\ &= \sum \alpha_s[\cdots[w, v_{i_1}]\cdots]v_{i_s}] + f(v_1, \cdots, v_{n-1}) \end{aligned}$$

Therefore, we decide whether there exists some $w \in A$ such that $\varphi(u) \in H$. This is equivalent to deciding whether the equation

$$y = \sum \alpha_s[\cdots[w, v_{i_1}]\cdots]v_{i_s}] + f(v_1, \cdots, v_{n-1})$$

on the variables w and y has a solution in F_n with $w \in A$ and $y \in H$. $\square$

Corollary 3.9. Given $u \in F_n$ and a subalgebra H of F_n. Then, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$.

PROOF.
Let $K_i = \langle x_1, \cdots, x_i \rangle$, $i \in \{1, \cdots, n\}$. It is known that K_{i-1} is a subalgebra of K_i and by [15]

$$K_i/\langle x_i \rangle \cong K_{i-1}, \quad i \in \{2, \cdots, n\}$$

Therefore, for an element u of K_n, we have $u+\langle x_n \rangle \in K_{n-1}$. By the same way $u+\langle x_n \rangle+\langle x_{n-1} \rangle \in K_{n-2}$ and with consecutive applications $u + \langle x_n \rangle + \cdots + \langle x_{n-r} \rangle \in K_r$ are obtained. Hence,

$$u = \sum \alpha_{n_s}[\cdots[x_n, y_{n_1}]\cdots], y_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[x_{n-r}, y_{(n-r)_1}], \cdots], y_{(n-r)_s}] + f(x_1, \cdots, x_r)$$

where $y_{j_1}, \cdots, y_{j_s} \in \{x_1, \cdots, x_{j-1}\}$ and $j = n-r, \cdots, n$. Let H be a subalgebra of F_n freely generated by a set $\{v_1, \cdots, v_r\}$, $r < n$, and φ be an automorphism of F_n. Assume that $\varphi(x_i) = v_i$, for $1 \leq i \leq r$, and $\varphi(x_i) = w_i$, for $r+1 \leq i \leq n$. Hence, we compute

$$\varphi(u) = \sum \alpha_{n_s}[\cdots[w_n, v_{n_1}]\cdots]v_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}]\cdots], v_{(n-r)_s}] + f(v_1, \cdots, v_r)$$

Therefore, we decide whether the equation

$$y = \sum \alpha_{n_s}[\cdots[w_n, v_{n_1}], \cdots], v_{n_s}] + \cdots + \sum \alpha_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}], \cdots], v_{(n-r)_s}] + f(v_1, \cdots, v_r)$$

has a solution on the variables $w_{n-r}, \cdots w_n$ of F_n and $y \in H$. □

Corollary 3.10. Let H be a subalgebra of F_n. Then, it is decidable whether or not H contains a primitive element.

Theorem 3.11. Let $u_1, u_2, \cdots, u_m \in F_n$ and H and G be subalgebras of F_n. The following problems are decidable

i. whether, $\varphi(u_1), \cdots, \varphi(u_m) \in H$, for some $\varphi \in \mathrm{Aut}F_n$

ii. whether, $\varphi(G) \subseteq H$, for some $\varphi \in \mathrm{Aut}F_n$

PROOF.
Let $u_1, u_2, \cdots, u_m \in F_n$ and H and G be subalgebras of F_n.

i. Let

$$y_i = \sum \alpha^{(i)}_{n_s}[\cdots[w_n, v_{n_1}]\cdots], v_{n_s}] + \cdots + \sum \alpha^{(i)}_{(n-r)_s}[\cdots[w_{n-r}, v_{(n-r)_1}]\cdots], v_{(n-r)_s}] + f_i(v_1, \cdots, v_r)$$

such that $i \in \{1, \cdots, m\}$. If this equation on the variables $w_n, \cdots w_{n-r}, y_1, \cdots, y_m$ has a solution in F_n, then $\varphi(u_i) = y_i \in H$ by Corollary 3.9.

ii. This statement is a particular case of i, when $\{u_1, u_2, \cdots, u_k\}$ is a generating set of G. Then, it is decidable whether or not $\varphi(u_i) \in H$, for some $\varphi \in \mathrm{Aut}F_n$. Hence, $\varphi(G) \subseteq H$. □

Example 3.12. Let $H = \langle x_1, x_2 \rangle$ be a subalgebra of F_n. Given $u = [x_3, x_2] \in F_n$. It is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \mathrm{Aut}F_n$. By [8],

$$\frac{\partial u}{\partial x_2} = x_3, \quad \frac{\partial u}{\partial x_3} = -x_2, \quad \text{and} \quad \mathrm{rank}u = 2$$

Since M_u is left $U(F_n)-$module generated by $\frac{\partial u}{\partial x_2}$ and $\frac{\partial u}{\partial x_3}$, $\mathrm{rank}M_u = 2$. Given an automorphism φ of F_n defined by

$$\begin{aligned} \varphi \quad : \quad & x_i \to x_i \\ & x_2 \to x_2 + x_1 \\ & x_3 \to x_3 - x_1 \end{aligned}$$

where $i \notin \{2, 3\}$. Then,

$$\varphi(u) = [x_3 - x_1, x_2 + x_1] = [x_3, x_2] + [x_3, x_1] - [x_1, x_2]$$

We calculate

$$\frac{\partial\varphi(u)}{\partial x_1} = x_3 + x_2, \quad \frac{\partial\varphi(u)}{\partial x_2} = x_3 - x_1, \quad \text{and} \quad \frac{\partial\varphi(u)}{\partial x_3} = -x_2 - x_1$$

Since

$$\frac{\partial\varphi(u)}{\partial x_3} = -\frac{\partial\varphi(u)}{\partial x_1} + \frac{\partial\varphi(u)}{\partial x_2}$$

then

$$\text{rank}u = \text{rank}M_u = \text{rank}M_{\varphi(u)} = 2$$

By [11], $\varphi(u)$ belongs to a subalgebra which has rank 2. It seems that $\varphi(u)$ involves the generator x_3, therefore, $\varphi(u) \notin H = \langle x_1, x_2\rangle$. However, it is verified that $\sigma(\varphi(u)) \in H$ for some automorphism σ of F_n by Lemma 2.2. Therefore,

$$\sigma(\varphi(u)) = \sigma([x_3 - x_1, x_2 + x_1]) = [\sigma(x_3) - \sigma(x_1), \sigma(x_2) + \sigma(x_1)] \in H$$

for some automorphism σ of F_n. Hence, solving the equation

$$[\sigma(x_3) - \sigma(x_1), \sigma(x_2) + \sigma(x_1)] = [x_1, x_2]$$

for an appropriate automorphism σ,

$$\sigma(x_3) - \sigma(x_1) = x_1$$

and

$$\sigma(x_2) + \sigma(x_1) = x_2$$

Choose $\sigma(x_1) = x_3$ and $\sigma(x_i) = x_i, i \notin \{1, 2, 3\}$. Then, $\sigma(x_3) = x_3 + x_1$ and $\sigma(x_2) = x_3 + x_2$. Hence, for the automorphism σ of F_n, we obtain $\sigma(\varphi(u)) \in H$.

Example 3.13. Let $H = \langle x_1+[x_2, x_3], x_2, x_3, x_4\rangle$ be a subalgebra of F_n and $u = [x_1, x_2]+[x_3, x_4] \in F_n$. It is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$. Given an automorphism φ of F_n defined by

$$\begin{aligned} \varphi : x_1 &\to x_1 + [x_2, x_3] \\ x_i &\to x_i \end{aligned}$$

such that $i \neq 1$. Therefore,

$$\begin{aligned} \varphi(u) &= [x_1, x_2] + [[x_2, x_3], x_2] + [x_3, x_4] \\ &= [x_1 + [x_2, x_3], x_2] + [x_3, x_4] \end{aligned}$$

Clearly, $\varphi(u)$ belongs to a subalgebra generated by $\{x_1 + [x_2, x_3], x_2, x_3, x_4\}$.

4. Conclusion

In this study, the orbit problem for free Lie algebras of finite rank n such that $n \geqslant 2$ is solved. In this context, we prove that for a given element u and a subalgebra H of F_n, it is decidable whether or not $\varphi(u) \in H$, for some $\varphi \in \text{Aut}F_n$. In addition, we get the decidability of the problem for given primitive elements of free Lie algebras of finite rank. Furthermore, in future research, the decidability of the orbit problem for relatively free Lie algebras can be investigated.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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