İSTANBUL TİCARET ÜNİVERSİTESİ
FEN BİLİMLERİ DERGİSİ

İstanbul Commerce University Journal of Science

http://dergipark.org.tr/ticaretfbd

Research Article / Araştırma Makalesi

ON THE SOLITARY WAVE SOLUTIONS OF DIFFERENT VERSIONS OF FRACTIONAL 3D- WAZWAZ -BENJAMIN-BONA-MAHONY EQUATIONS

KESİRLİ 3D- WAZWAZ -BENJAMIN-BONA-MAHONY DENKLEMLERİNİN FARKLI VERSİYONLARININ SOLİTARY DALGA ÇÖZÜMLERİ ÜZERİNE

Ulviye DEMİRBİLEK[1]

https://doi.org/10.55071/ticaretfbd.1285053

Corresponding Author / Sorumlu Yazar
udemirbilek@mersin.edu.tr

Received / Geliş Tarihi
18.04.2023

Accepted / Kabul Tarihi
04.09.2023-

Abstract

Nonlinear fractional Wazwaz -Benjamin-Bona-Mahony (WBBM) equations play an important role in physics. The equations form an important model for studying the approximately unidirectional propagation of small amplitude long waves in certain nonlinear distribution systems as an alternative to Kortweg and de Vries (KDV). In this study, the fractional 3D-WBBM equations are solved by using the Improved Bernoulli Sub-Equation Function (IBSEF) method. 3D, 2D and contour plots are given to show the physical properties of the solutions. The main aim of this method is to clarify obvious the exact solutions to the equations. Moreover, the effectiveness of the method is demonstrated by the findings presented in this paper.

Keywords: Exact solution, fractional derivative, WBBM equations.

Öz

Lineer olmayan kesirli Wazwaz-Benjamin-Bona-Mahony (WBBM) denklemleri fizikte önemli bir rol oynar. Bu denklemler, Kortweg ve de Vries'e (KDV) alternatif olarak belirli doğrusal olmayan dağıtım sistemlerinde küçük genlikli uzun dalgaların yaklaşık olarak tek yönlü yayılmasını incelemek için önemli bir model oluşturur. Çalışmada, kesirli 3D-WBBM denklemleri, Geliştirilmiş Bernoulli Alt Denklem Fonksiyonu (IBSEF) yöntemi kullanılarak çözülmüştür. Çözümlerin fiziksel özelliklerinin gösterilmesi için 3D, 2D ve kontur çizimleri verilmiştir. Bu yöntemin temel amacı, bu denklemlerin kesin çözümlerini açıklığa kavuşturmaktır. Ayrıca yöntemin etkinliği, bu makalede sunulan bulgularla gösterilmektedir.

Anahtar Kelimeler: Kesirli türev, WBBM denklemleri, tam çözüm. .

[1]Mersin University. The Faculty of Sciences, Department of Mathematics, Mersin, Türkiye.
udemirbilek@mersin.edu.tr, Orcid.org/0000-0002-5767-1089.

1. INTRODUCTION

Nonlinear partial differential equations (PDEs) are important for modeling many daily life problems in fields of optical fibers, biology, chemical physics, plasma physics, quantum mechanics and so on. It is difficult to understand these complex structures used in non-linear branches such as engineering, physics and mathematics. For this reason, many mathematical methods have been developed in this field. So, many exact solution methods are used to understand the physical behavior of equations. Some of these methods are the dynamical system method by (Fu & Lie, 2017), the Hirota's direct method (Wazwaz, 2018), the generalized bilinear transformation method (Ma, 2011), the improved Bernoulli sub-equation function method (Ala et al., 2021), the double (G'/G,1/G)-expansion method (Ünal & Ekici, 2021), the rational (G′/G)-expansion method (Ekici &Ünal, 2022), the (m+1/ G′) -expansion method (Atas et al., 2022), the extended (G′/G)-expansion method (Roshid et al., 2014), the Sine-Gordon expansion method (Baskonuş et al., 2019), the Ricatti-Bernoulli sub-ODE method (Yusuf et al., 2019), the modified exponential function method (Aktürk & Kudali, 2022), and so on.

The Benjamin-Bona-Mahony (BBM) equation is defined in (Benjamin et al.,1972) by:

$$\upsilon_x + \upsilon_t + \upsilon\upsilon_x - \upsilon_{xxt} = 0.$$

This equation has a boundless diffusion relationship with the KDV equation. So, it is a powerful alternative to the KDV equation.

Also, various modifications of the BBM equations have been studied by many researchers. Wazwaz developed a novel model which is called Wazwaz -Benjamin-Bona-Mahony (WBBM) equations in (Wazwaz, 2017) as:

$$\upsilon_x + \upsilon_t + \upsilon^2\upsilon_y - \upsilon_{xzt} = 0,$$

$$\upsilon_z + \upsilon_t + \upsilon^2\upsilon_x - \upsilon_{xyt} = 0,$$

$$\upsilon_y + \upsilon_t + \upsilon^2\upsilon_z - \upsilon_{xxt} = 0.$$

These three recently derived equations are redefined by Wazwaz, and the resulting version of these 3D fractional WBBM equations is considered in this article (Mamun et al., 2020):

$$D_t^{\eta}\upsilon + D_x^{\eta}\upsilon + D_y^{\eta}\upsilon^3 - D_{xzt}^{3\eta}\upsilon = 0, \quad t \geq 0,\ 0 < \eta \leq 1, \tag{1}$$

$$D_t^{\eta}\upsilon + D_z^{\eta}\upsilon + D_x^{\eta}\upsilon^3 - D_{xyt}^{3\eta}\upsilon = 0, \quad t \geq 0,\ 0 < \eta \leq 1, \tag{2}$$

$$D_t^{\eta}\upsilon + D_y^{\eta}\upsilon + D_z^{\eta}\upsilon^3 - D_{xxt}^{3\eta}\upsilon = 0, \quad t \geq 0,\ 0 < \eta \leq 1, \tag{3}$$

where $\upsilon(x, y, z, t)$ is differentiable functions in the Eq. (1)-(3) in which independent variables x, y, z, and $t, D_t^{\eta}, D_x^{\eta}, D_y^{\eta}$ and D_z^{η} represent the conformable fractional derivative of order η.

Various methods have been used to arrive at different solutions for the family (1)-(3); for example, the modified extended tanh-function method yields hyperbolic and trigonometric function solutions (Mamun et al., 2020), the tanh–coth method is utilized to derive singular, shock, periodic, and bell-shaped soliton solutions as presented (Mamun et al., 2022a), the Sine-Gordon expansion method yields hyperbolic function solutions (Mamun et al., 2022b), the improved modified extended tanh-function method is used to obtain bright solitons, dark solitons, bright-dark solitons, single solitons, and multiple solitons (Mamun et al., 2022c).

This article is structured as follows: Sect. 1 is dedicated to the introduction. Sect. 2 includes the basic definitions of fractional derivatives and describes the proposed method. In Sect. 3, the applications of the IBSEF method to Eqs. (1)-(3) are presented. Sect. 4 provides graphical representations of the solutions. Finally, Sect. 5 presents the main conclusions.

2. MATERIALS AND METHODS

2.1. Conformable Fractional Derivative

This part covers the basic concepts of the conformable fractional derivative discussed in this study.

Definition: Let be a function $\upsilon = \upsilon(\xi): [0, \infty) \to \mathbb{R}$. The conformable derivative of order η is defined by (Khalil et al., 2014) :

$$D_t^{\eta}\big(\upsilon(t)\big) = \lim_{\tau \to \infty} \frac{\upsilon(t+\tau t^{1-\eta}) - \upsilon(t)}{\tau},$$

for all $t > 0, \eta \epsilon (0,1]$.

Theorem 1. Suppose that $\upsilon = \upsilon(t)$ and $\omega = \omega(t)$ are $\eta \epsilon (0,1]$ −differentiable. Then

i. $D_t^{\eta}(k_1\upsilon + k_2\omega) = k_1 D_t^{\eta}(\upsilon) + k_2 D_t^{\eta}(\omega), \quad \forall k_1, k_2 \epsilon \mathbb{R}$.

ii. $D_t^{\eta}(t^k) = k t^{k-\eta},\ k \epsilon \mathbb{R}$.

iii. $D_t^{\eta}(\alpha) = 0$, for all constant function $\upsilon(\xi) = \alpha$.

iv. $D_t^{\eta}(\upsilon\omega) = \upsilon D_t^{\eta}(\omega) + \omega D_t^{\eta}(\upsilon)$.

v. $D_t^{\eta}\left(\frac{\upsilon}{\omega}\right) = \frac{\omega T_t^{\eta}(\upsilon) - \upsilon T_t^{\eta}(\omega)}{\omega^2},\ \omega \neq 0,$

for all positive t.

Theorem 2. Let $\upsilon = \upsilon(t)$ be an $\eta -$ conformable differentiable function and assume that ω is differentiable and defined in the range of υ. Then,

$$D_t^{\eta}(\upsilon o \omega) = t^{1-\eta}\omega'(t)\upsilon'\big(\omega(t)\big).$$

For the proofs, see (Atangana et al., 2015; Abdeljawad, 2015).

2.2. Description of the Proposed Method

In this part, we give outline the steps for the method to be used (Başkonuş & Bulut, 2015).

Step 1. To start with, we consider the conformable fractional PDE equation given by

$$Q\big(\upsilon, \upsilon_x, D_t^{\eta}\upsilon, \upsilon_{xx}, \ldots\big) = 0, \quad 0 < \eta \leq 1, \tag{4}$$

where D_t^{η} is the conformable derivative operator, η is fractional order. We define a transformation

$$\upsilon(x,t) = V(\xi),\ \xi = (x - ct^{\eta}\eta^{-1}), \tag{5}$$

where c is an arbitrary constant and not zero. Then substituting Equation (5) into Eq.(4), gives the following of ordinary differential equation

$$N(V, V', V'', \dots) = 0, \tag{6}$$

where N is function of $V, V', V''$and its derivatives with respect to ξ. We assume that the solution of (6) can be represented as:

$$V(\xi) = \frac{\sum_{i=0}^{n} a_i H^i(\xi)}{\sum_{i=0}^{m} b_i H^i(\xi)}, \tag{7}$$

where $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_m$ are real or complex constants. m and n are calculated with the help of the balance principle. The form of the following Bernoulli differential equation is taken into account:

$$H'(\xi) = \sigma H(\xi) + dH^M(\xi), \tag{8}$$

where $d \neq 0, \sigma \neq 0, M \in \mathbb{R}\backslash\{0,1,2\}$ and $H(\xi)$ is polynomial.

Step 2: m, n, M is found by balance principle and different from zero. Balance princple is both nonlinear term and the highest order derivative term of Eq. (4). Considering Eqs. (7) , (8) in (6), we get

$$\Theta\big(H(\xi)\big) = \rho_s H(\xi)^s + \cdots + \rho_1 H(\xi) + \rho_0 = 0, \tag{9}$$

the coefficients $\rho_i,\ i = \overline{0, s}$ will be determined later. The coefficients of $\Theta(H(\xi))$ which will give us a system of algebraic equations, whole be zero.

Step 3. The function $H(\xi)$ is a solution of the Bernoulli differential Eq. (8). Therefore, $H(\xi)$ is given by

a) For $d \neq \sigma, \varepsilon \in \mathbb{R}$,

$$H(\xi) = \left[\frac{-de^{\sigma(\varepsilon-1)} + \varepsilon\sigma}{\sigma e^{\sigma(\varepsilon-1)\xi}}\right]^{\frac{1}{1-\varepsilon}}. \tag{10}$$

b) For $d = \sigma, \varepsilon \in \mathbb{R}$,

$$H(\xi) = \left[\frac{(\varepsilon-1) + (\varepsilon+1)\tanh\left(\sigma(\varepsilon-1)\frac{\xi}{2}\right)}{1-\tanh\left(\sigma(\varepsilon-1)\frac{\xi}{2}\right)}\right]. \tag{11}$$

With the solving of this algebraic equation system, the values of the constants used in the solution function and the soliton solutions of (4) are obtained via Mathematica.

3. IMPLEMENTATION

3.1.The First Fractional WBBM Equation and Its Solitons

In this section, using the IBSEF method, we will realize the exact solutions in terms of some variables for Eq. (1).

Let's consider the traveling wave transformation by

$$v(x,y,z,t) = V(\xi),\ \xi = \alpha\left(\frac{x^\eta}{\eta}\right) + \beta\left(\frac{y^\eta}{\eta}\right) + \gamma\left(\frac{z^\eta}{\eta}\right) - \kappa\left(\frac{t^\eta}{\eta}\right). \quad (12)$$

Using the basic properties fractional derivative and substituting Eqs. (12) into (1), it becomes:

$$(-\kappa+\alpha)V' + \beta(V^3)' + \alpha\gamma\kappa V''' = 0.$$

Integrating above the equations concerning ξ, we get ordinary differential equation (ODE) as:

$$(-\kappa+\alpha)V + \beta V^3 + \alpha\gamma\kappa V'' + c_0 = 0, \quad (13)$$

where c_0 is the integrating constant and $c_0 = 0$ is chosen for simplicity. Considering that the nonlinear term of the highest algebraic power is V^3 and the highest derivative term is V'', we obtain the relation

$$M + m = n + 1.$$

This relationship of m, n and M give us different types of the solutions of Eq. (13). Using homogeneous balance principle, $M = n = 3$ and $m = 1$ are chosen. Therefore we get,

$$V(\xi) = \frac{\sum_{i=0}^{3} a_i H^i(\xi)}{\sum_{i=0}^{1} b_i H^i(\xi)} = \frac{a_0 + a_1 H(\xi) + a_2 H^2(\xi) + a_3 H^3(\xi)}{b_0 + b_1 H(\xi)} = \frac{\psi(\xi)}{\varphi(\xi)}, \quad (14)$$

$$V'(\xi) = \frac{\psi'(\xi)\varphi(\xi) - \psi(\xi)\varphi\prime(\xi)}{\varphi^2(\xi)},$$

$$V''(\xi) = \frac{\psi'(\xi)\varphi(\xi) - \psi(\xi)\varphi'(\xi)}{\varphi^2(\xi)} - \frac{[\psi(\xi)\varphi'(\xi)]'\varphi^2(\xi) - 2\psi(\xi)[\varphi']^2\varphi(\xi)}{\varphi^4(\xi)}. \quad (15)$$

Substituting Eqs. (15) along with (14) into (13) as well as equating the like power of H and solving the algebraic system as:

$$H^0: \beta a_0^3 + \alpha a_0 b_0^2 - \kappa a_0 b_0^2 = 0,$$

$$H^1: 3\beta a_0^2 a_1 + \alpha a_1 b_0^2 - \kappa a_1 b_0^2 + \alpha\gamma\kappa\sigma^2 a_1 b_0^2 + 2\alpha a_0 b_0 b_1 - 2\kappa a_0 b_0 b_1 - \alpha\gamma\kappa\sigma^2 a_0 b_0 b_1 = 0\ ,$$

$$H^2: 3\beta a_0 a_1^2 + 3\beta a_0^2 a_2 + \alpha a_2 b_0^2 - \kappa a_2 b_0^2 + 4\alpha\gamma\kappa\sigma^2 a_2 b_0^2 + 2\alpha a_1 b_0 b_1 - 2\kappa a_1 b_0 b_1 - \alpha\gamma\kappa\sigma^2 a_1 b_0 b_1 + \alpha a_0 b_1^2 - \kappa a_0 b_1^2 + \alpha\gamma\kappa\sigma^2 a_0 b_1^2 = 0,$$

$$H^3: \beta a_1^3 + 6\beta a_0 a_1 a_2 + 3\beta a_0^2 a_3 + 4d\alpha\gamma\kappa\sigma a_1 b_0^2 + \alpha a_3 b_0^2 - \kappa a_3 b_0^2 + 9\alpha\gamma\kappa\sigma^2 a_3 b_0^2 - 4d\alpha\gamma\kappa\sigma a_0 b_0 b_1 + 2\alpha a_2 b_0 b_1 - 2\kappa a_2 b_0 b_1 + 3\alpha\gamma\kappa\sigma^2 a_2 b_0 b_1 + \alpha a_1 b_1^2 - \kappa a_1 b_1^2 = 0,$$

$$H^4: 3\beta a_1^2 a_2 + 3\beta a_0 a_2^2 + 6\beta a_0 a_1 a_3 + 12 d\alpha\gamma\kappa\sigma a_2 b_0^2 + 2\alpha a_3 b_0 b_1 - 2\kappa a_3 b_0 b_1 + 11\alpha\gamma\kappa\sigma^2 a_3 b_0 b_1 + \alpha a_2 b_1^2 - \kappa a_2 b_1^2 + \alpha\gamma\kappa\sigma^2 a_2 b_1^2 = 0,$$

$$H^5: 3\beta a_1 a_2^2 + 3\beta a_1^2 a_3 + 6\beta a_0 a_2 a_3 + 3d^2\alpha\gamma\kappa a_1 b_0^2 + 24 d\alpha\gamma\kappa\sigma a_3 b_0^2 - 3d^2\alpha\gamma\kappa a_0 b_0 b_1 + 12 d\alpha\gamma\kappa\sigma a_2 b_0 b_1 + \alpha a_3 b_1^2 - \kappa a_3 b_1^2 + 4\alpha\gamma\kappa\sigma^2 a_3 b_1^2 = 0,$$

$$H^6: \beta a_2^3 + 6\beta a_1 a_2 a_3 + 3\beta a_0 a_3^2 + 8d^2\alpha\gamma\kappa a_2 b_0^2 + d^2\alpha\gamma\kappa a_1 b_0 b_1 + 32 d\alpha\gamma\kappa\sigma a_3 b_0 b_1 - d^2\alpha\gamma\kappa a_0 b_1^2 + 4 d\alpha\gamma\kappa\sigma a_2 b_1^2 = 0,$$

$$H^7: 3\beta a_2^2 a_3 + 3\beta a_1 a_3^2 + 15 d^2\alpha\gamma\kappa a_3 b_0^2 + 9d^2\alpha\gamma\kappa a_2 b_0 b_1 + 12 d\alpha\gamma\kappa\sigma a_3 b_1^2 = 0,$$

$$H^8: 3\beta a_2 a_3^2 + 21 d^2\alpha\gamma\kappa a_3 b_0 b_1 + 3d^2\alpha\gamma\kappa a_2 b_1^2 = 0,$$

$$H^9: q\beta a_3^3 + 8d^2\alpha\gamma\kappa a_3 b_1^2 = 0.$$

The above algebraic equations yield the following coefficients:

Result 1.

$$a_0 = -\frac{i\sqrt{\alpha-\kappa}\, b_0}{\sqrt{\beta}};\ a_1 = -\frac{i\sqrt{\alpha-\kappa}\, b_1}{\sqrt{\beta}};\ a_2 = -\frac{2id\sqrt{\alpha-\kappa}\, b_0}{\sqrt{\beta}\sigma};$$
$$a_3 = -\frac{2id\sqrt{\alpha-\kappa}\, b_1}{\sqrt{\beta}\sigma};\quad \gamma = \frac{\alpha-\kappa}{2\alpha\kappa\sigma^2}.$$

Substituting these coefficients along with Eqs. (10) in (14), we obtain the following solution of (12) as follows;

$$v_1(x,y,z,t) = -\frac{i\sqrt{\alpha-\kappa}\left(1-\frac{2d}{d-e^{\frac{\left(z^\eta(-\alpha+\kappa)-2\alpha\kappa\sigma^2(x^\eta\alpha+y^\eta\beta-t^\eta\kappa)\right)\sigma}{\alpha\kappa\sigma^2\eta}}\epsilon\sigma}\right)}{\sqrt{\beta}}. \tag{16}$$

Result 2.

$$a_0 = \frac{\sqrt{\alpha-\beta}\, a_2}{2\sqrt{2}d\sqrt{\alpha}\sqrt{\gamma}\sqrt{\kappa}};\ a_1 = \frac{\sqrt{\alpha-\beta}\, a_2 b_1}{2\sqrt{2}d\sqrt{\alpha}\sqrt{\gamma}\sqrt{\kappa}\, b_0};\ a_3 = \frac{a_2 b_1}{b_0};\ \sigma = \frac{\sqrt{\alpha-\kappa}}{\sqrt{2}\sqrt{\alpha}\sqrt{\gamma}\sqrt{\kappa}};\ q = -\frac{8d^2\alpha\gamma\kappa b_0^2}{a_2^2}.$$

If the coefficients given above are used, the following solution functions are obtained as follows:

$$v_2(x,y,z,t) = \frac{(4(e^{-\frac{(-1+M)\sqrt{\alpha-\kappa}(x^\eta\alpha+y^\eta\beta+z^\eta\gamma-t^\eta\kappa)}{\sqrt{2}\sqrt{\alpha}\sqrt{\gamma}\eta\sqrt{\kappa}}}\epsilon-\frac{\sqrt{2}d\sqrt{\alpha}\sqrt{\gamma}\sqrt{\kappa}}{\sqrt{\alpha-\kappa}})^{-\frac{2}{-1+M}}+\frac{\sqrt{2}\sqrt{\alpha-\beta}}{d\sqrt{\alpha}\sqrt{\gamma}\sqrt{\kappa}})a_2}{4b_0}. \tag{17}$$

3.2. The Second Fractional WBBM Equation and Its Soliton

When we apply the wave transformation (12) to the second fractional WBBM equation (2), we get the following ODE:

$$(-\kappa+\gamma)V + \alpha V^3 + \alpha\beta\kappa V'' = 0. \tag{18}$$

Proceeding the same way as the above steps and with the help of Mathematica, we yield the following values of Eq. (18):

Result 1.

$$a_0 = i\sqrt{2}\sqrt{\beta}\sqrt{\kappa}\sigma b_0; \qquad a_1 = i\sqrt{2}\sqrt{\beta}\sqrt{\kappa}\sigma b_1;\ a_2 = 2i\sqrt{2}d\sqrt{\beta}\sqrt{\kappa}b_0;$$

$$a_3 = 2i\sqrt{2}d\sqrt{\beta}\sqrt{\kappa}b_1; \qquad \gamma = \kappa + 2\alpha\beta\kappa\sigma^2.$$

Using these solutions sets, we obtain the solutions for the Eq. (2)

$$v_3(x,y,z,t) = i\sqrt{2}\sqrt{\beta}\sqrt{\kappa}\left(\frac{2d}{e^{-\frac{2\sigma\left(x^\eta\alpha+y^\eta\beta-t^\eta\kappa+z^\eta\kappa(1+2\alpha\beta\sigma^2)\right)}{\eta}}\epsilon-\frac{d}{\sigma}}+\sigma\right). \tag{19}$$

3.3. The Third Fractional WBBM Equation and Its Soliton

Introducing the traveling wave transform (12) into Eq. (3), the ODE is obtained as follows:

$$(-\kappa+\beta)V + \gamma V^3 + \alpha^2\kappa V'' = 0. \tag{20}$$

Proceeding in the same manner as the above steps, and utilizing Mathematica, we obtain the following values for Eq. (20):

Result 1.

$$a_0 = \frac{\sigma a_2}{2d};\ a_1 = \frac{\sigma a_2 b_1}{2db_0};\ a_3 = \frac{a_2 b_1}{b_0};\ \kappa = \beta - 2\alpha^2\sigma^2;\ \gamma = -\frac{8d^2\alpha^2 b_0^2}{a_2^2}.$$

Using these solutions sets, we obtain the solutions for the Eq. (3):

$$v_4(x,y,z,t) = \frac{\sigma\left(\frac{1}{d}-\frac{2}{d-e^{\frac{2\sigma\left(-x^\eta\alpha-y^\eta\beta+t^\eta(\beta-2\alpha^2\sigma^2)+\frac{8d^2z^\eta\alpha^2b_0^2}{a_2^2}\right)}{\eta}}\epsilon\sigma}\right)a_2}{2b_0}. \tag{21}$$

4. RESULTS AND DISCUSSION

In this section 3D, 2D and contour plot graphical representations of the solutions $v_1(x,y,z,t), v_2(x,y,z,t)$, $v_3(x,y,z,t)$, and $v_4(x,y,z,t)$ are given, respectively.

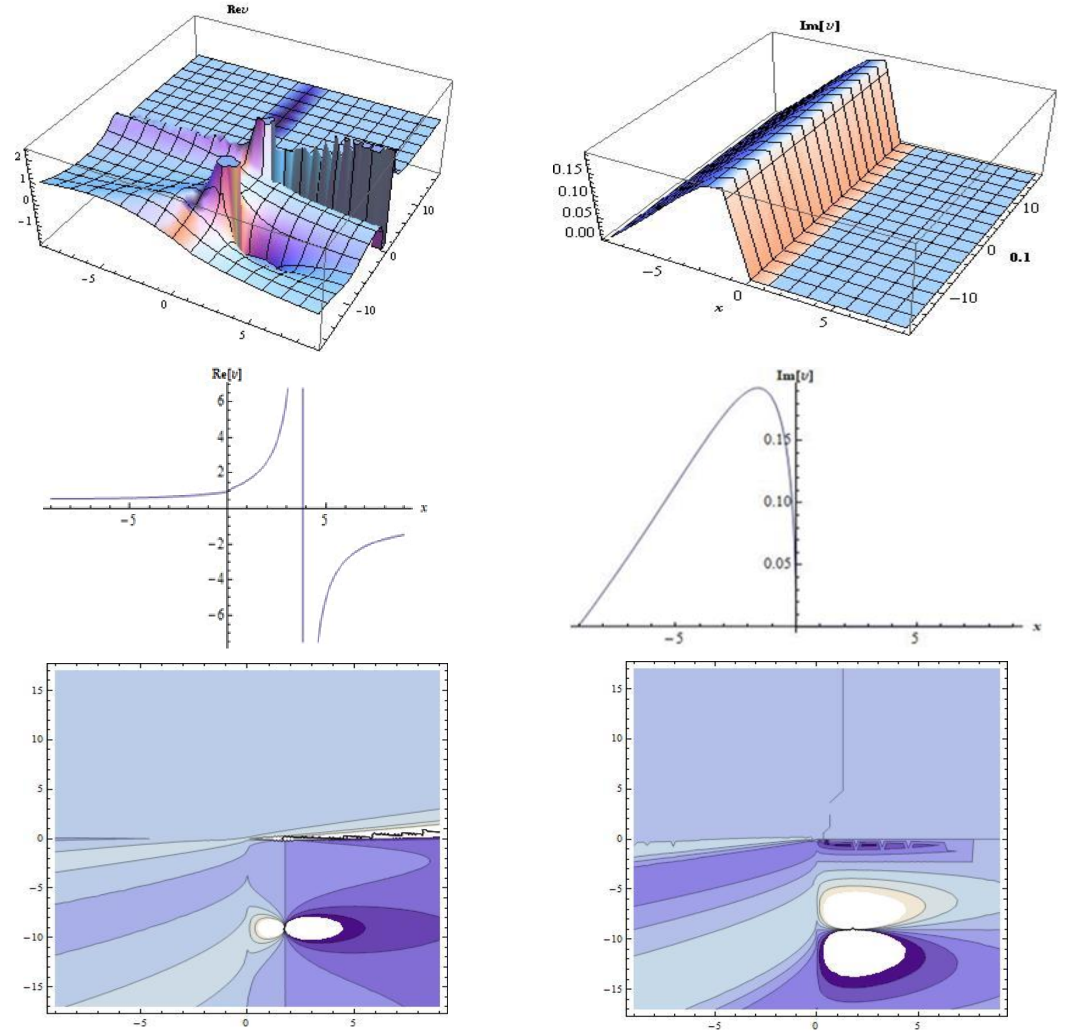

Figure1. 3D, 2D and contour plots of $\upsilon_1(x, y, z, t)$ for the values $\alpha = 0{,}3; \beta = 0{,}56;$ $\kappa = 0{,}6; \eta = 0{,}5; \epsilon = 0{,}44; \gamma = 0{,}1;\ \sigma = 0{,}87; d = 0{,}86; -9 < x < 9, -17 < t < 17$

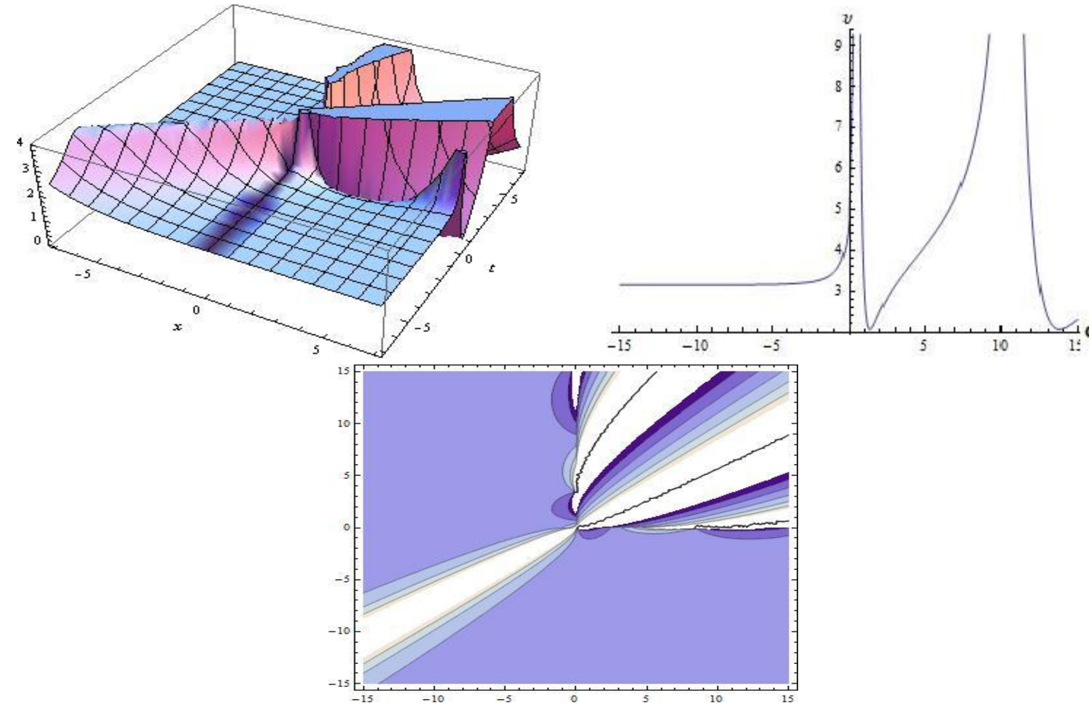

Figure2. 3D, 2D and contour plots of $v_2(x, y, z, t)$ for the values $\alpha = 0{,}5; \beta = 0{,}2; \kappa = 0{,}59; \gamma = 0{,}1; \eta = 0{,}5; \epsilon = 0{,}55; d = 0{,}75; a_2 = 0{,}44; b_0 = 0{,}21; -15 < x < 15, -15 < t < 15$

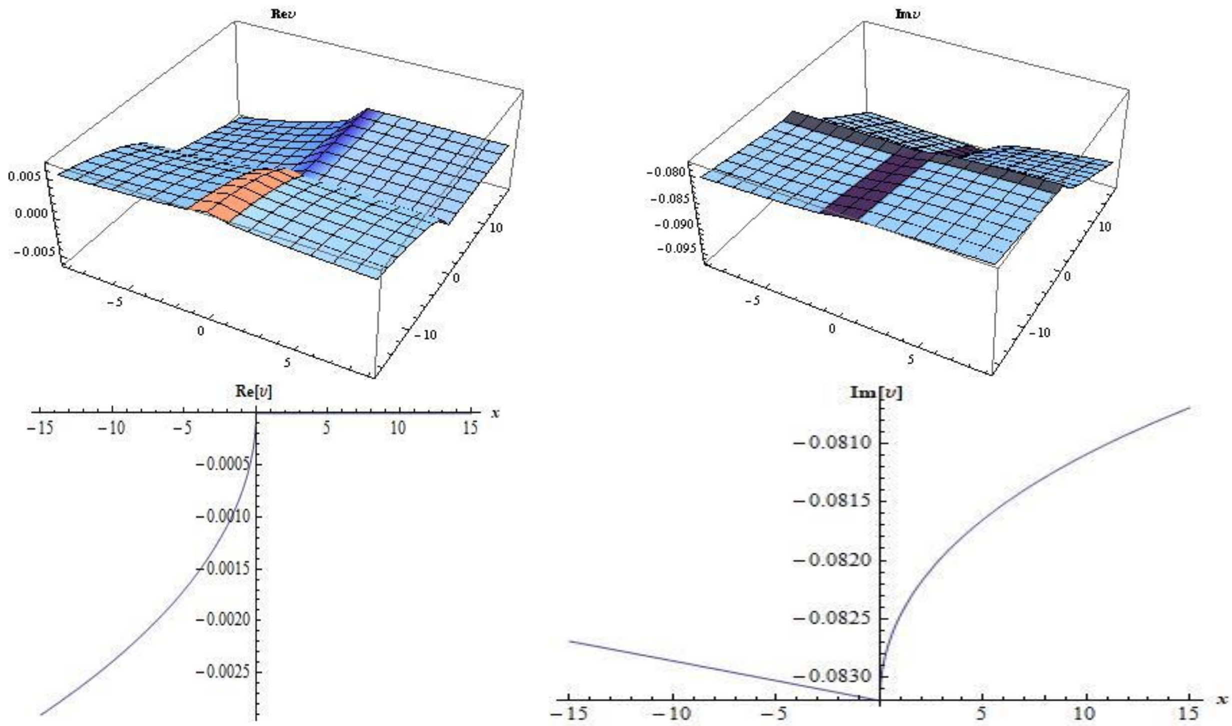

Figure3. 3D of $v_3(x, y, z, t)$ for the values $\alpha = 0{,}2; \beta = 0{,}5;\ \kappa = 0{,}55;\ \eta = 0{,}5; \epsilon = 0{,}3; d = 0{,}49; -9 < x < 9, -17 < t < 17$ and 2D plots of $v_3(x, y, z, t)$ for the values $\alpha = 0{,}2; \beta = 0{,}5;\ \kappa = 0{,}55;\ \eta = 0{,}5; \epsilon = 0{,}3; d = 0{,}49; -15 < x < 15$

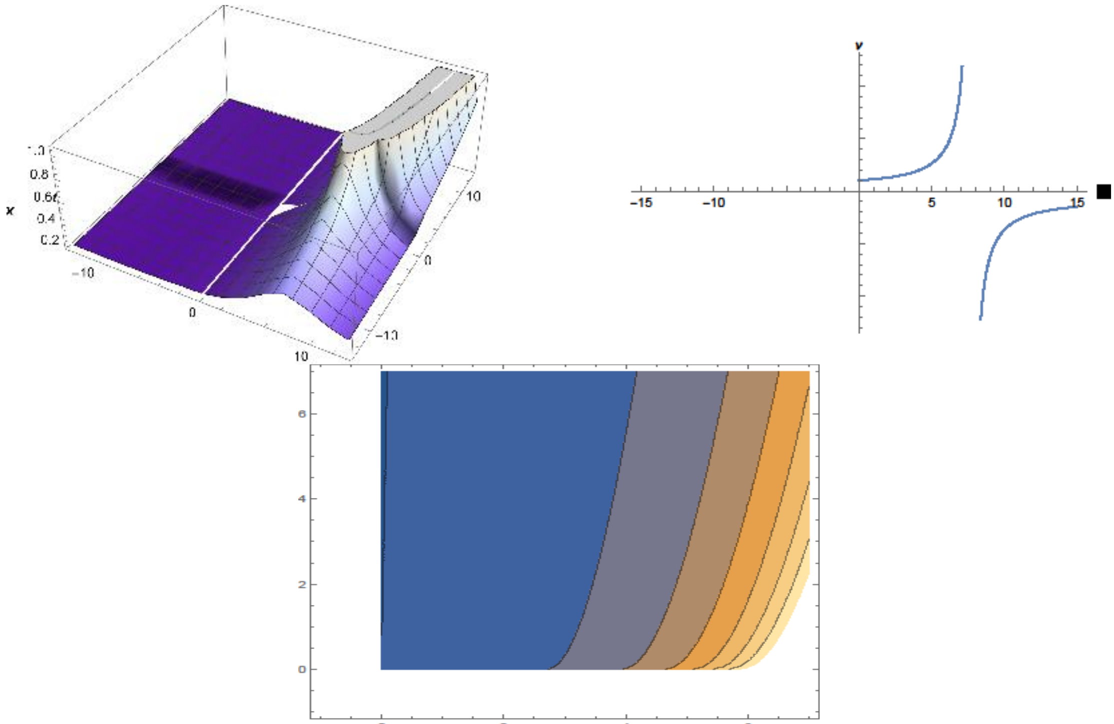

Figure 4. 3D of $\upsilon_4(x, y, z, t)$ for the values $\alpha = 0{,}5; \beta = 0{,}32;\ \eta = 0{,}5; \epsilon = 0{,}1; d = 0{,}4;$ $a_2 = 0{,}11; b_0 = 0{,}5; -10 < x < 10, -10 < t < 10$ and 2D plots of $\upsilon_3(x, y, z, t)$ for the values $\alpha = 0{,}5; \beta = 0{,}32;\ \eta = 0{,}5; \epsilon = 0{,}1; d = 0{,}4; a_2 = 0{,}11; b_0 = 0{,}5;\ -15 < x < 15$

5. CONCLUSION

In this study, the IBSEF method was employed to obtain exact solutions for fractional 3D-fractional WBBM equations (1)-(3). The method demonstrated its effectiveness, reliability, and conformability in obtaining accurate solutions for nonlinear conformable time-fractional derivative partial differential equations. Utilizing Wolfram Mathematica, we employed graphical representations such as 3D plots, 2D plots, and contour plots to visually present some of the solutions. The exponential function solutions, outlined in Eqs. (16), (17), (19), and (21) and depicted in Figs. 1, 2, 3, and 4, respectively, capture natural processes of growth and decay. These solutions find broad application in modeling population growth, radioactive decay, financial investments, and other scenarios characterized by quantities changing proportionally to their current values. Furthermore, within the field of quantum mechanics, exponential functions surface as descriptors of wave function behavior and probability amplitudes in quantum systems. So, these conclusions could prove important to further research into these systems in order to address the nonlinear problems encountered in applied sciences.

Statement of Research and Publication Ethics
Research and publication ethics were observed in the study.

REFERENCES

Abdeljawad, T. (2015). On conformable fractional calculus. *J Comput Appl Math*, 279, 57-66.

Aktürk, T. & Kubal, Ç. (2022). The behavior of plasma and space-charge waves represented by Nonlinear mathematical models. *Journal of Ocean Engineering and Science*, 23, 50, https://doi.org/10.1016/j.joes.2022.06.031.

Ala, V., Demirbilek, U. & Mamedov, Kh. R. (2021). On the exact solutions to conformable equal width wave equation by improved Bernoulli sub-equation function method. *Bulletin of the South Ural State University Ser. Mathematics. Mechanics. Physics*, 13(3), 5–13.

Atangana, A., Baleanu, D. & Alsaedi, A. (2015). New properties of conformable derivative. *Open Math*, 13, 1-10.

Atas, S.S., Ali, K.K., Sulaiman, T.A. & Bulut, H. (2022). Optical solitons to the Fokas system equation in monomode optical fibers. *Opt Quant Electron*, 54(11), 1-13.

Baskonus, H. M. & Bulut, H. (2015). On the complex structures of Kundu-Eckhaus equation via improved Bernoulli sub-equation function method. *Waves in Random and Complex Media*, 66, 720-728.

Baskonus, H.M., Bulut, H. & Sulaiman, T.A. (2019). New complex hyperbolic structures to the Lonngrenwave equation by using sine-gordon expansion method. *Applied Mathematics and Nonlinear Sciences,* 4, 129-138.

Benjamin, T., Bona, J. & Mahony, J. (1972). Model equations for long waves in nonlinear dispersive systems. *Philos Trans. R. Soc. London*, Ser A 272(1220), 47.

Ekici, M. & Ünal, M. (2022). Application of the rational (G′/G)-expansion method for solving some coupled and combined wave equations. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 71(1), 116-132.

Fu, Y. & Li, J. (2017). Exact stationary -wave solutions in the standart model of the Kerr-nonlinear optical fiber with the Bragggrating. *Journal of Applied Analysis and Computation*, 7, 1177-1184.

Khalil, R., Al Horani, A., Yousef, A. & Sabadheh, M. (2014). A new definition of fractional derivative. *J. Comput. Appl. Math*., 264, 65-70.

Ma, W.X. (2011). Generalized bilinear differential equations. *Stud. Nonlinear Sci.*, 2, 140–144.

Mamun, A.A., An, T., Shahen, N. H. M., Ananna, S. N., Hossain, M. F., & Muazu, T. (2020). Exact and explicit travelling-wave solutions to the family of new 3D fractional WBBM equations in mathematical physics. *Results in Physics*, 19, 103517.

Mamun, A.A., Ananna, S. N., An, T., Asaduzzaman, M., & Miah, M. M. (2022a). Solitary wave structures of a family of 3D fractional WBBM equation via the tanh–coth approach. *Partial Differential Equations in Applied Mathematics*, 5, 100237.

Mamun, A.A., Ananna, S.N., An, T., Asaduzzaman, Md. & Rana, M.S. (2022b.) Sine-Gordon expansion method to construct the solitary wave solutions of a family of 3D fractional WBBM equations, *Results in Physics*, 40, 105845.

Mamun, A.A., Ananna, S.N., Gharami, PP., An, T. & Md. Asaduzzaman. (2022c). The improved modified extended tanh-function method to develop the exact travelling wave solutions of a family of 3D fractional WBBM equations, *Results in Physics*, 41, 105969.

Roshid, H.O., Akbar, M.A., Alam Md. N., Hoque, Md. F.& Rahman, N. (2014). New extended (G'/G)-expansion method to solve nonlinear evolution equation: The (3 + 1)-dimensional potential-YTSF equation. *SpringerPlus*, 3(122), 6.

Ünal, M. & Ekici, M. (2021). The double (G'/G,1/G)-expansion method and its applications for some nonlinear partial differential equations. *Journal of the Institute of Science and Technology*, 1(11), 599-608.

Wazwaz, A.M. (2008). The Hirota's direct method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Ito seventh-order equation. *Appl Math Comput.*, 199,133-8.

Wazwaz, A.M. (2017). Exact soliton and kink solutions for new (3+1)-dimensional nonlinear modifed equations of wave propagation. *Open Eng*., 7,169–174.

Yusuf, A., Inc, M., Aliyu, A. & Baleanu, D. (2019). Optical solitons possessing beta derivative of the Chen Lee-Liu equation in optical fiber. *Front Phys.*, 7(34), doi:10.3389/fphy.2019.00034.