Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (6) (2024), 1560 – 1574
DOI : 10.15672/hujms.1285069

Research Article

Unique recovery of the initial state of distributed order time fractional diffusion equation

Hui Liu[1], Zhiyuan Li*[1], Zhousheng Ruan[2]

[1] School of Mathematics and Statistics, Ningbo University
[2] School of Science, East China University of Technology

Abstract

We use Phragmén-Lindelöf-Liouville argument to prove the uniqueness for the determining the initial state of solution for the time fractional diffusion equation with distributed order derivative. Several numerical experiments are presented to show the accuracy and efficiency of the algorithm.

Mathematics Subject Classification (2020). 35K20, 35R30, 35B53

Keywords. ultraslow diffusion, inverse problem, regularization method

1. Introduction and main results

In recent years, the time-fractional diffusion equations (tFDEs) are found to be more adequate than the integer-order models in describing problems in many fields including biology, polymer physics, chemistry and biochemistry, see e,g.,[5, 12]. For example, the tFDEs achieved great success in modeling the diffusion in the heterogeneous anomalous medium. Moreover tFDEs can be used to model some anomalous diffusion processes in a lightly heterogeneous aquifer [2], and well capture the long-tailed profile which cannot be described by Gaussian processes, see, e.g., Bouchaud and Georges [3], Hatano [9] and the references therein. Regarding physical and practical importance, in this paper, we consider the following one dimensional tFDE

$$\partial_t^w u - \partial_x^2 u = 0, \quad \text{in } (0,1)\times(0,T) \tag{1.1}$$

with the distributed order derivative ∂_t^w which is usually defined as follows

$$\partial_t^w g(t) := \int_0^1 w(\alpha)\partial_t^\alpha g(t)d\alpha,$$

where ∂_t^α is the Caputo derivative of order $\alpha \in (0,1)$, which defined as follows

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(\tau)}{(t-\tau)^\alpha}d\tau, \quad t>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

*Corresponding Author.
Email addresses: 15935611004@gmail.com (Hui Liu), lizhiyuan@nbu.edu.cn (Zhiyuan Li), zhshruan@126.com(Zhousheng Ruan)
Received: 18.04.2023; Accepted: 18.11.2023

The above distributed order time-fractional diffusion equations are usually used in describing the ultraslow diffusion in e.g., polymer physics, kinetics of particles moving in the quenched random force fields, iterated map models where the mean square displacement has a logarithmic growth, see e.g., Caputo [4], Chechkin et al. [6], Mainardi et al. [23] and the references therein.

For various properties of the Caputo derivative, we refer to Kilbas, Srivastava and Trujillo [13], Podlubny [1] and the references therein.

In this paper, we are concerned with the recovery of the initial state of one-dimensional fractional diffusion equation with distributed order derivative from lateral Cauchy data.

Inverse Problem: We assume that u is a solution to the equation (1.1). $u(x,0)$ is unknown to be determined from the lateral Cauchy observation

$$u(1,t),\ u_x(1,t), \quad 0 < t < T.$$

This is the inverse problem considered in this paper, and we have the following important topics related to this problem:

(1) (Uniqueness) Whether the additional lateral Cauchy data $u(1,t)$, $u_x(1,t)$ $(0 < t < T)$ determine $u(x,0)$ uniquely.

(2) (Reconstruction) We need propose an efficient algorithm to reconstruct the unknown initial value.

Inverse problems have received a lot of attention from the mathematical community over the last decade, owing it to the major impact they made in many areas, including for time-fractional equations, we can refer to many works: Sakamoto and Yamamoto [24], Liu and Yamamoto [19], Li and Yamamoto [21], Tuan, Huynh, Ngoc, and Zhou [27]. As for numerical approaches, see Tuan, Long and Tatar [28], Gong, Li, Wang and Xu [8], Wang and Liu [29], Wang, Wei and Zhou [30] and the references therein. It reveals that most of the existing literature concerned with the backward problem by using the final overdermination data. For other kind of inverse problems, we refer to e.g., in Jiang, Li, Liu and Yamamoto [11]. Liu, Hu and Yamamoto [15] determining moving source profile functions in evolution equations with a derivative order in time. Zhang and Xu [32], [14] and [25] for the inverse problems in determining the spatially varying source term from the data at a fixed time, and we refer to Zhang [31] and Liu, Rundell and Yamamoto [22] for the determination of the temporally component of the source term, Cheng, Nakagawa, Yamamoto and Yamazaki [7] and Li and Yamamoto [20] for the inverse problems of identification of the fractional orders. Although there has been a lot of research done for the inverse problems of FDEs, there is little work on the determining the initial state of solution for the FDEs.

Different from integer order equation, the non-local nature of fractional derivative makes it generally not satisfy the Leibniz formula and chain rule of classical differential, which makes some powerful tools for dealing with integer order equation unable to be used, such as Carleman estimates. And then, leads to the difficulty in the study of the fractional diffusion equation.

To the first topic, we have the following main result.

Theorem 1.1. Let $T > 0$ be a fixed constant and $u \in L^\infty(0,T;H^2(0,1))$ be a solution to the fractional diffusion equation (1.1) with the initial value $u(\cdot,0) \in H^1(0,1)$. Then we have $u(x,0) = 0$ for $x \in (0,1)$ provided that

$$u(1,t) = u_x(1,t) = 0, \quad t \in [0,T].$$

When the highest order time derivative of the equation under consideration is one and the lower order time derivative is less than one, we refer to the paper from Huang, Li and Yamamoto [10] in which the stability of the inverse source problem was established by using the Carleman estimates. However, their methods heavily relies on the first order time-derivative so that cannot work for our case. To the best of the authors' knowledge, this is the first result for the inverse problem of the determination of the initial value for the distributed order fractional diffusion equation with lateral Cauchy data.

The remainder of the paper is divided into two sections. In Section 2, we set some notations, lemmas which will be useful for the later discussion. Section 3 is devoted to the proof of the main result. Section 4 is concerned with the second topic of this paper, that is, the numerical algorithm to reconstruct the unknown initial value of the system (1.1). Section 5 ends this paper with a brief conclusion.

2. Preliminary materials

2.1. Initial-boundary value problem

Let $w \in L^\infty(0,1)$ be non-negative and suppose that

$$\exists \alpha_0 \in (0,1),\ \exists \delta \in (0,\alpha_0),\ \forall \alpha \in (\alpha_0-\delta,\alpha_0),\ w(\alpha) \geq \frac{w(\alpha_0)}{2} > 0.$$

We consider the following initial-boundary value problem (IBVP)

$$\begin{cases} \partial_t^w u - \partial_x^2 u = 0 & \text{in } (0,1)\times(0,\infty),\\ u(\cdot,0) = u_0 & \text{in } (0,1),\\ u_x(0,\cdot) = g_0,\ u_x(1,\cdot) = 0 & \text{in } (0,\infty). \end{cases} \tag{2.1}$$

We have the estimate for the solution u.

Lemma 2.1. *Assuming $u_0 \in H^1(0,1)$ and $g_0 \in W^{1,\infty}(0,\infty)$, then the IBVP (2.1) admits a unique solution $u \in L^2(0,T;H^1(0,1))$. Moreover, let $\beta \in (\frac{3}{2},2)$ and $m > 4$, there exists a positive constant $C = C(\beta,\mu,m)$ such that*

$$\|J^m u(\cdot,t)\|_{H^\beta(0,1)} \leq Ce^{Ct}\|u_0\|_{L^2(0,1)} + Ce^{Ct}\|g_0\|_{W^{1,\infty}(0,t)}, \quad t > 0.$$

Proof. **Step 1** Recalling the definition of the Caputo derivative $\partial_t^\alpha := J^{-\alpha}$ in $H_\alpha(0,T)$, where $\alpha \in (0,1)$, we see that $J^{m-\alpha} = J^{-\alpha}J^m$ in $H_\alpha(0,T)$, and we operate J^m to the both sides of (2.1) to obtain

$$\partial_t^w(J^m u) - \partial_x^2(J^m u) = \int_0^1 w(\alpha) J^{m-\alpha} u_0 d\alpha.$$

Setting $v := J^m u$, we obtain

$$\partial_t^w v - \partial_x^2 v = \int_0^1 w(\alpha) J^{(m-\alpha)} u_0 d\alpha \text{ in } (0,1)\times(0,\infty).$$

By an argument similar to the proof of Theorem 1.2 in Li, Liu and Yamamoto [17], we can obtain $v(\cdot,0) = 0$, $h_0 := v_x(0,\cdot) \in H_{(m-1)}(0,T) \subset C^2[0,T]$ and $h_0(0) = 0$. In place of u, we consider the solution v to

$$\begin{cases} \partial_t^w v - \partial_x^2 v = \int_0^1 w(\alpha) J^{m-\alpha} u_0 d\alpha & \text{in } (0,1)\times(0,\infty),\\ v_x(0,t) = h_0(t), \quad v_x(1,t) = 0, & t > 0,\\ v(x,0) = 0, \quad x \in (0,1). \end{cases} \tag{2.2}$$

Letting $V := u - \frac{1}{2}(x-1)^2 h_0(t)$, then V formally reads the following IBVP

$$\begin{cases} \partial_t^w V - \partial_x^2 V = F, & 0 < x < 1,\ t > 0,\\ V(x,0) = 0, & 0 < x < 1,\\ V_x(0,t) = V_x(1,t) = 0, & t > 0. \end{cases} \tag{2.3}$$

Here

$$F(x,t) := 2h_0(t) - (x-1)^2 \partial_t^w h_0(t) + \int_0^1 w(\alpha) J^{m-\alpha} u_0 d\alpha.$$

Let $\{\lambda_n, \varphi_n\}_{n\in\mathbb{N}}$ be the eigensystem of ∂_x^2 with the domain $D = \{\eta \in H^2(0,1);\, \eta_x(0) = \eta_x(1) = 0\}$. Moreover, for $\varepsilon \in (0,+\infty)$ and $\theta \in (\frac{\pi}{2},\pi)$, we define the following contour in $\mathbb{C}$,

$$\gamma(\varepsilon,\theta) := \gamma_-(\varepsilon,\theta) \cup \gamma_c(\varepsilon,\theta) \cup \gamma_+(\varepsilon,\theta).$$

Here

$$\gamma_\pm(\varepsilon,\theta) := \{s \in \mathbb{C},\ \arg s = \pm\theta,\ |s| \ge \varepsilon\},$$
$$\gamma_c(\varepsilon,\theta) := \{s \in \mathbb{C},\ |\arg s| \le \theta,\ |s| = \varepsilon\}.$$

Now, we define operator with the domain $L^2(0,1)$ and the range in itself by

$$S(t)\psi := \sum_{n=1}^{+\infty} \frac{1}{2\pi i} \int_{\gamma(\varepsilon,\theta)} \frac{1}{sw(s)+\lambda_n} e^{st} ds\, (\psi,\varphi_n)\varphi_n, \quad \psi \in L^2(0,1),\ t>0. \tag{2.4}$$

We can derive the unique existence of solution to the initial-boundary value problem by following the argument used in the proof of Theorems 1.1 and 1.3 in [16]. For any $T \in (0,\infty)$, we can get a unique weak solution $V \in L^2(0,T,H^2(0,1))$ to the IBVP (2.3) and it admits the following representation

$$V(t) = \int_0^t S(t-\tau)F(\tau)d\tau,\ t \in [0,T].$$

Step 2 In this step, we estimate V. Together with the existence of a solution V to (2.3), we will estimate $\|V(\,\cdot\,,t)\|$ and $\|V_{xx}(\,\cdot\,,t)\|$. Firstly, we have the following properties concerning $S(t)$.

Lemma 2.2 ([16, Lemmas 3.1 and 3.2]). *Let $S(t)$ be defined in (2.4). Then for $a \in H^1(0,1)$ and $t>0$, there hold:*

$$\|S(t)a\|_{H^{2-\varepsilon}(0,1)} \le C\, t^{-\beta} e^{Ct} \|a\|_{L^2(0,1)}.$$

Here $\varepsilon \in (0,1)$ and $\beta \in (1-\frac{\varepsilon\alpha_0}{2}, 1)$.

Henceforth, $C>0$ denote generic constants independent of t but may depend on μ.

On the basis of the above estimate for the operator $S(t)$, it is not difficult to establish the estimate of the function V as follows.

$$\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le C \int_0^t e^{C(t-\tau)}(t-\tau)^{-\beta}\|F(\tau)\|_{L^2(0,1)}d\tau.$$

Recalling the definition of the function F and Riemann-Liouville fractional integral operator $J^{m-\alpha}$, from the boundedness of the weight function w, we conclude that

$$\|F(t)\|_{L^2(0,1)} \le C\Big(\|h_0\|_{L^\infty(0,t)} + \int_0^1 \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}|\frac{d}{d\tau}h_0(\tau)|d\tau d\alpha + \int_0^1 t^{m-\alpha}d\alpha \|u_0\|_{L^2(0,1)}\Big).$$

Moreover, a direct calculation yields

$$\int_0^1 \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}|\frac{d}{d\tau}h_0(\tau)|d\tau d\alpha \le C\frac{|1-t|}{|\log t|}\|\frac{d}{dt}h_0\|_{L^\infty(0,t)})$$

and

$$\int_0^1 t^{m-\alpha}d\alpha \le C\frac{t^{m-1}|t-1|}{|\log t|}.$$

Collecting all the above estimates, we find that

$$\begin{aligned}\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le& C\int_0^t e^{C(t-\tau)} \frac{|1-\tau| h_0(t-\tau)^{m-1-\beta}}{|\log\tau|}\|u_0\|_{L^2(0,1)}d\tau \\ &+ C\int_0^t e^{C(t-\tau)}\frac{(|1-\tau|)(t-\tau)^{-\beta}}{|\log\tau|}d\tau \|\frac{d}{dt}h_0\|_{L^\infty(0,t)}.\end{aligned}$$

In the case of $t \in (0,1)$, by using the fact that $\frac{1-t}{|\log t|} \le Ct^{-\gamma}$ with $\beta < \gamma + \beta < 1$, we arrive at the inequalities

$$\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le Ce^{Ct}t^{m-\beta-\gamma}\|u_0\|_{L^2(0,1)} + Ct^{1-\gamma-\beta}e^t\|\frac{d}{dt}h_0\|_{L^\infty(0,t)}.$$

By choosing $C > 1$, we further see that

$$\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le Ce^{Ct}\|u_0\|_{L^2(0,1)} + Ce^{Ct}\|h_0\|_{W^{1,\infty}(0,t)}, \quad 0 < t < 1.$$

For $t \ge 1$, we see first that

$$\int_0^t e^{C(t-\tau)}\frac{|1-\tau|(t-\tau)^{-\beta}}{|\log \tau|}d\tau$$

can be analyzed as follows

$$\begin{aligned}&\int_0^t e^{C(t-\tau)}\frac{|1-\tau|(t-\tau)^{t-\beta}}{|\log \tau|}d\tau\\ =&\int_0^1 e^{C(t-\tau)}\frac{|1-\tau|(t-\tau)^{-\beta}}{|\log \tau|}d\tau + \int_1^t e^{C(t-\tau)}\frac{|1-\tau|(t-\tau)^{-\beta}}{|\log \tau|}d\tau \le Ce^t + Cte^t,\end{aligned}$$

the last inequality is due to $\frac{t-1}{\log t} \le Ct$, $t > 1$. Similarly, we have

$$\int_0^t e^{C(t-\tau)}\frac{|1-\tau|(t-\tau)^{m-1-\beta}}{|\log \tau|}d\tau \le Ce^t + Ct^{m-\beta}e^t.$$

Again by choosing $C > 1$, we see that

$$\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le Ce^{Ct}\|u_0\|_{L^2(0,1)} + Ce^{Ct}\|h_0\|_{W^{1,\infty}(0,t)}, \quad t \ge 1.$$

Finally, we obtain

$$\|V(t)\|_{H^{2-\varepsilon}(0,1)} \le Ce^{Ct}\|u_0\|_{L^2(0,1)} + Ce^{Ct}\|h_0\|_{W^{1,\infty}(0,t)}, \quad t > 0.$$

This completes the proof of the lemma. □

2.2. Some results from complex analysis

Lemma 2.3 (Phragmén-Lindelöf's principle)**.** *Let $F(z)$ be a holomorphic function in a sector $S = \{z \in \mathbb{C}; \theta_1 < arg\, z < \theta_2\}$ of angle $\pi/\beta = \theta_1 - \theta_2$, $-\pi < \theta_1 < \theta_2 < \pi$, and continuous on the closure $\overline{S}$. If*

$$|F(z)| \le 1 \tag{2.5}$$

for $z \in \partial S$: the boundary of S, and

$$|F(z)| \le Ce^{C|z|^\gamma}$$

for all $z \in S$, where $0 \le \gamma < \beta$ and $C > 0$, then (2.5) holds also for all z in S.

The proof of the above lemma can be found in Stein and Shakarchi [26].

3. Proof of the main result

We first set $v = J^m u$ and $g_0(t) := v_x(0,t)$. From the conclusion of the above sections, it follows that $g_0 \in C^2[0,T]$. Therefore we can smoothly extend g_0 for $t > T$ such that $g_0 = 0$ in $(T+1,\infty)$. If no conflict occurs, we still denote the extension as g_0. Thus the initial-boundary value problem (2.1) and the lateral Cauchy observation can be rephrased into the initial-boundary value problem (2.2) with the additional condition

$$v(x,t) = 0, \quad t \in (0,T).$$

Taking Laplace transform with respect to t on both sides of the equation in (2.1) implies

$$\begin{cases} -\partial_x^2 \hat{v}(x;s) + \int_0^1 w(\alpha)s^\alpha d\alpha \hat{v}(x;s) = \hat{F}(x;s), & x \in (0,1),\ s>0, \\ \widehat{v}_x(0;s) = \int_0^{T+1} g_0(t)e^{-st}dt, \quad \widehat{v}_x(1;s)=0, & s>0. \end{cases} \tag{3.1}$$

Here

$$\hat{F} := \int_0^1 w(\alpha)s^{\alpha-m}d\alpha u_0.$$

Lemma 3.1. *Let* $\varphi(x,\zeta) := \cos(\zeta x)$, *then*

$$\begin{aligned} &\int_0^1 (-\partial_x^2 \widehat{v}(x;s))\varphi(x,\zeta)dx \\ &= \int_0^{T+1} g(t)e^{-st}dt + \zeta^2 \int_0^1 \widehat{v}(x;s)\varphi(x,\zeta)dx + \varphi'(1,\zeta)\widehat{v}(1;s). \end{aligned}$$

Proof. This can be directly done by integration by parts twice and taking the conditions $\widehat{v}_x(1;s)=0$, $\widehat{v}_x(0;s) = \int_0^{T+1} g_0(t)e^{-st}dt$ and $\varphi'(0,\zeta)=0$ into account. ☐

Now multiplying $\varphi(x,\zeta)$ on the equation in (3.1), and integrating from $x=0$ to $x=1$, from the above lemma, we find

$$\begin{aligned} &\int_0^1 w(\alpha)s^{\alpha-m}d\alpha \int_0^1 u_0(x)\varphi(x,\zeta)dx \\ &= \left(\int_0^1 w(\alpha)s^\alpha d\alpha + \zeta^2\right)\int_0^1 \widehat{v}(x;s)\varphi(x,\zeta)dx + \varphi'(1,\zeta)\widehat{v}(1;s) + \int_0^{T+1} g_0(t)e^{-st}dt. \end{aligned}$$

Letting $\zeta = i\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}$, hence $\int_0^1 w(\alpha)s^\alpha d\alpha + \zeta^2 = 0$, we see that

$$\int_0^1 w(\alpha)s^{\alpha-m}d\alpha \int_0^1 u_0(x)\varphi(x,\zeta)dx = \varphi'(1,\zeta)\widehat{v}(1;s) + \int_0^{T+1} g_0(t)e^{-st}dt. \tag{3.2}$$

Based on this integral equation involved the initial value and boundary condition, we can derive an estimate for the initial value as follows.

Lemma 3.2. *Let $T>0$ be a fixed constant and $u \in L^2(0,T;H^2(0,1))$ be a solution to the problem* (2.1). *Then the exists a sufficiently large integer N such that the inequality*

$$\left|\int_0^1 u_0(x)e^{sx}dx\right| \le C$$

holds true for any $s>0$, where the constant C is independent of s.

Proof. For $\zeta = i\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}$, we see that

$$|\varphi'(1,\zeta)| = |\zeta|\,|\sin(\zeta)| \le C|\zeta|e^{|\zeta|},$$

which combined with (3.2) implies

$$\begin{aligned} &\left|\int_0^1 u_0(x)\varphi(x,\zeta)dx\right| \\ &\le \frac{C}{\int_0^1 w(\alpha)s^{m-\alpha}d\alpha}|\zeta|e^{|\zeta|}\left|\int_0^\infty v(1,t)e^{-st}dt\right| + \frac{1}{\int_0^1 w(\alpha)s^{m-\alpha}d\alpha}\left|\int_0^{T+1} g_0(t)e^{-st}dt\right|, \end{aligned}$$

for $|\zeta|^2 = \int_0^1 w(\alpha)s^\alpha d\alpha$. From Lemma 2.1, we see that $|v(1,t)| \le Ce^{Ct}$, and using the fact that $v(1,t)=0$ for $t\in[0,T]$, we have

$$\int_T^{+\infty} |v(1,t)|e^{-st}dt \le \int_T^M Ce^{(C-s)t}dt = \frac{Ce^{CT}}{s-C}e^{-sT}, \quad s>2C.$$

From the choice of g, we see that

$$\left|\int_0^{T+1} g_0(t)e^{-st}dt\right| \le \|g_0\|_{L^1(0,T+1)}, \quad s>0.$$

On the basis of the above calculation, we further arrive at the following inequality

$$\begin{aligned}&\left|\int_0^1 u_0(x)\varphi(x,\zeta)dx\right| \\ \le &\frac{C}{\sqrt{\int_0^1 w(\alpha)s^{m-\alpha}d\alpha}} e^{\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}} \frac{e^{CT}}{s-C}e^{-sT} + \frac{1}{\int_0^1 w(\alpha)s^{m-\alpha}d\alpha}\|g_0\|_{L^1(0,T+1)}, \quad s>2C.\end{aligned}$$

Since $\alpha \in (0,1)$ and m is large enough, we take s being sufficiently large and then we see that

$$\left|\int_0^1 u_0(x)\varphi(x,\zeta)dx\right| \le C, \quad s >> 1.$$

From the Euler formula, it follows that

$$\left|\int_0^1 u_0(x)\frac{e^{i\zeta x}-e^{-i\zeta x}}{2}dx\right| \le C, \quad s>>1.$$

Which further implies that

$$\left|\int_0^1 u_0(x)e^{x\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}}dx\right| \le C + \left|\int_0^1 u_0(x)e^{-x\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}}dx\right| \le C, \quad s>>1.$$

By changing $\sqrt{\int_0^1 w(\alpha)s^\alpha d\alpha}$ to s, we finally obtain

$$\left|\int_0^1 u_0(x)e^{sx}dx\right| \le C, \quad s>>1.$$

This completes the proof of the lemma. □

Setting

$$H(z) := \int_0^1 u_0(x)\varphi(x,z)dx \tag{3.3}$$

for z in the complex plane $\mathbb{C}$. From Phragmén-Lindelöf principle in Lemma 2.3, it is not very difficult to see that the function $H(z)$ defined by (3.3) is holomophic on the complex plane, and we conclude from Liouville theorem (see, e.g., Stein and Shakarchi[26].) that $H(z)$ must be constant on the whole complex plane. Moreover, we see that

Corollary 3.3. *The function $H(\cdot)$ defined by* (3.3) *is identically vanished on the whole complex plane.*

Proof. From the above corollary, we can assume that $H(z) = C$, $z \in \mathbb{C}$. Letting $z > 0$, we consider the limit of $H(z)$ as $z \to +\infty$. From the asymptotic behavior of the function $\varphi(x,z)$, we see that

$$\lim_{z\to+\infty} H(z) = \lim_{z\to-\infty}\int_0^1 u_0(x)e^{zx}dx = 0.$$

That is, $H(z) \equiv 0$, $z \in \mathbb{C}$. □

Now we are ready to give the proof of the uniqueness of our inverse problem.

Proof of Theorem 1.1. We first extend the function u_0 by letting $u_0 = 0$ outside of $x \in (0,1)$, that is we have

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & x\in(0,1), \\ 0, & x \in (1,\infty). \end{cases}$$

Based on the above notation, it follow easily that

$$\int_0^1 u_0(x)e^{zx}dx = \int_0^\infty \tilde{u}_0(x)e^{zx}dx = 0.$$

Letting $z = -s$, $s \in \mathbb{C}$, we see that

$$\int_0^\infty \tilde{u}_0(x)e^{-sx}dx = 0, \quad s \in \mathbb{C}.$$

From the uniqueness of the Laplace transform, we must have $\tilde{u}_0 = 0$, which finishes the proof of the theorem. □

4. Numerical Simulation

In this section, we are devoted to developing an effective numerical method for the numerical reconstruction of the unknown initial value in $(0, 1)$ from the addition data $u(1, t)$, $u_x(1, t)$ in $(0, T)$.

4.1. Inversion algorithm description

In this subsection, we give a direct description for our proposed inversion algorithm. Obviously, the solution $u(x, t)$ to the direct problem (2.1) is relative to the boundary condition $g_0(t)$ and initial value $u_0(x)$. We denote by $u(g_0, u_0)(x, t)$ the solution to the corresponding direct problem. Similarly, the corresponding observation data $u(g_0, u_0)(1, t)$ is denoted by $H(t)$. As we know, the observation errors are inevitable. We assume the noise contaminated measurement data $H^\delta(t)$ satisfy the following estimate

$$\|H(\cdot) - H^\delta(\cdot)\|_{L^2(0,T)} \le \delta.$$

Based on the optimal ideal for solving the inverse problem, we transform the inverse problem into the following least square functional optimization problem

$$\min_{(g_0,u_0)\in\mathcal{A}} \frac{1}{2}\|u(g_0, u_0)(1, \cdot) - H^\delta(\cdot)\|^2_{L^2(0,T)}, \tag{4.1}$$

where $\mathcal{A} = C[0, T] \times C[0, 1]$. In order to conveniently describe the inversion algorithm, we adopt the principle of superposition to make the boundary condition homogeneous. Let $\bar{u}(x, t) = \frac{g_1 - g_0}{2}x^2 + g_0x$. By direct derivation, $v(x, t) = u(x, t) - \bar{u}(x, t)$ satisfies the following homogeneous Neumann boundary value problem

$$\begin{cases} \partial_t^w v - \partial_x^2 v = f(x, t) & \text{in } (0, 1) \times (0, \infty), \\ v(\cdot, 0) = v_0, & \text{in } (0, 1), \\ v_x(0, \cdot) = 0,\ v_x(1, \cdot) = 0 & \text{in } (0, \infty). \end{cases}$$

Where

$$v_0 = u_0 - \frac{g_1(0) - g_0(0)}{2}x^2 - g_0(0)x, \tag{4.2}$$

$$f(x, t) = g_1 - \frac{x^2}{2}\partial_t^w g_1 - g_0 + (\frac{x^2}{2} - x)\partial_t^w g_0. \tag{4.3}$$

Similarly, based on the principle of superposition we have $v(x, t) = V(f)(x, t)+W(v_0)(x, t)$, where $V(f)(x, t)$ and $W(v_0)(x, t)$ meet the following subproblem **P1** and subproblem **P2**, respectively.

$$\mathbf{P1)} \quad \begin{cases} \partial_t^w V - \partial_x^2 V = f(x, t) & \text{in } (0, 1) \times (0, \infty), \\ V(\cdot, 0) = 0, & \text{in } (0, 1), \\ V_x(0, \cdot) = 0,\ V_x(1, \cdot) = 0 & \text{in } (0, \infty), \end{cases} \tag{4.4}$$

$$\mathbf{P2)}\quad \begin{cases} \partial_t^w W - \partial_x^2 W = 0 & \text{in } (0,1)\times(0,\infty), \\ W(\cdot,0) = v_0, & \text{in } (0,1), \\ W_x(0,\cdot) = 0,\ W_x(1,\cdot) = 0 & \text{in } (0,\infty). \end{cases} \tag{4.5}$$

Therefore, we can rewrite the functional optimization problem (4.1) as follows

$$\min_{(g_0,u_0)\in\mathcal{A}} \frac{1}{2}\|V(f)(1,\cdot) + W(v_0)(1,\cdot) + \bar{u}(1,t) - H(\cdot)\|^2_{L^2(0,T)}. \tag{4.6}$$

Next, we change the infinite dimensional optimization problem into the finite dimensional optimization problem by taking function approximation technique. Here we take the finite element interpolation technique to approximate the $u_0(x)$ and the termed Fourier series expansion method to approach functions $g_0(t)$. We firstly divide the space domain $[0,1]$ with regular partitions S_h. The regular partition S_h is composed of M_1 equal space subintervals. Let $\{x_\iota\}_{\iota=0}^{M_1}$ be the set of the space nodes, i.e., $x_\iota = \iota * h$, where $h = \frac{1}{M_1}$. We denote $\mathcal{W}_h$ the continuous piecewise linear finite element space defined over S_h, i.e.,

$$\mathcal{W}_h = \{\ s : s \in C[0,1],\ s\,|_{\Delta_h} \in P_1(\Delta_h),\ \forall\ \Delta_h \in\ S_h\ \}.$$

Then any $s_h \in \mathcal{W}_h$ can be repeated as $s_h(x) = \sum\limits_{\iota=0}^{M_1} s_\iota \phi^\iota(x)$, where s_ι is the value of $s_h(x)$ at point x_ι. $\phi^\iota(x)$ are the piecewise linear basis functions defined as follows,

$$\phi^0(x) = \begin{cases} -\frac{x-x_1}{h}, & x\in[x_0,x_1], \\ 0, & \text{othewise}, \end{cases} \quad \phi^{M_1}(x) = \begin{cases} \frac{x-x_{M_1-1}}{h}, & x\in[x_{M_1-1},x_{M_1}], \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi^\iota(x) = \begin{cases} \frac{x-x_{\iota-1}}{h}, & x\in[x_{\iota-1},x_\iota], \\ -\frac{x-x_{\iota+1}}{h}, & x\in(x_\iota,x_{\iota+1}], \\ 0, & \text{otherwise}. \end{cases} \quad \iota = 1,\cdots,M_1-1.$$

Applying the piecewise linear interpolation technique, we can approximate the initial value $u_0(x)$ in the finite element space $\mathcal{W}_h$ as follows

$$u_0(x) \approx u_{0,h}(x) = \sum_{\iota=0}^{M_1} u_{0,\iota}\phi^\iota(x),$$

where $u_{0,\iota} := u_0(x_\iota),\ \iota = 0,\cdots,M_1$. Based on the idea of subspace projection, here we adopt the truncated Fourier series expansion method to finitely approximate the boundary condition $g_0(t)$, i.e.,

$$g_0(t) \approx g_{0,\tau}(t) = \sum_{\iota=0}^{K_1} g_{0,\iota}\varrho^\iota(t),$$

where $g_{0,\iota} := (g_0(t),\varrho^\iota(t)),\ \iota = 0,\ 1,\cdots,K_1$, the symbol $(\cdot,\cdot)$ represents the scalar product on the interval $[0,T]$ and

$$\varrho^\iota(t) = \begin{cases} \sqrt{\frac{1}{2}}\sin\frac{(\iota+1)\pi t}{2}, & \iota \text{ is odd number}, \\ \sqrt{\frac{1}{2}}\cos\frac{\iota\pi t}{2}, & \iota \text{ is even number}. \end{cases}$$

Denoted by $\bar{f}(x,t) = g_1(t) - \frac{x^2}{2}\partial_t^w g_1(t)$, $f_l(x,t) = (\frac{x^2}{2}-x)\partial_t^w\varrho^\iota(t) - \varrho^\iota(t)$, $\bar{u}_0(x) = \frac{x^2}{2}-x$. Substituting $u_0(x)$ and $g_0(t)$ by $u_{0,h}(x)$ and $g_{0,\tau}(t)$ in expression (4.2) and (4.3), we get the following approximating relationships

$$\begin{aligned} v_0(x) &\approx \sum_{\iota=0}^{M_1} u_{0,\iota}\phi^\iota(x) - \frac{x^2}{2}g_1(0) + \bar{u}_0(x)\sum_{\iota=0}^{[\frac{K_1}{2}]} g_{0,2\iota}, \\ f(x,t) &\approx \sum_{\iota=0}^{K_1} f_l(x,t)g_{0,\iota} + \bar{f}(x,t). \end{aligned}$$

Therefore, the infinite dimensional optimization problem (4.6) can be approximated by the following finite dimensional optimization problem

$$\min_{\substack{\vec{g}_0 \in \mathbb{R}^{K_1+1} \\ \vec{u}_0 \in \mathbb{R}^{M_1+1}}} \frac{1}{2}\Big\|\sum_{\iota=0}^{K_1} g_{0,\iota}(V(f_\iota)(1,\cdot)+\frac{1}{2}\varrho^\iota)+\sum_{\iota=0}^{M_1} u_{0,\iota}W(\phi^\iota)(1,\cdot)+\sum_{\iota=0}^{[\frac{K_1}{2}]} g_{0,2\iota}W(\bar{u}_0)(1,\cdot)-\bar{H}\Big\|^2_{L^2(0,T)},$$

where $\bar{H}(t) = H(t) - V(\bar{f})(1,t) + W(\frac{x^2}{2})(1,t)g_1(0) - \frac{1}{2}g_1(t)$, $\vec{g}_0 = (g_{0,0},\cdots,g_{0,K_1})^T$, $\vec{u}_0 = (u_{0,0},\cdots,u_{0,M_1})^T$. For briefness, we introduce the following notation

$$a_\iota(t) = W(\phi^\iota)(1,t),\ \iota = 0,1,\cdots,M_1,$$
$$a_{M_1+\iota+1}(t) = V(f_\iota)(1,t) + \frac{1}{2}\varrho^\iota(t) + \chi_\iota W(\bar{u}_0)(1,t),\ \iota = 0,1,\cdots,K_1,$$

where χ_ι is defined by

$$\chi_\iota = \begin{cases} 1, & \iota \text{ is even}, \\ 0, & \iota \text{ is odd}. \end{cases}$$

Then the above finite dimensional optimization problem can be rewritten into the following form

$$\min_{\substack{\vec{g}_0 \in \mathbb{R}^{K_1+1} \\ \vec{u}_0 \in \mathbb{R}^{M_1+1}}} \frac{1}{2}\|\sum_{\iota=0}^{M_1} u_{0,\iota}a_\iota(\cdot) + \sum_{\iota=0}^{K_1} g_{0,\iota}a_{M_1+1+\iota}(\cdot) - \bar{H}(\cdot)\|^2_{L^2(0,T)}, \tag{4.7}$$

By the necessary conditions of the extremum problem, we can obtain the normal equations of quadratic functional (4.7) as follows

$$A\Phi = \mathcal{F}, \tag{4.8}$$

where

$$\begin{cases} A_{i,j} = (a_i, a_j),\ i,j = 0,1,\cdots,M_1+K_1+1, \\ \Phi = (u_{0,0},u_{0,1},\cdots,u_{0,M_1},g_{0,0},g_{0,1},\cdots,g_{0,K_1})^T, \\ \mathcal{F}_i = (a_i,\bar{H}(t)),\ i = 0,1,\cdots,M_1+K_1+1. \end{cases}$$

As we all know, the initial value identification is ill-posed problem, which means the linear algebra equations are ill-conditioned. Therefore, some regularized technique is necessary. Here we adopt the truncated singular value decomposition method to solve the system (4.8), the corresponding regularized solution is denoted by $\Phi_{\mathbf{N}}$, where symbol $\mathbf{N}$ represents the number of the truncated term. The number of the truncated term $\mathbf{N}$ is determined by the following Morozov discrepancy principle

$$\|A\Phi_{\mathbf{N}} - \mathcal{F}\| = \bar{\tau}\delta,$$

where $\bar{\tau} \geq 1$ is a positive constant. In numerical implementations, we choose $\mathbf{N} = \mathbf{N}_{\bar{\tau}}$ satisfying the following inequality:

$$e_{\mathbf{N}_{\bar{\tau}}} \leq \bar{\tau}\delta \leq e_{\mathbf{N}_{\bar{\tau}}-1}, \tag{4.9}$$

where $e_{\mathbf{N}} = \|A\Phi_{\mathbf{N}} - \mathcal{F}\|$. Hereto, we can formulate the inversion algorithm for the initial function as follows

Algorithm 1 inversion method for the initial function $u_0(x)$

1: Input the observation data $H^\delta(t)$, the constant $\bar{\tau}$ and set $\mathbf{N} = 1$.
2: Compute the subproblem (4.4) with $f(x,t) = f_i, (i = 0, \cdots, K_1)$ in parallel and compute subproblem (4.5) with $v_0(x) = \bar{u}_0$, $\phi^\iota(\iota = 0, \cdots, M_1)$ in parallel to get the elements $A_{i,j}$ of the matrix A and the elements $\mathcal{F}_i$ of the vector $\mathcal{F}$.
3: Obtain the regularization solution $\Phi_\mathbf{N}$ by solving system (4.8) with truncated singular value decomposition method.
4: Check whether the stopping criterion (4.9) is met. If the stopping criterion (4.9) is met, then go to step 5. Otherwise, update the number of the truncated term $\mathbf{N} := \mathbf{N} + 1$, return step 3.
5: Output the regularized solution $u_{0,h}^{\mathbf{N}_{\bar{\tau}}}(x) = \sum_{\iota=0}^{M_1} \Phi_{\mathbf{N}_{\bar{\tau}}}(\iota)\phi^\iota(x)$.

4.2. Numerical examples

In this subsection, several numerical examples are implemented to illustrate the effectiveness of the inversion algorithm. Without loss of generality, we take $T = 1$ and $g_1(t) = 0$ for all the numerical simulations. To obtain the (noisy) observation data $H^\delta(t)$, we first endow the problem (2.1) with true initial function $u_0^\star(x)$ and exact boundary condition $g_0^\star(t)$, and solve the direct problem (2.1) by finite difference method proposed in [18], then add pointwise noise by $H^\delta(t_i) = u(g_0^\star, u_0^\star)(1, t_i)(1 + \delta\xi)$, where t_i is the discretization time nodal point, δ is the noise level and ξ is a uniform random variable on $[-1, 1]$.

Example 4.1. Let $\omega(\alpha) = \Gamma(6 - \alpha)$ We take $u_0(x) = -x^3 + \frac{3}{2}x^2 + 10$ as the exact initial condition and $g_0(t) = \sin(t)$ as the true boundary condition. In the numerical implementations, we take $M_1 = 50$, $K_1 = 2$.

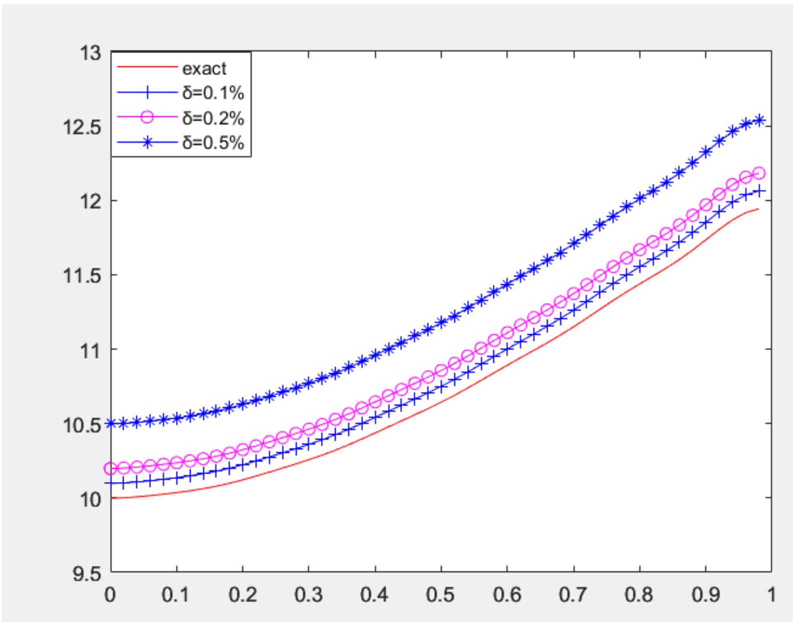

Figure 1. inversion solution for example 1 with different error levels

Example 4.2. Let $\omega(\alpha) = 10 + \alpha$. We take $u_0(x) = x^4 - \frac{4}{3}x^3 + 20 - 10\cos(\pi x)$ as the exact initial condition and $g_0(t) = \sin(t) + 5\sin(3t)$ as the true boundary condition. In the numerical procedure, we take $M_1 = 50$, $K_1 = 4$.

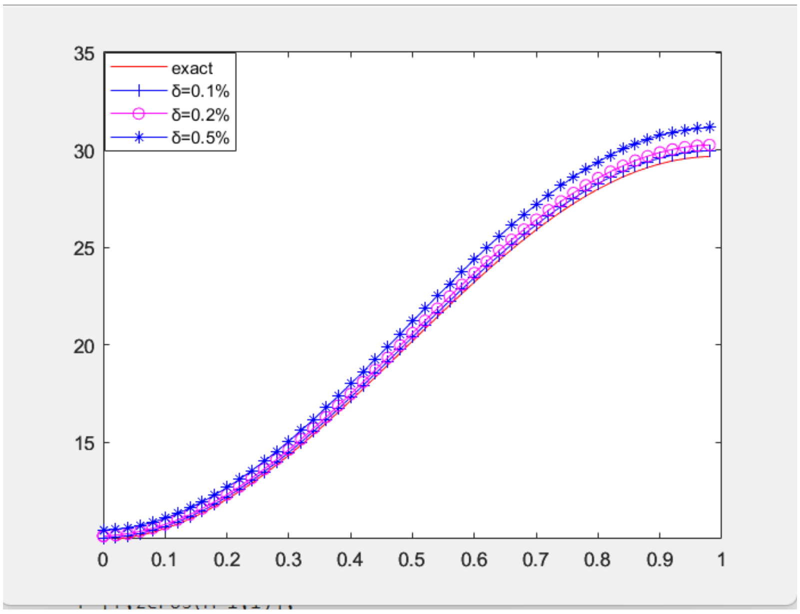

Figure 2. inversion solution for example 2 with different error levels

Example 4.3. Let $\omega(\alpha) = 10 + 5\alpha^2$. We take $g_0(t) = \sin(t) + 2\sin(5t)$ as the true boundary condition, and the exact initial function as the following piece-wise linear form

$$u_0(x) = \begin{cases} 20x, \ x \in [0, \frac{1}{2}), \\ 20(1-x), \ x \in [\frac{1}{2}, 1]. \end{cases}$$

We take $M_1 = 50, K_1 = 5$ in the numerical inversion procedure.

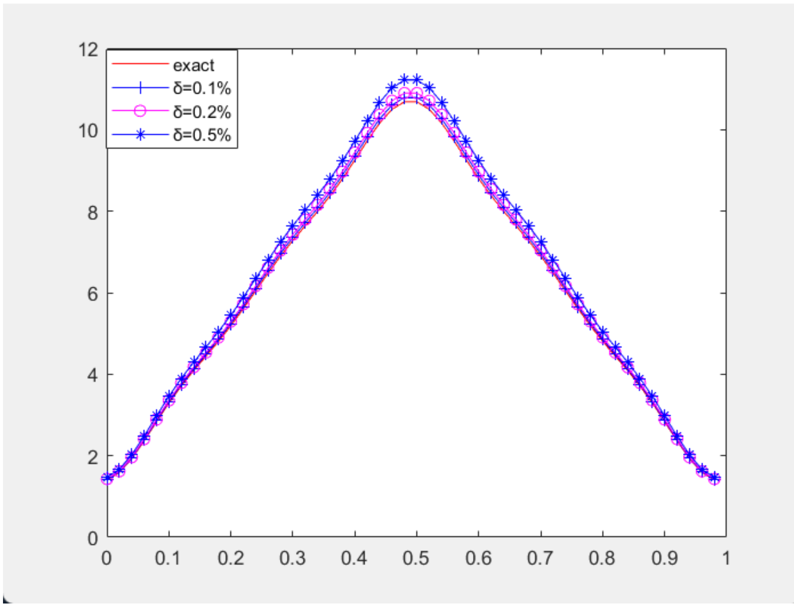

Figure 3. inversion solution for example 3 with different error levels

The reconstructions for examples 4.1-4.3 are indicated in figure 1-3. From the three figures, we can find that the smaller error level δ is, the better the computed approximation is. In all, the numerical experiment shows the proposed method work effectively.

5. Concluding remarks

In this paper, we considered the one-dimensional time-fractional diffusion equation with distributed order derivative. By using Fourier and Laplace transforms argument, we changed the inverse problem to an integral identity involving the initial value and measurement data. On the basis of the above identity, the uniqueness of the inverse problem was proved by using the Phragmén-Lindelöf principle and the Liouville theorem (Phragmén-Lindelöf-Liouville argument for short). It should be pointed out that our method cannot work for the case of the fractional order $\alpha \in (1,2)$. The main reason is because two initial conditions will lead to the appearance of multivalued functions in the discussion. We will consider it in the next paper.

Let us mention that the Phragmén-Lindelöf-Liouville argument used heavily relies on the dimension of the space. It would be interesting to investigate what happens about the inverse problem with lateral Cauchy data in the general dimensional case.

In the numerical aspect, we have also applied the classical Tikhonov regularization method to transform the inverse problem into a minimization problem, which is solved by an iterative thresholding algorithm. Several numerical examples are presented to show the accuracy and effectiveness of the proposed algorithm.

Acknowledgement

The first author thanks National Natural Science Foundation of China 12271277. The third author is supported by National Natural Science Foundation of China (12061008). This work is (partly) supported by the Open Research Fund of Key Laboratory of Nonlinear Analysis & Applications (Central China Normal University), Ministry of Education, P. R. China.

References

[1] D. Araya and C. Lizama, *Almost automorphic mild solutions to fractional differential equations*, Nonlinear Analysis Theory Methods and Applications, **69**(11), 3692–3705, 2008.

[2] J.M. Boggs and E.E. Adams, *Field study of dispersion in a heterogeneous aquifer: 4. investigation of adsorption and sampling bias*, Water Resources Research, **28**(12), 3325–3336, 1992.

[3] J.P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Physics reports, **195**(4-5), 127–293, 1990.

[4] M. Caputo, *Mean fractional-order-derivatives differential equations and filters*, Annali dellUniversita di Ferrara, **41**(1), 73–84, 1995.

[5] J.M. Carcione, F.J. Sanchez-Sesma, F. Luzón, and J.J. Perez Gavilán, *Theory and simulation of time-fractional fluid diffusion in porous media*, Journal of Physics A: Mathematical and Theoretical, **46**(34), 345501, 2013.

[6] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, and V.Y. Gonchar, *Distributed order time fractional diffusion equation*, Fractional Calculus and Applied Analysis, **6**(3), 259–280, 2003.

[7] J. Cheng, J. Nakagawa, M. Yamamoto, and T. Yamazaki, *Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation*, Inverse problems, **25**(11), 115002, 2009.

[8] Y. Gong, P. Li, X. Wang and X. Xu, *Numerical solution of an inverse random source problem for the time fractional diffusion equation via phaselift*, Inverse Problems, **37**(4), 045001, 2021.

[9] Y. Hatano and N. Hatano, *Dispersive transport of ions in column experiments: An explanation of long-tailed profiles*, Water resources research, **34**(5), 1027–1033, 1998.

[10] X. Huang, Y. Kian, E. Soccorsi, and M. Yamamoto, *Determination of source and initial values for acoustic equations with a time-fractional attenuation*, arXiv preprint arXiv:2111.05240, 2021.

[11] D. Jiang, Z. Li, Y. Liu, and M. Yamamoto, *Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations*, Inverse Problems, **33**(5), 055013, 2017.

[12] B. Jin and W. Rundell, *A tutorial on inverse problems for anomalous diffusion processes*, Inverse problems, **31**(3), 035003, 2015.

[13] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, volume 204. elsevier, 2006.

[14] M. Kirane and S.A. Malik, *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time*, Applied Mathematics and Computation, **218**(1), 163–170, 2011.

[15] Y. Liu, G. Hu, and M. Yamamoto, *Inverse moving source problem for time-fractional evolution equations: determination of profiles*, Inverse Problems, **37**(8), 084001, 2021.

[16] Z. Li, Y. Kian and E. Soccorsi, *Initial-boundary value problem for distributed order time-fractional diffusion equations*, Asymptotic Analysis, **115**(1-2), 95–126, 2019.

[17] Z. Li, Y. Liu and M. Yamamoto, *Inverse source problem for a one-dimensional time-fractional diffusion equation and unique continuation for weak solutions*, arXiv e-prints arXiv:2112.01018, 2021.

[18] X. Li and H. Rui, *A block-centered finite difference method for the distributed-order time-fractional diffusion-wave equation*, Applied Numerical Mathematics, **131**, 123–139, 2018.

[19] J.J. Liu and M. Yamamoto, *A backward problem for the time-fractional diffusion equation*, Applicable Analysis, **89**(11), 1769–1788, 2010.

[20] Z. Li and M. Yamamoto, *Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation*, Applicable Analysis, **94**(3), 570–579, 2015.

[21] Z. Li and M. Yamamoto, *Unique continuation principle for the one-dimensional time-fractional diffusion equation*, Fractional Calculus and Applied Analysis, **22**, 644–657, 2019.

[22] Y. Liu, W. Rundell, and M. Yamamoto, *Strong maximum principle for fractional diffusion equations and an application to an inverse source problem*, Fractional Calculus and Applied Analysis, **19**(4), 888–906, 2016.

[23] F. Mainardi, A. Mura, G. Pagnini and R. Gorenflo, *Time-fractional diffusion of distributed order*, Journal of Vibration and Control, **14**(9-10), 1267–1290, 2008.

[24] K. Sakamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, Journal of Mathematical Analysis and Applications, **382**(1), 426–447, 2011.

[25] K. Sakamoto and M. Yamamoto, *Inverse source problem with a final overdetermination for a fractional diffusion equation*, Math. Control Relat. Fields, **1**(4), 509–518, 2011.

[26] E.M. Stein and R. Shakarchi, *Complex analysis*, volume 2. Princeton University Press, 2010.

[27] N.H. Tuan, L.N. Huynh, T.B. Ngoc and Y. Zhou, *On a backward problem for nonlinear fractional diffusion equations*, Applied Mathematics Letters, **92**, 76–84, 2019.

[28] N.H. Tuan, L.D. Long and S. Tatar, *Tikhonov regularization method for a backward problem for the inhomogeneous time-fractional diffusion equation*, Applicable Analysis, **97**(5), 842–863, 2018.

[29] L. Wang and J. Liu, *Total variation regularization for a backward time-fractional diffusion problem*, Inverse problems, **29**(11), 115013, 2013.

[30] J.G. Wang, T. Wei, and Y.B. Zhou, *Tikhonov regularization method for a backward problem for the time-fractional diffusion equation*, Applied Mathematical Modelling, **37**(18-19), 8518–8532, 2013.

[31] Z. Zhang, *An undetermined coefficient problem for a fractional diffusion equation*, Inverse Problems, **32**(1), 015011, 2015.

[32] Y. Zhang and X. Xu, *Inverse source problem for a fractional diffusion equation*, Inverse problems, **27**(3), 035010, 2011.