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VARIOUS RESULTS ON SOME ASYMMETRIC TYPES OF DENSITY

Nezakat JAVANSHIR[1] and Filiz YILDIZ[2]

[1]Ankara Yıldırım Beyazıt University, Department of Mathematics, Ankara, TÜRKİYE
[2]Hacettepe University, Faculty of Science, Department of Mathematics, Beytepe, Ankara, TÜRKİYE

Abstract. The structures of symmetric connectedness and dually, antisymmetric connectedness were described and studied before, especially in terms of graph theory as the corresponding counterparts of the connectedness of a graph and the connectedness of its complementary graph. By taking into consideration the deficiencies of topological density in the context of symmetric and antisymmetric connectedness, two special kinds of density in the theory of non-metric T_0-quasi-metrics were introduced in the previous studies under the names symmetric density and antisymmetric density. In this paper, some crucial and useful properties of these two types of density are investigated with the help of the major results and (counter)examples peculiar to the asymmetric environment. Besides these, many further observations about the structures of symmetric and antisymmetric-density are dealt with, especially in the sense of their combinations such as products and unions through various theorems in the context of T_0-quasi-metrics. Also, we examine the question of under what kind of quasi-metric mapping these structures will be preserved.

1. Introduction

In [11], symmetrically connected and dually, antisymmetrically connected T_0-quasi-metric spaces were described and studied in detail. These theories were especially discussed in the sense of the notions peculiar to graph theory [1, 4, 10] as the suitable counterparts of the connectedness for a graph and complementary graph of it, respectively. In particular, it was shown that there were natural relationships

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[1] nezakat.javanshir@aybu.edu.tr; 0000-0002-1780-4684;

[2] yfiliz@hacettepe.edu.tr- Corresponding author; 0000-0002-2112-8949

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between the theory of symmetrically connected - antisymmetrically connected T_0-quasi-metric spaces and the theory of connected graphs - connected complementary graphs.

Following these theories, some new types of density specific to T_0-quasi-metric subspaces in the asymmetric environment due to the apparent inadequacy of topological density in the transfer of properties symmetric and antisymmetric connectedness to the subspaces or superspaces are described in [9] under the names symmetric density and antisymmetric density.

As for the subject of this study, we observed some combinations of symmetrically / antisymmetrically-dense subspaces such as products, unions and intersections in the context of T_0-quasi-metrics. With this viewpoint, it is also natural to inquire whether the images of symmetrically / antisymmetrically-dense subspaces under an isometric isomorphism have the same property or not. Hereby, we will also obtain some crucial and useful results within this framework.

In the light of all these considerations, the content of paper is as follows:

Some necessary background material for the remaining of paper is presented in Section 2. After recalling the preliminary information, as one of the purposes of the paper, in Section 3 we discussed some properties and new (counter)examples of the symmetric density theory in the context of asymmetric topology. In addition, we presented some observations about the products, unions, intersections... of the symmetrically-dense T_0-quasi-metric subspaces and their preservation under the specific mappings peculiar to quasi-metrics.

Following these, in Section 4 some future properties and asymmetric aspects of antisymmetrically-dense subspaces are investigated with the help of many useful (counter)examples. The remainder of this section is devoted to discussing preservation of antisymmetric density under the specific mappings in the context of quasi-metrics as well as some combinations such as unions, products,... of the antisymmetrically-dense T_0-quasi-metric subspaces.

Consequently, Section 5 as the last part of the paper gives a conclusion about the whole of the work.

2. Background

This section will present some background material on T_0-quasi-metrics and particularly, it consists of the required information related to the theories of symmetrically connected and antisymmetrically connected spaces as well as antisymmetric spaces which are a kind of opposite to metric spaces.

All preliminary information presented in this section is taken from the references [3, 5–8, 12].

T_0-quasi-metrics:

Definition 1. *Let X be a set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a* T_0-quasi-metric *on X if*

(a) $d(x, x) = 0$
(b) $d(x, y) = 0 = d(y, x) \;\Rightarrow\; x = y$
(c) $d(x, z) \leq d(x, y) + d(y, z)$

whenever $x, y, z \in X$. Thus, (X, d) is called T_0-quasi-metric space.

Here the notation τ_{d^s} will be used to denote the topology induced by the symmetrization metric $d^s = d \vee d^{-1}$ where $d^{-1}(x, y) = d(y, x)$.

Example 1. *On $\mathbb{R}$, take*

$$u(x, y) = max\{x - y, 0\}$$

whenever $x, y \in \mathbb{R}$.

It is easy to prove that u satisfies the conditions of Definition 1, and u is called the standard *T_0-quasi-metric on $\mathbb{R}$.*

Now, let us recall some important notions and (counter)examples related to the theories constructed in [11]:

Symmetrically connected spaces:

Definition 2. *Let (X, d) be a T_0-quasi-metric space.*

i) *A pair $(x, y) \in X \times X$ is called symmetric pair if $d(x, y) = d(y, x)$.*
ii) *A finite sequence of points in X, starting at x and ending with y, is called a (finite)* symmetric path *$P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ (where $n \in \mathbb{N}$) from x to y provided that all the pairs (x_i, x_{i+1}) are symmetric where $i \in \{0, \ldots, n-1\}$.*

For a T_0-quasi-metric space (X, d), we take

$$Z_d = \{(x, y) \in X \times X : d(x, y) = d(y, x)\}$$

as the set of symmetric pairs in (X, d). Note that this relation is reflexive and symmetric.

Incidentally, note that

$$d^s(x, y) = d(x, y) = d^{-1}(x, y)$$

for $(x, y) \in Z_d$.
Also,

$$Z_d(x) = \{y \in X \mid (x, y) \in Z_d\}$$

is called *symmetry set* of $x \in X$.

Definition 3. *If (X, d) is a T_0-quasi-metric space and $x, y \in X$ then $x \in X$ is* symmetrically connected *to $y \in X$ whenever there is a symmetric path $P_{x,y}$, starting at the point x and ending at the point y.*

Clearly, "symmetric connectedness" is an equivalence relation on X, by definition.

Definition 4. *The equivalence class of a point $x \in X$ with respect to the symmetric connectedness relation is called the* symmetry component *of x.*

More clearly, if C_d denotes the *symmetric connectedness relation* then the symmetry component of $x \in X$ is

$$C_d(x) = \{y \in X : \text{there is a symmetric path from } x \text{ to } y\}.$$

We are now in a position to recall the following crucial notion:

Definition 5. *A T_0-quasi-metric space (X, d) such that $C_d(x) = X$ for all $x \in X$, is called* symmetrically connected*.*

Therefore, (X, d) is symmetrically connected if and only if for all $x, y \in X$, x and y are symmetrically connected by Definition 3, obviously.

At this stage, we will turn our attention to the dual counterparts of some notions described above.

Antisymmetrically connected spaces:

Definition 6. *Let (X, d) be a T_0-quasi-metric space, and $x, y \in X$. Then*

i) $(x, y) \in X \times X$ *is called* antisymmetric pair *if $d(x, y) \neq d(y, x)$*

ii) *A finite sequence of points in X, starting at x and ending with y, is called a (finite)* antisymmetric path $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ *(where $n \in \mathbb{N}$) from x to y provided that all the pairs (x_i, x_{i+1}) are antisymmetric where $i \in \{0, \ldots, n-1\}$.*

Definition 7. *In a T_0-quasi-metric space (X, d), two points $x, y \in X$ are called* antisymmetrically connected *if there is an antisymmetric path $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$, or $x = y$.*

Now, if we consider the relation

$$T_d := \{(x, y) \in X \times X : x \text{ and } y \text{ are antisymmetrically connected in } (X, d)\}$$

then T_d describes an equivalence relation on X, trivially.

Let us recall some other notions from [11]:

Definition 8. i) *The equivalence class of a point $x \in X$ with respect to T_d is called the* antisymmetry component *and it is denoted by*

$$T_d(x) = \{y \in X : \textit{there is an antisymmetric path from } x \textit{ to } y\}.$$

ii) *If $T_d(x) = X$ for each $x \in X$, then the space (X, d) is called* antisymmetrically connected.

Hence, (X, d) is antisymmetrically connected if and only if for all $x, y \in X$, x and y are antisymmetrically connected by Definition 7.

Example 2. *The T_0-quasi-metric space $(\mathbb{R}, u)$ given in Example 1 is antisymmetrically connected but not symmetrically connected.*

Antisymmetric spaces:

The following notion is described in [11] as opposite to that of "metric" :

Definition 9. *[11] A T_0-quasi-metric space (X, d) is called* antisymmetric *if $Z_d = \{(x, x) : x \in X\} = \Delta_X$, that is, if $d(x, y) = d(y, x)$ then $x = y$, for all $x, y \in X$.*

Symmetric-Antisymmetric points:

Definition 10. *Let (X, d) be a T_0-quasi-metric space and $x \in X$.*

i) *x is called* symmetric point *if $d(x, y) = d(y, x)$ for each $y \in X$.*
ii) *x is called* antisymmetric point *if $d(x, y) \neq d(y, x)$ for each $y \in X \setminus \{x\}$.*

According to the above descriptions, the next statements will be obvious:

Proposition 1. a) *A T_0-quasi-metric space which has a symmetric point will be symmetrically connected and not an antisymmetric space.*
b) *A T_0-quasi-metric space which has an antisymmetric point will be antisymmetrically connected and not a metric space.*

3. Some Further Properties and Examples of Symmetric Density

Firstly, let us recall the following notion from [9].

Definition 11. *Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. If for $x \in X \setminus A$, there exists $a_x \in A$ such that $d(x, a_x) = d(a_x, x)$ then A is called* symmetrically-dense *in (X, d).*

Example 3. *Let us define a T_0-quasi-metric p on the set $X = \{1, 2, 3\}$ via the matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & 9 & 8 \\ 9 & 0 & 1 \\ 10 & 1 & 0 \end{pmatrix}.$$

That is, $P = (p_{ij})$ where $p(i, j) = p_{ij}$ for $i, j \in X$. It is easy to prove that p is a T_0-quasi-metric on X. Specifically, now we will check the triangle inequality:

$p(1, 2) = 9 \leq 8 + 1 = p(1, 3) + p(3, 2)$, $p(3, 1) = 10 \leq 1 + 9 = p(3, 2) + p(2, 1)$
$p(1, 3) = 8 \leq 9 + 1 = p(1, 2) + p(2, 3)$, $p(2, 1) = 9 \leq 1 + 10 = p(2, 3) + p(3, 1)$
$p(2, 3) = 1 \leq 9 + 8 = p(2, 1) + p(1, 3)$, $p(3, 2) = 1 \leq 10 + 9 = p(3, 1) + p(1, 2)$.

Thus p satisfies the triangle inequality.

Here also note that $p(1, 2) = p(2, 1)$. Therefore, the subset $A = \{2, 3\}$ of X is symmetrically-dense in X. In addition, the subset $B = \{1\}$ is not symmetrically-dense since $p(3, 1) \neq p(1, 3)$.

Proposition 2. *Let (X, d) be a T_0-quasi-metric space with at least two-elements and let $x \in X$ be an antisymmetric point. Thus, the subsets $\{x\}$ and $X \setminus \{x\}$ are not symmetrically-dense in X.*

Proof. By Definition 10 ii), we have $d(x, y) \neq d(y, x)$ whenever $y \in X \setminus \{x\}$. Then, $\{x\}$ is not symmetrically-dense in X. Similarly, $y = x$ whenever $y \in X \setminus (X \setminus \{x\})$ and since x is antisymmetric point, $d(x, a) \neq d(a, x)$ for $a \in X \setminus \{x\}$. That is, $X \setminus \{x\}$ is not symmetrically-dense. □

Example 4. *Let $Y = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and define $f : Y \to [0, \infty)$ as follows:*

$$f(x,y) = \begin{cases} |x-y| & ;\ x < y \text{ and } (x,y) \neq (\frac{1}{2^{n+1}}, \frac{1}{2^n}),\ \forall n \in \mathbb{N} \\ \\ 2|x-y| & ;\ \text{otherwise} \end{cases}$$

for $x, y \in Y$.

The fact that f is a T_0-quasi-metric on Y is proved in [5].

Also, because of the inequality $f(a, 0) \neq f(0, a)$ for each $a \in Y \setminus \{0\}$ the point 0 is antisymmetric point. Thus (Y, f) is antisymmetrically connected by Proposition 1. Additionally, the space (Y, f) has symmetrically-dense subsets $T_f(\frac{1}{2})$, $T_f(\frac{1}{4})$... since they are antisymmetry components.

At this stage, we can compute $C_f(\frac{1}{2})$: For each $n \in \mathbb{N}$, we have that $\frac{1}{2^n} \in C_f(\frac{1}{2})$, since $(\frac{1}{2^n}, \dots, \frac{1}{2})$ is a symmetric path in (Y, f) from $\frac{1}{2^n}$ to $\frac{1}{2}$. But $0 \notin C_f(\frac{1}{2})$, since $f(0, \frac{1}{2^k}) = \frac{1}{2^k}$, $f(\frac{1}{2^k}, 0) = \frac{1}{2^{(k+1)}}$, for all $k \in \mathbb{N}$. Thus, $C_f(\frac{1}{2}) = Y \setminus \{0\} = V$, and so (Y, f) is not symmetrically connected. A similar argument shows that in V, $C_{f_V}(x) = V$ for every $x \in V$, and so (V, f_V) will be symmetrically connected.

Moreover, the subsets $\{0\}$ and V are not symmetrically-dense from Proposition 2.

Incidentally, we can recall from [9] the following characterization of the metrics via symmetric density, in the context of T_0-quasi-metrics:

Proposition 3. *Each nonempty subset of a T_0-quasi-metric space (X, d) is symmetrically-dense in X if and only if d is metric.*

As a result of Proposition 3 the next corollary is obvious.

Corollary 1. *Let (X, d) be a T_0-quasi-metric space. If each nonempty subset of (X, d) is symmetrically-dense in X then (X, d) is symmetrically connected.*

The converse of Corollary 1 may not be true by virtue of the following space.

Example 5. *Consider the Star Space (X, d) constructed in [5, Example 2.12], as follows:*

On $X = [0, \infty)$, take

$$d(x,y) = \begin{cases} x - y & ; \quad x \geq y \\ x + y & ; \quad x < y \end{cases}$$

for each $x, y \in X$. *Trivially,* 0 *is symmetric point since* $d(x, 0) = d(0, x)$ *for all* $x \in X$, *according to Definition 10. Thus,* (X, d) *is symmetrically connected by Proposition 1. Now consider the subset* $B = \{1\}$ *of* X. *Then it is easy to verify that* B *is not symmetrically-dense. Indeed, take* $2 \in X \setminus B$, *and note that* $d(1, 2) = 3 \neq 1 = d(2, 1)$.

Incidentally, the fact that any subspace of an antisymmetric T_0-quasi-metric space is antisymmetric is trivial by Definition 9. Nevertheless, a T_0-quasi-metric space (X, d) may not be antisymmetric even though (X, d) has a symmetrically-dense and antisymmetric subspace, as the following example shows:

Example 6. *It can be easily shown that Star Space* (X, d) *given in Example 5 is not antisymmetric by the fact that* $d(1, 0) = d(0, 1)$. *It is also easy to see* $X \setminus \{0\}$ *is symmetrically dense since* 0 *is a symmetric point. Moreover,* $X \setminus \{0\}$ *is an antisymmetric subspace since* $x = y$ *whenever* $d(x, y) = d(y, x)$ *for all* $x, y \in (0, \infty)$.

We are now in a position to recall a T_0-quasi-metric function described on the product of two T_0-quasi-metric spaces:

Remark 1. *Let* (X, d) *and* (Y, q) *be* T_0*-quasi-metric spaces. The function defined by*

$$D((x, y), (a, b)) = d(x, a) \vee q(y, b)$$

for each $(x, y), (a, b) \in X \times Y$ *gives a* T_0*-quasi-metric on the product set* $X \times Y$.

The fact that D is a T_0-quasi-metric can be verified since d and q are T_0-quasi-metrics.

Proposition 4. *Let* (X, d), (Y, q) *be* T_0*-quasi-metric spaces and* $A \subseteq X$, $B \subseteq Y$. *If* A *is symmetrically-dense in* X *and* B *is symmetrically- dense in* Y *then* $A \times B$ *is symmetrically-dense in* $X \times Y$.

Proof. Take $(x, y) \in (X \times Y) \setminus (A \times B)$. Since A is symmetrically-dense in (X, d) and B is symmetrically-dense in (Y, q), respectively there exist $a \in A$ and $b \in B$ such that $d(x, a) = d(a, x)$ and $q(y, b) = q(b, y)$. Now by the definition of product T_0-quasi-metric D on $X \times Y$, we have

$$D((x, y), (a, b)) = d(x, a) \vee q(y, b) = d(a, x) \vee q(b, y) = D((a, b), (x, y))$$

that is $A \times B$ is symmetrically-dense in $X \times Y$. □

The following result will be trivial via induction by Proposition 4.

Corollary 2. *For a* T_0*-quasi-metric space* (X, d), *the finite product of symmetrically-dense subsets of* X *is symmetrically-dense.*

Even though Proposition 4, we have the following example which states that if A is symmetrically-dense and B is not symmetrically-dense then $A \times B$ may not be symmetrically-dense.

Example 7. *Let us consider Star Space* (X, d) *from Example 5. Clearly the set* $A = \{0\}$ *is symmetrically-dense, and the set* $B = \{1\}$ *is not symmetrically-dense in* (X, d). *If we take the product* T_0*-quasi-metric* D *given in Remark 1 as* $D((x, y), (a, b)) = d(x, a) \vee d(y, b)$ *on* $X \times X$ *then the product set* $A \times B$ *is not symmetrically-dense since* $D((3, 5), (0, 1)) \neq D((0, 1), (3, 5))$ *for* $(3, 5) \in (X \times X) \setminus (A \times B)$ *and* $(0, 1) \in A \times B$.

At this stage, let us turn our attention to the following natural question:

Under which mappings is the symmetric density property preserved in the context of T_0-quasi-metric spaces ?

Proposition 5. *Let* (X, d), (Y, e) *be* T_0*-quasi-metric spaces and* $f : X \to Y$ *an isometric isomorphism. In this case,* $A \subseteq X$ *is symmetrically-dense in* X *if and only if* $f(A)$ *is symmetrically-dense in* Y.

Proof. Take $y \in Y \setminus f(A)$. So we have $x \in X \setminus A$ such that $f(x) = y$ since f is onto. By considering the symmetric density of A in X, there exists $a \in A$ such that $d(x, a) = d(a, x)$. Clearly, $f(a) \in f(A)$. Also, we have $e(f(x), f(a)) = d(x, a) = d(a, x) = e(f(a), f(x))$ since f is an isometry. Thus, $f(A)$ is symmetrically-dense in Y.

Conversely, if $a \in X \setminus A$ then $f(a) \notin f(A)$ as f is one-to-one. In this case, there exists $b \in f(A)$ satisfying the equality $e(b, f(a)) = e(f(a), b)$ since $f(A)$ is symmetrically-dense. Thus, there exists $z \in A$ such that $f(z) = b$ via the fact that f is onto. Now, by considering the isometry property of f the expression $d(z, a) = d(a, z)$ is obtained. This shows that A is symmetrically-dense. □

Proposition 6. *Let* (X, d) *be a* T_0*-quasi-metric space. If* A *is symmetrically dense in* X *and* $B \subseteq X$ *then* $A \cup B$ *is symmetrically-dense in* X.

Proof. If $x \in X \setminus (A \cup B)$ then there exists $a \in A$ such that $d(x, a) = d(a, x)$ since A is symmetrically-dense in X. Thus, $A \cup B$ is also symmetrically-dense in X due to $A \subseteq A \cup B$. □

As the consequence of Proposition 6 the following fact will be trivial.

Corollary 3. *The union of all subsets of a* T_0*-quasi-metric space which has at least one symmetrically-dense subset is symmetrically-dense.*

Despite the above fact, we have:

Remark 2. *The intersection of two symmetrically-dense subsets of a* T_0*-quasi-metric space may not be symmetrically-dense.*

Example 8. *On the set* $X = \{\frac{1}{2}, 1, 2, 3\}$ *consider the Sorgenfrey (bounded)* T_0*-quasi-metric* $b : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ *as*

$$b(x, y) = \begin{cases} \min\{1, x - y\} & ; \ x \geq y \\ 1 & ; \ x < y \end{cases}.$$

Now let us take $A = \{\frac{1}{2}, 1\} \subseteq X$. *In this case, since* $b(x, 1) = b(1, x)$ *for each* $x \in X \setminus A$ *the set* A *is symmetrically-dense in* X. *Similarly,* $b(x, 3) = b(3, x)$ *for each* $x \in X \setminus B$ *where* $B = \{\frac{1}{2}, 3\} \subseteq X$, *and so the set* B *is symmetrically-dense in* X. *But the intersection set* $A \cap B = \{\frac{1}{2}\}$ *is not symmetrically-dense in* X *because of the facts that* $b(1, \frac{1}{2}) \neq b(\frac{1}{2}, 1)$ *and* $1 \in X \setminus (A \cap B)$.

4. Some Further Properties and Examples of Antisymmetric Density

Definition 12. *[9] Let* (X, d) *be a* T_0*-quasi-metric space and* $A \subseteq X$. *If for* $x \in X \setminus A$, *there exists* $a_x \in A$ *such that* $d(x, a_x) \neq d(a_x, x)$ *then* A *is called* antisymmetrically-dense *in* (X, d).

Example 9. *Consider a* T_0*-quasi-metric on the set* $Y = \{1, 2, 3, 4\}$ *by the matrix*

$$\mathbf{Q} = \begin{pmatrix} 0 & 8 & 4 & 1 \\ 9 & 0 & 6 & 7 \\ 4 & 6 & 0 & 5 \\ 3 & 7 & 5 & 0 \end{pmatrix}.$$

That is, $Q = (q_{ij})$ *where* $q(i, j) = q_{ij}$ *for* $i, j \in Y$. *The function* q *will be a* T_0*-quasi-metric on* Y. *Indeed, it satisfies the other conditions of Definition 1, so we will prove just the triangle inequality:*

$q(1, 2) = 8 \leq 4 + 6 = q(1, 3) + q(3, 2)$, $q(1, 2) = 8 \leq 1 + 7 = q(1, 4) + q(4, 2)$
$q(1, 3) = 4 \leq 8 + 6 = q(1, 2) + q(2, 3)$, $q(1, 3) = 4 \leq 1 + 5 = q(1, 4) + q(4, 3)$
$q(1, 4) = 1 \leq 8 + 7 = q(1, 2) + q(2, 4)$, $q(1, 4) = 1 \leq 4 + 5 = q(1, 3) + q(3, 4)$
$q(2, 3) = 6 \leq 9 + 4 = q(2, 1) + q(1, 3)$, $q(2, 3) = 6 \leq 7 + 5 = q(2, 4) + q(4, 3)$
$q(2, 4) = 7 \leq 9 + 1 = q(2, 1) + q(1, 4)$, $q(2, 4) = 7 \leq 6 + 5 = q(2, 3) + q(3, 4)$
.
.
.
Other cases can be shown in a similar way.

Now, let us consider the subset $B = \{1, 2, 3\}$ *of* Y. *It is easy to verify that* B *is antisymmetrically-dense in* Y*: For each* $y \in Y \setminus B$ *we must find* $b \in B$ *such that* $q(y, b) \neq q(b, y)$. *Here, if* $y \in Y \setminus B$ *then* $y = 4$, *and clearly* $q(1, 4) \neq q(4, 1)$ *for* $1 \in B$.

Proposition 7. *Let* (X, d) *be a* T_0*-quasi-metric space with at least two-elements and* $x \in X$ *a symmetric point. Then the subsets* $\{x\}$ *and* $X \setminus \{x\}$ *cannot be antisymmetrically-dense in* X.

Proof. By the definition of symmetric point (see Definition 10 i)), $d(x, y) = d(y, x)$ whenever $y \in X \setminus \{x\}$. Then, $\{x\}$ cannot be antisymmetrically-dense in X. In a similar way, clearly $y = x$ whenever $y \in X \setminus (X \setminus \{x\})$. Thus, $d(x, a) = d(a, x)$ whenever $a \in X \setminus \{x\}$ since x is symmetric point. That is, $X \setminus \{x\}$ is not antisymmetrically-dense. □

The following proposition will be very useful in this context.

Proposition 8. *Let (X,d) be a T_0-quasi-metric space. If each nonempty subset of (X,d) is antisymmetrically-dense in X then (X,d) will be an antisymmetrically connected space.*

Proof. Assume that each nonempty subset of (X,d) is antisymmetrically-dense in X. Now we can prove that (X,d) is indeed an antisymmetrically connnected space by showing that, for $x \in X$, if $y \in X$ and $y \notin T_d(x)$ then $y \in X \setminus T_d(x)$ and $y \neq x$. Thus there exists $a \in T_d(x)$ such that $d(y,a) \neq d(a,y)$ since $T_d(x)$ is antisymmetrically-dense in X. Now, if $a = x$ then $d(y,x) \neq d(x,y)$, that is $y \in T_d(x)$ contradiction. If $a \neq x$ then $x \in X \setminus \{a\}$. Also, note that the subset $\{a\}$ is antisymmetrically-dense by the hypothesis. Thus $d(x,a) \neq d(a,x)$. In this case, $P_{x,y} = (x,a,y)$ will be an antisymmetric path from x to y. That is, $y \in T_d(x)$ which is a contradiction. Finally, $T_d(x) = X$, so (X,d) will be antisymmetrically connected. □

The converse of Proposition 8 is not true always. Example 10 below is a counterexample. For it, first of all let us recall the notion *asymmetric norm* by Cobzaş [2].

Definition 13. *Let X be a real vector space equipped with a given map $\|\cdot| : X \to [0,\infty)$ satisfying the conditions:*

(a) $\|x| = \|-x| = 0$ *if and only if* $x = \mathbf{0}$.
(b) $\|\lambda x| = \lambda \|x|$ *whenever* $\lambda \geq 0$ *and* $x \in X$.
(c) $\|x+y| \leq \|x| + \|y|$ *whenever* $x,y \in X$.

Then $\|\cdot|$ is called an asymmetric norm *and $(X,\|\cdot|)$ is said to be an* asymmetrically normed real vector space. *(Here, $\mathbf{0}$ denotes the zero vector of the vector space X.)*

Obviously, an asymmetric norm induces a T_0-quasi-metric on X with the equality $d_{\|\cdot|}(x,y) = \|x-y|$ for each $x,y \in X$, where $(X,\|\cdot|)$ is an asymmetrically normed real vector space. But, naturally some T_0-quasi-metrics may not be induced by an asymmetric norm.

Note also that each norm is an asymmetric norm. However, the function $\|\cdot|$ described by the equality $\|(x_1,x_2)| = x_1 \vee x_2 \vee 0$ on $\mathbb{R}^2$, satisfies the above conditions and thus, it is an asymmetric norm which is not a norm.

Example 10. *Consider the plane $\mathbb{R}^2$ with the T_0-quasi-metric d induced by the maximum asymmetric norm $\|(x,y)| = x \vee y \vee 0$. It is easy to see that for each $(a_1,a_2) \in \mathbb{R}^2$ the symmetry component $C_d((a_1,a_2)) = \{(x+a_1, -x+a_2) \mid \ x \in \mathbb{R}\} \neq \mathbb{R}^2$ where $d = d_{\|\cdot|}$ and so the space $(\mathbb{R}^2,d)$ is not symmetrically connected by Definition 5. In this case, $(\mathbb{R}^2,d)$ is antisymmetrically connected from [11, Proposition 58] which states the fact that a T_0-quasi-metric space is symmetrically connected or antisymmetrically connected by virtue of graph theory.*

Now take the subset $G = \{(x, -x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$*. Clearly, the subspace* (G, d_C) *is a metric space, and so symmetrically connected. But it is not antisymmetrically connected. On the other hand,* G *is antisymmetrically-dense: if* $(x, y) \in \mathbb{R}^2 \setminus G$ *then* $(x, y) \notin G$ *and so* $y \neq -x$*, that is* $x + y \neq 0$*. Now, let us consider the point* $(x, -x) \in G$*. In this case* $d((x, -x), (x, y)) = (x - x) \vee (-x - y) \vee 0 = -(x + y) \vee 0$ *and* $d((x, y), (x, -x)) = (x - x) \vee (x + y) \vee 0$*, thus* $d((x, -x), (x, y)) \neq d((x, y), (x, -x))$*.*

In addition, note that the point $(0, 0)$ *is neither symmetric point nor antisymmetric point. Morover, the singleton set* $\{(0, 0)\}$ *is neither symmetrically-dense nor antisymmetrically-dense in* $\mathbb{R}^2$*.*

It is well-known that any subspace of an antisymmetric space is antisymmetric. Nevertheless, a T_0-quasi-metric space (X, d) may not be antisymmetric even though (X, d) has an antisymmetrically-dense and antisymmetric subspace, as the following example shows:

Example 11. *Let us define a* T_0*-quasi-metric on the set* $X = \{1, 2, 3\}$ *via the matrix*

$$\mathbf{W} = \begin{pmatrix} 0 & 9 & 8 \\ 7 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix}.$$

That is, $W = (w_{ij})$ *where* $w(i, j) = w_{ij}$ *for* $i, j \in X$*. It is easy to prove that* w *is a* T_0*-quasi-metric on* X*. Because of the fact that* $w(2, 3) = w(3, 2)$*, the space* (X, w) *is not antisymmetric. Consider the subset* $B = \{1, 3\}$ *of* X*. It is easy to show that* B *is antisymmetric subspace w.r.t the induced* T_0*-quasi-metric* w_B *on* B*. In addition,* B *is antisymmetrically-dense: If take* $x \in X \setminus B$ *then* $x = 2$*. In this case, for* $1 \in B$ *we have* $w(1, 2) = 9$ *and* $w(2, 1) = 7$*. That is,* $w(1, 2) \neq w(2, 1)$*.*

Incidentally, there is an extension space which has an antisymmetrically-dense subspace not antisymmetrically connected even though the space itself is antisymmetrically connected.

Example 12. *Take a metric space* (X, m) *with at least two-elements. By [5, Corollary 3.4],* (X, m) *has an antisymmetrically connected* T_0*-quasi-metric one-point extension* (Y, v) *such that* $Y = X \cup \{\infty\}$*, and* ∞ *is antisymmetric point. In this case, if we delete the added (antisymmetric) point, then the remaining metric space is no longer antisymmetrically connected. Moreover,* X *is antisymmetrically-dense in* Y*: if* $y \in Y \setminus X$ *then* $y = \infty$*. Also, because* $X \neq \emptyset$ *there is at least* $a \in X$*. Clearly,* $a \neq \infty$ *and* $v(\infty, a) \neq v(a, \infty)$ *since* ∞ *is an antisymmetric point in* Y*.*

Now we can turn our attention to the following natural question:
Under which mappings is the antisymmetric density property preserved in the context of T_0-quasi-metric spaces ?

Proposition 9. *Let* (X, d)*,* (Y, e) *be* T_0*-quasi-metric spaces and* $f : X \to Y$ *an isometric isomorphism. Then* $A \subseteq X$ *is antisymmetrically-dense in* X *if and only if* $f(A)$ *is antisymmetrically-dense in* Y*.*

Proof. If $y \in Y \setminus f(A)$ then there exists $x \in X \setminus A$ such that $f(x) = y$ since f is onto. On the other hand, there exists $a \in A$ such that $d(x, a) \neq d(a, x)$ since A is antisymmetrically-dense. Clearly, $f(a) \in f(A)$. Also, we have $e(f(x), f(a)) = d(x, a) \neq d(a, x) = e(f(a), f(x))$ as f is an isometry. Thus, $f(A)$ is antisymmetrically-dense in Y.

Conversely, if $a \in X \setminus A$ then $f(a) \notin f(A)$ as f is one-to-one. In this case, there exists $b \in f(A)$ satisfying the inequality $e(b, f(a)) \neq e(f(a), b)$ since $f(A)$ is antisymmetrically-dense. Thus, there exists $z \in A$ such that $f(z) = b$ via the fact that f is onto. Now, by considering the isometry property of f the expression $d(z, a) \neq d(a, z)$ is obtained. This shows that A is antisymmetrically-dense. □

In contrast to Proposition 4, we have the following remark for antisymmetric density:

Remark 3. *For any T_0-quasi-metric spaces (X, d), (Y, q), the product of any antisymmetrically-dense subsets of X and Y may not be antisymmetrically-dense in $X \times Y$.*

Actually, we have a counterexample:

Example 13. *Consider the standard T_0-quasi-metric space $(\mathbb{R}, u)$ where $u(x, y) = max\{x - y, 0\}$, given in Example 1 and the T_0-quasi-metric space $(\mathbb{R}, d)$ with the function*

$$d(x, y) = \begin{cases} 0 & ;\ x \leq y \\ 1 & ;\ y < x \end{cases}$$

on $\mathbb{R}$.

If we take the subset $A = \{0\}$ of $\mathbb{R}$ then A is antisymmetrically dense in $(\mathbb{R}, u)$ because of the fact that $u(x, 0) \neq u(0, x)$ whenever $x \in \mathbb{R} \setminus A$.

Similarly, let us take the subset $B = \{1\}$ of $\mathbb{R}$. Clearly, B is antisymmetrically dense in $(\mathbb{R}, d)$ since the inequality $d(x, 1) \neq d(1, x)$ holds whenever $x \in \mathbb{R} \setminus B$.

On the other hand, by taking into consideration the definition of product T_0-quasi-metric D given in Definition 1 we have the following fact:

If we consider the product T_0-quasi-metric space $(\mathbb{R} \times \mathbb{R}, D)$ then for $(1, 0) \in (\mathbb{R} \times \mathbb{R}) \setminus (A \times B)$ we have the equality

$$D((1, 0), (a, b)) = u(1, a) \vee d(0, b) = 1 \vee 0 = 1 = u(a, 1) \vee d(b, 0) = D((a, b), (1, 0))$$

whenever $a \in A$, $b \in B$, that is $a = 0$ and $b = 1$. It means that the subset $A \times B$ is not antisymmetrically-dense in $(\mathbb{R} \times \mathbb{R}, D)$ even though A is antisymmetrically-dense in $(\mathbb{R}, u)$ and B is antisymmetrically-dense in $(\mathbb{R}, d)$.

Proposition 10. *Let (X, d) be a T_0-quasi-metric space. If A is antisymmetrically-dense in X and $B \subseteq X$ then $A \cup B$ is antisymmetrically-dense in X.*

Proof. Take $x \in X \setminus (A \cup B)$. So, by the antisymmetric density of A, there exists $a \in A$ such that $d(x,a) \neq d(a,x)$. Thus, $A \cup B$ is antisymmetrically-dense in X as well, since $A \subseteq A \cup B$. □

Hence, the following result will be trivial as a consequence of Proposition 10.

Corollary 4. *The union of all subsets of a T_0-quasi-metric space which has at least one antisymmetrically-dense subset is antisymmetrically-dense.*

Despite the above result, we have:

Remark 4. *The intersection of two antisymmetrically-dense subsets of a T_0-quasi-metric space may not be antisymmetrically-dense.*

There is a counterexample:

Example 14. *On the set $X = \{1,2,3,4,5\}$ consider the Sorgenfrey T_0-quasi-metric as*

$$s(x,y) = \begin{cases} x - y & ;\ y \leq x \\ 1 & ;\ y > x \end{cases}.$$

Now let us take $A = \{1,5\}$. In this case, since $s(3,1) \neq s(1,3)$, $s(1,4) \neq s(4,1)$ and $s(2,5) \neq s(5,2)$, the set A is antisymmetrically-dense in X. Similarly, the set $B = \{1,4\}$ is antisymmetrically-dense in X. But the intersection set $A \cap B = \{1\}$ is not antisymmetrically-dense in X because of the facts that $s(1,2) = s(2,1)$ and $2 \in X \setminus (A \cap B)$.

5. CONCLUSION

As opposed to the negative result ([5, Example 2.11]) which occurs in case τ_{d^s}-density, a T_0-quasi-metric space which has a symmetrically connected and symmetrically-dense subspace will be symmetrically connected by [9, Theorem 2.15]. That is, it is possible to carry the symmetric connectedness of the subspace to the space, provided that the subspace is symmetrically-dense in the space.

However, we have a counterexample in [9] showing that "any symmetrically-dense subspace of a symmetrically connected space need not be symmetrically connected". Similarly, in this paper we have an example which shows that "any antisymmetrically-dense subspace of an antisymmetrically connected space need not be antisymmetrically connected".

In the light of above considerations, for future work, let us state a few new questions using the notions introduced in [6] and [12], as follows.

If (X,d) is locally symmetrically connected and it has a symmetrically-dense subset A, then is the subspace (A, d_A) locally symmetrically connected? If (X,d) has a symmetrically-dense and locally symmetrically connected subspace then is the space (X,d) locally symmetrically connected?

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