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Research Article

Investigation of Hyperbolic Type Solutions of the Fitzhugh-Nagumo Model in Neuroscience

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Abstract: This article aims to obtain analytical solutions of the Fitzhugh – Nagumo model, which has an important place in neuroscience. The $\frac{1}{G'}$- expansion method is used to obtain the solutions. Hyperbolic type travelling wave solutions are produced by using the $\frac{1}{G'}$- expansion method, which is an effective and efficient method in solving nonlinear partial differential equations (NLPDEs). Then 3D, 2D and contour graphs are presented using a computer program.

Keywords: Fitzhugh – Nagumo model, hyperbolic type solution, $\frac{1}{G'}$- expansion method

Araştırma Makalesi

Sinirbilimde Fitzhugh-Nagumo Modelinin Hiperbolik Tip Çözümlerinin İncelenmesi

Öz: Bu makale sinirbilimde önemli bir yere sahip olan Fitzhugh – Nagumo modelinin analitik çözümlerini elde etmeyi amaçlamaktadır. Çözümleri elde etmek için $\frac{1}{G'}$- açılım yöntemi kullanılır. Lineer olmayan kısmi diferansiyel denklemlerin (NLPDE) çözümünde etkili ve verimli bir yöntem olan $\frac{1}{G'}$-açılım yöntemi kullanılarak hiperbolik tip gezici dalga çözümleri üretilmektedir. Daha sonra bir bilgisayar programı kullanılarak 3 boyutlu, 2 boyutlu ve kontur grafikleri sunulur.

Anahtar Kelimeler: Fitzhugh – Nagumo modeli, hiperbolik tip çözüm, $\frac{1}{G'}$-açılım yöntemi

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1. Introduction

Fitzhugh – Nagumo (FN) model, which is one of the NLPDEs used in many fields such as mathematics, physics, biology and physiology, has attracted great interest from physicists and mathematicians in recent years due to its place in science [1]. The FN model is a simple equation of the Hodgkin-Huxley model and is generally used in the transmission and stimulation of nerve impulses in physiology [2]. Its appearance in population genetics, circuit theory and optics shows that it has a very important place in our lives [3]. In addition, the FN model is known as a nonlinear reaction-diffusion equation or nonlinear evolution equation [4-8].

There are various studies in the literature in which numerical and analytical solution methods of the FN model are obtained. On the FN model; Huaying and Yucui have been found their exact solutions using the first integral method [3], Yokus has been compared the numerical solutions they found by applying the finite forward difference method and the automatic backloud transformation method and the approximate solutions [2], Nourazar et al. have been found exact solution by applying the the homotopy perturbation method [9], Li and Guo have been reached exact solutions of FN model using the first integral method [10]. Dehghan et al. (2010) have been compared the results of numerical methods applied to the FN model with the results of analytical methods [4].

Today, there are various studies with analytical methods. For example, Durur et al. have been found analytical solutions for the system that models nematicons [11]. Subaşı and Durur have been reached the travelling wave solutions of the Shallow Water-Like equation [12]. Li et al. have been found travelling wave solutions of Zakharov equations [13], Duran has been found the solitary wave solutions of the coupled Konno-Oono equation [14] and the (2+1)-D Boiti Leon Pempinelli system [15], Zayed et al. have been found wave solutions of the (3+1)-D Kadomtsev-Petviashvili equation [16], Yokus et al. have been found exact solutions of Bogoyavlenskii equation [17].

In this study, the FN model, which is a neuroscience model, is as follows:

$$u_t = u_{xx} - u\left(\alpha - u\right)\left(1-u\right). \tag{1.1}$$

The purpose of this study is to obtain hyperbolic wave solutions by applying the $\frac{1}{G'}$ method, one of the analytical solution methods, on the FN model. In the second part of the study $\frac{1}{G'}$ method will be explained, in the third part $\frac{1}{G'}$ method will be applied on the FN model and solutions will be presented with contour, 3D and 2D graphs, and in the fourth part, the result will be.

2. $\frac{1}{G'}$ - Expansion Method

Data were collected using the $\frac{1}{G'}$- expansion method. The $\frac{1}{G'}$- expansion method is a new model developed by Yokus in his doctoral thesis in 2011, inspired by the $\frac{G'}{G}$- expansion method. Since the $\frac{1}{G'}$ method is an easier and uncomplicated method than the $\frac{G'}{G}$- expansion method, it is a more advantageous method in terms of processing intensity [18]. The $\frac{1}{G'}$ method is an effective method for finding analytical solutions of NLPDEs.

The general form of the family of NLPDEs is as follows.

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{2.1}$$

$u=u(x,t)=U(\xi), \xi=kx+vt$, $v\neq 0.$ v is the wave velocity and is constant. We can convert it to the following nonlinear ODE for $\mathrm{U}(\xi)$:

$$K\left(u,u',u'',u''',...\right)=0. \tag{2.2}$$

Solution of equation (2.2);

$$u(\xi)=a_0+\sum_{i=1}^{n}a_i\left(\frac{1}{G'}\right)^i. \tag{2.3}$$

Here a_i, $i=(1,2,3,...,n)$ a scalar transform into $G=G(\xi)$ a quadratic ordinary differential equation. The n is balancing term between the highest order linear term and the highest order nonlinear term in the Equation.

$$G''+\lambda G'+\mu=0. \tag{2.4}$$

Here λ and μ are constants,

$$\frac{1}{G'(\xi)}=\frac{1}{-\frac{\mu}{\lambda}+A\cosh[\xi\lambda]-A\sinh[\xi\lambda]}. \tag{2.5}$$

The desired derivatives of equation (2.3) were calculated and a polynomial was obtained by $\frac{1}{G'}$ method by substituting in equation (2.2). Setting the polynomial's coefficients to zero created a system of algebraic equations. This equation was solved using the Mathematica package program and the default equation (2.2) was replaced in the solution function. Finally (2.1) the solutions of the equation were found.

3. Solutions of the FN Equation

In this section, the traveling wave solutions of equation (1.1) will be generated using the $\frac{1}{G'}$-expansion method. In general, expansion methods are used to transform the partial differential equation into an ordinary differential equation using the classical wave transform. The state of equation (1.1) after $u=u(x,t)=U(\xi)$ $\xi=kx+vt$, $v\neq 0$ transformation is as follows

$$vU'-k^2U''+U(1-U)(\alpha-U)=0. \tag{3.1}$$

v represents the wave velocity and k wave number in this transformation [19].

The balancing term between the highest order linear term U'' and the highest order nonlinear term U^3 in the equation comes with $n=1$.

$$U(\xi)=a_0+a_1\left(\frac{1}{G'}\right). \tag{3.2}$$

It is calculated by substituting the equation given in equation (3.2) in equation (3.1).

After some mathematical operations, a polynomial equation based on the $\frac{1}{G'}$- expansion method is created. The coefficient of each term in this polynomial is zero. This gives the following system of equations.

$$\begin{aligned}
&Const: \alpha a_0-a_0^2-aa_0^2+a_0^3=0,\\
&\frac{1}{G'[\xi]}: aa_1+v\lambda a_1-k^2\lambda^2a_1-2a_0a_1-2aa_0a_1+3aa_0^2a_1=0,\\
&\frac{1}{G'[\xi]^2}: v\mu a_1-3k^2\lambda\mu a_1-a_1^2-aa_1^2+3a_0a_1^2=0,\\
&\frac{1}{G'[\xi]^3}: -2k^2\mu^2a_1+a_1^3=0.
\end{aligned} \tag{3.3}$$

Using a software application, the constants $v, k, \lambda, \alpha, \mu, a_0$ and a_1 are obtained in equation (2.3).

Case 1. If

$$a_0 = 0, \quad a_1 = \sqrt{2}k\mu, \quad v = -\frac{k}{\sqrt{2}} + \sqrt{2}k\alpha, \quad \lambda = -\frac{1}{\sqrt{2}k}, \tag{3.4}$$

the constants given in the equation (3.2) are replaced by the values in the equation (3.4), a hyperbolic solution of the equation (1.1) is obtained.

$$u_1(x,t) = \frac{\sqrt{2}k\mu}{\sqrt{2}k\mu + A\cosh\left[\frac{kx + t\left(-\frac{k}{\sqrt{2}} + \sqrt{2}k\alpha\right)}{\sqrt{2}k}\right] + A\sinh\left[\frac{kx + t\left(-\frac{k}{\sqrt{2}} + \sqrt{2}k\alpha\right)}{\sqrt{2}k}\right]}. \tag{3.5}$$

In expansion methods, traveling wave solutions are produced under some restrictive conditions. In the (3.5) traveling wave solution the restrictive condition is $k \neq 0$.

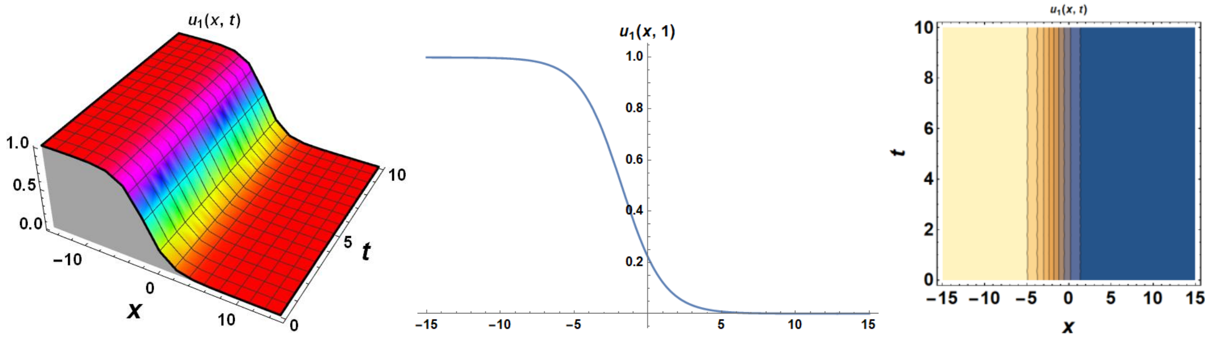

Figure1. Graphs of (3.5) equation are also given in Figure1 for $k = 0.2, \quad \mu = 1, \quad \alpha = 0.5, \quad A = 1.$

Case 2. If

$$\alpha = \frac{1}{2}, \quad a_0 = \frac{1}{2}, \quad a_1 = \sqrt{2}k\mu, \quad v = \frac{3k}{2\sqrt{2}}, \quad \lambda = \frac{1}{2\sqrt{2}k}, \tag{3.6}$$

the constants given in equation (3.2) are replaced by the values in equation (3.6), a hyperbolic solution of equation (1.1) is obtained.

$$u_2(x,t) = \frac{1}{2} + \frac{\sqrt{2}k\mu}{-2\sqrt{2}k\mu + A\cosh\left[\frac{\frac{3kt}{2\sqrt{2}} + kx}{2\sqrt{2}k}\right] - A\sinh\left[\frac{\frac{3kt}{2\sqrt{2}} + kx}{2\sqrt{2}k}\right]}. \tag{3.7}$$

Similarly, the restrictive condition in the (3.7) solution is $k \neq 0$.

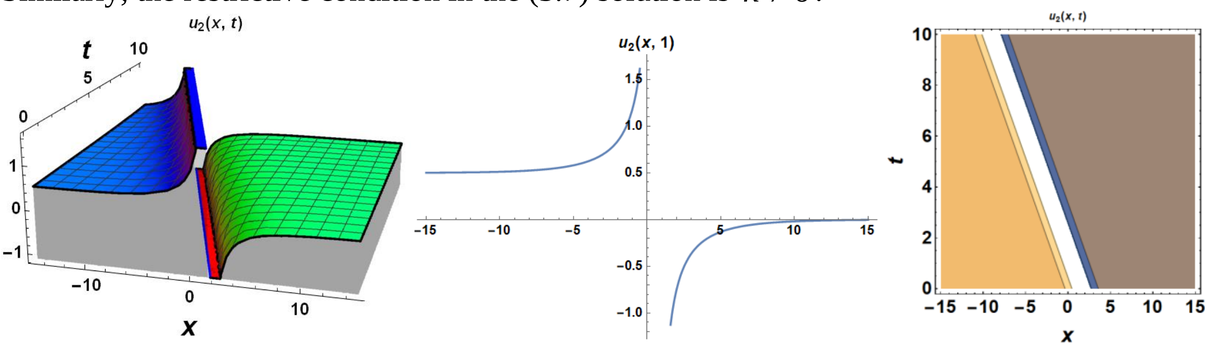

Figure2. Graphs of (3.7) equation are also given in Figure2 for $k = 0.2, \quad \mu = 1, \quad \alpha = 0.5, \quad A = 1.$

4. Conclusion

The aim of this study was to obtain exact solutions of the FN model. The $\frac{1}{G'}$ - expansion method, which is frequently used in analytical solutions, was applied. The computer package program was used to create the solutions. For the FN model, the $\frac{1}{G'}$- expansion method was found to be easy and effective to apply. Then, the hyperbolic type solutions of the FN model were obtained from the computer ready package program. As a result, these solutions are available in 3D, 2D and contour graphics.

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