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Analysis of the n-Term Klein-Gordon Equations in Cantor Sets

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ABSTRACT The effectiveness of the local fractional reduced differential transformation method (LFRDTM) for the approximation of the solution related to the extended n-term local fractional Klein-Gordon equation is the main aim of this paper in which fractional complex transform and local fractional derivative have been employed to analyze the n-term Klein-Gordon equations, and Cantor sets. The proposed method, along with the existence of the solutions demonstrated through some examples, provides a powerful mathematical means in solving fractional linear differential equations. Considering these points, the paper also provides an accurate and effective method to solve complex physical systems that display fractal or self-similar behavior across various scales. In conclusion, the fractional complex transform with the local fractional differential transform method has been proven to be a robust and flexible approach towards obtaining effective approximate solutions of local fractional partial differential equations.

KEYWORDS
Local fractional calculus
Local fractional differential equations
Reduced differential transformation method
Fractals
Fractional Klein-Gordon equation

INTRODUCTION

Various fields of study have used the Klein–Gordon equation in the last few decades, including quantum field theory, nonlinear optics, thermodynamics, and solid-state physics Kanth and Aruna (2009). The Klein-Gordon equation is a fundamental quantum field theory that describes particle behavior with spin 0. It was first introduced as a relativistic wave equation for a free scalar particle. There are a variety of approaches to solving problems of this nature. Recent years have seen significant use of a fractional modification involving the Caputo fractional derivative. However, fractional techniques for the Riemann-Liouville and Caputo derivatives are inadequate when smooth functions cannot represent the study area. In this situation, the local fractional calculus is a useful tool to simulate these physical problems.

Since its inception, the Klein-Gordon equation has been extensively studied and applied in various fields of physics, including particle physics, condensed matter physics, and cosmology. However, the standard form of the Klein-Gordon equation only considers integer-order derivatives, which restricts its applicability to specific physical systems that exhibit non-local behavior or fractal geometry. To overcome this limitation, local fractional calculus has been introduced to generalize the Klein-Gordon equation to accommodate fractional-order derivatives. Local fractional calculus is a mathematical framework that extends classical calculus to nondifferentiable and fractal functions by introducing the concept of fractional derivatives, which capture the behavior of these functions at small scales It is discovered that local fractional calculus, which Kolwankar and Gangal (1996) first proposed in the 1990s, is a practical tool in fields ranging from fundamental research to engineering. There, they describe the behavior of a continuous but non-differentiable function.

Over the past 20 years, the significance and attractiveness of local fractional calculus have increased due to its application to functions in the real world that involve fractals and are not continuously differentiable. The application of local fractional calculus to the Klein-Gordon equation has led to the development of a new class of equations known as local fractional Klein-Gordon equations. These equations have been used to model various physical phenomena, such as quantum wave propagation in fractal media, fractional quantum mechanics, and non-local interactions in quantum field theory Dubey *et al.* (2022).

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In recent years, the study of local fractional Klein-Gordon equations has gained significant attention, with researchers exploring their theoretical properties and applications in various areas of physics. This includes the development of numerical methods for solving these equations and their application to complex physical systems, such as the behavior of particles in non-local media and the study of fractal structures in condensed matter physics. For example, In Sun (2018), the author explored a mathematical model involving fractional derivatives to describe a porous structure. This model is called the Harry Dym fractal equations and incorporates the Burgers fractal nonlinear equation. Furthermore, reference Wang *et al.* (2019) introduced the concept of a local fractional KdV-Burgers-Kuramoto (KBK) equation within the context of fractal space. Here, the concept of a local fractional derivative and a differential transform method has been used. Keskin and Oturanc (2009) were the first to propose the Reduced Differential Transform Method (RDTM). Many scholars use this method to explore fractional, non-fractional, linear, and nonlinear PDEs. The method provides a reliable and efficient technique for a wide range of scientific, industrial, and many other applications in physics, encompassing linear, nonlinear, homogeneous, and nonhomogeneous, fractional, and non-fractional PDEs, and so on. Solutions to significant mathematical problems are explored using this derivative. In 2016, Yang and Tenreiro Machado (2019) introduced the new local fractional differential transform technique (LFDTM) by combining the local fractional derivative (LFD) and the differential transform method (DTM). He offered several fundamental theorems and some examples of the use of this strategy. According to Jafari *et al.* (2016) theory of local fractional calculus, He combined LFD and RDTM in 2016 to produce the local fractional reduced differential transform method (LFRDTM). Moreover, he presented several fundamental theorems and applications of this method.

The classical KGE can be converted in its local fractional form in cantor sets using the fractional complex transform and the local fractional derivative. Yang *et al.* (2014) developed a continuous but nondifferentiable solution in cantor sets for the local fractional linear KGE was developed by Yang *et al.* (2014) using the technique of local fractional series technique under the local fractional differential operator to produce a nondifferentiable solution. Using the local fractional Sumudu transform approach and the standard homotopy perturbation technique, Kumar *et al.* (2017) researched linear KGE in Cantor sets.
The motivation for using local fractional calculus to examine the solution of the n-term fractional Klein-Gordon equation came from the need to develop a more effective approach to modeling complex physical phenomena. Traditional calculus methods fail to accurately describe many real-world systems that exhibit fractal or self-similar behavior at various length scales. Local fractional calculus provides a framework for analyzing such systems considering the non-local and non-differentiable properties of fractals Chu *et al.* (2023).

The n-term fractional Klein-Gordon equation is a specific example of a physical system that can benefit from applying local fractional calculus. This equation describes the behavior of a scalar field in space-time and has important applications in quantum mechanics, field theory, and condensed matter physics. By using local fractional calculus to solve the n-term fractional Klein-Gordon equation, we can gain a deeper understanding of the behavior of scalar fields in complex systems, leading to improved models and better predictions of physical phenomena.

The following paper aims to demonstrate the effectiveness of the local fractional reduced differential transformation method (LFRDTM) in approximating the solution of the extended n-term local fractional Klein-Gordon equation. Here, we also discussed the existence of the solution, followed by a few examples. The ultimate goal of this study is to provide an effective and accurate method for modeling complex physical systems that exhibit fractal or self-similar behavior at various length scales. By demonstrating the effectiveness of the LFRDTM in approximating the solution of the local fractional Klein-Gordon equation of term n, we hope to encourage its use in a wide range of applications in physics and engineering. The article is organized as follows: (1) Introduction (2) Definitions and preliminary (3) Existence and uniqueness of the solution of the local fractional Klein-Gordon equations (4) Approximate analytical solutions of the local fractal Klein-Gordon equations of term n (5) Results and Discussion.

DEFINITION AND PRELIMINARIES

The definitions of fractional operators, the transformation method, and their properties are as follows.

Definition 1. *Jafari et al. (2016) Let $\psi : [a,b] \times R \to R$ be a local fractional continuous function, then the local fractional partial derivative operator of $\psi(y,t)$ of order ν where $0 < \nu \leq 1$ concerning t at the point (y,t_0) expressed as*

$$\begin{aligned} D^{\nu}\psi(y,t_0) &= \frac{\partial}{\partial t}\psi(y,t_0) \\ &= \lim_{t\to t_0}\frac{\Delta^{\nu}[\psi(y,t)-\psi(y,t_0)]}{(t-t_0)^{\nu}} \end{aligned} \tag{1}$$

where $\Delta^{\nu}[\psi(y,t)-\psi(y,t_0)] \cong \Gamma(1+\nu)\,[\psi(y,t)-\psi(y,t_0)]$

with that in view the local fractional partial derivative operator of $\psi(y,t)$ of order $k\nu$, $0 < \nu \leq 1$ is given as

$$\begin{aligned} D_t{}^{k\nu}\psi(y,t) &= \frac{\partial^{k\nu}}{\partial t}\psi(y,t) \\ &= \overbrace{\frac{\partial^{\nu}}{\partial t^{\nu}}\cdots\frac{\partial^{\nu}}{\partial t^{\nu}}}^{k\,times}\psi(y,t). \end{aligned} \tag{2}$$

Definition 2. *A continuous function $\psi : [a,b] \times R^{\nu} \to R^{\nu}$ which is local fractional is Lipschitz continuous if $\exists\, 0 < \eta < 1$ s.t. $\forall\, y \in [a,b]$*

$$|\psi(y,t_1)-\psi(y,t_2)| \leq \eta^{\nu}|t_1-t_2|, 0 < \nu < 1.$$

.

Definition 3. *On a Generalised Banach space $(X, ||.||_{\nu})$, a mapping V from X to X is said to be a contraction mapping if $\exists\, \eta^{\nu} \in (0^{\nu}, 1^{\nu})$ s.t. for $y_1{}^{\nu}, y_2{}^{\nu} \in X$*

$$||V(y_1^{\nu}) - V(y_2^{\nu})||_{\nu} \leq \eta^{\nu}||y_1^{\nu} - y_2^{\nu}||_{\nu}.$$

.

Also $||\psi y^{\nu} - y^{\nu}||_{\nu} = 0$ implies y^{ν} is said to be a fixed point of ψ.

Theorem 1. *A map $\psi : X \to X$ on a complete general Banach space $(X, ||.||_{\nu})$ has a unique fix point if $\exists\, k \geq 1$ s.t. ψ^k is contracting.*

Theorem 2. *Let $\psi : [a,b] \times R^{\nu} \to R^{\nu}$ be LFC map. Then ψ is Lipschitz continuous.*

Definition 4. *Jafari et al. (2016) Let $\Psi^{(k+1)\nu}(y) \in C_\nu(a,b)$, for $k = 0,1,2,...,n$ and $1 < \nu \le 1$, then, we have*

$$\psi(y) = \sum_{k=0}^{\infty} \psi^{k\nu}(0)\frac{(y-y_o)^{k\nu}}{\Gamma(1+k\nu)} \quad (3)$$

$$\text{and } \psi^{(k+1)\nu}(y) = \overbrace{D_y^\nu D_y^\nu ... D_y^\nu}^{(k+1)}\psi(y). \quad (4)$$

Definition 5. *The local fractional differential transform of a two-dimensional transform $\Psi_k(y)$ or $\Psi(y,k)$ of function $\psi(y,t)$ is*

$$\Psi_k(y) = \frac{1}{\Gamma(1+k\nu)}\left[\frac{\partial^{k\nu}}{\partial t^{k\nu}}\psi(y,t)\right]_{t=0} \quad (5)$$

$$k = 0,1,2,....n \text{ and } \nu \in (0,1]. \quad (6)$$

Definition 6. *Jafari et al. (2016) The inverse transform formula for a two-dimensional local fractional reduced differential of $\psi_k(y)$*

$$\psi(y,t) = \sum_{k=0}^{\infty} \Psi_k(y)t^{k\nu}$$

Some other properties of RTDM are as follows:

1. If $g(y,t) = a\psi(y,t)$ then $G_k(y) = a\Psi_k(y)$,
where a is a constant
2. If $\pi(y,t) = \psi(y,t) + \psi(y,t)$, then, $\Pi_k(y) = \psi_k(y) + \Psi_k(y)$.
3. If $\pi(y,t) = \psi(y,t)\psi(y,t)$, then, $\Pi_k(y) = \sum_{r=0}^{k}\Psi_r(y)\Psi_{k-r}(y)$.
4. If $g(y,t) = \frac{\partial^{n\nu}}{\partial t^{n\nu}}\psi(y,t)$, then,$G_k(y) = \frac{\Gamma(1+(k+n)\nu)}{\Gamma(1+k\nu)}\Psi_{k+n}(y)$
where $\Psi_{k+n}(y) = \frac{1}{\Gamma(1+(k+n)\nu)}\left[\frac{\partial^{(k+n)\nu}}{\partial t^{(k+n)\nu}}\psi(y,t)\right]_{t=0}$
$k = 0,1,2,....n$ $\nu \in (0,1]$ and $n \in \mathbb{N}$.
5.If $g(y,t) = \frac{\partial^{n\nu}}{\partial x^{n\nu}}\psi(y,t)$, then we have $G_k(y) = \frac{\partial^{n\nu}}{\partial x^{n\nu}}\Psi_k(y)$.

Lemma 1. *([Yang (2012) Zhang et al. (2015) Zhang and Yang (2016)]) Let φ_1, φ_2 be two non differential functions with Local fractional derivative operator $\nu \in (0,1]$, then*

1. $D^{(\nu)}(a\varphi_1 + b\varphi_2) = a(D^{(\nu)}\varphi_1) + b(D^{(\nu)}\varphi_2)$ *for $a,b \in \mathbb{R}$.*

2. $D^{(\nu)}(\varphi_1\varphi_2) = \varphi_1 D^{(\nu)}(\varphi_2) + \varphi_2 D^{(\nu)}(\varphi_1)$.

3. $D^{(\nu)}(\frac{\varphi_1}{\varphi_2}) = \frac{\varphi_2 D^{(\nu)}\varphi_1 - \varphi_1 D^{(\nu)}\varphi_2}{\varphi_2^2}$ *provided $\varphi_2 = 0$.*

Lemma 2. *Acan et al. (2017) Yang (2012) Zhang et al. (2015) Zhang and Yang (2016) Suppose that φ is a non-differential function and $\nu \in (0,1]$ is the order of local fractional derivative, then,*

1. $D^\nu(\varphi(y)) = 0$, *for all constant functions $\varphi(y) = k$.*
2. $D^{(\nu)}(\frac{y^{k\nu}}{\Gamma(k\nu+1)}) = \frac{y^{(k-1)\nu}}{\Gamma((k-1)\nu+1)}$
3. $D^{(\nu)}(E_\nu(y^\nu)) = E_\nu(y^\nu)$
4. $D^{(\nu)}(E_\nu(-y^\nu)) = -E_\nu(-y^\nu)$
5. $D^{(\nu)}(sin_\nu(y^\nu)) = cos_\nu(y^\nu)$
6. $D^{(\nu)}(cos_\nu(y^\nu)) = -sin_\nu(y^\nu)$

where $E_\nu(y^\nu) = \sum_{k=0}^{\infty}\frac{y^{k\nu}}{\Gamma(k\nu+1)}$,

$sin_\nu(y^\nu) = \sum_{k=0}^{\infty}(-1)^k\frac{y^{(2k+1)\nu}}{\Gamma((2k+1)\nu+1)}$

and $cos_\nu(y^\nu) = \sum_{k=0}^{\infty}(-1)^k\frac{y^{2k\nu}}{\Gamma(2k\nu+1)}$.

Local Fractional Klein- Gordon equation (on Cantor sets)

This section uses the fractional complex transform and the local fractional derivative to derive the local fractional KGE (Klein-Gordon equation) fractal model of the term n in cantor sets. We know that the classical Klein-Gordon Equation is

$$\frac{\partial\psi(\Omega,T)}{\partial T} = \frac{\partial^2\psi(\Omega,T)}{\partial\Omega^2} + a\psi(\Omega,T) + b\psi^2(\Omega,T) + c\psi^3(\Omega,T)$$

then the classical n-term Klein-Gordon equation is considered as

$$\frac{\partial\psi(\Omega,T)}{\partial T} = \frac{\partial^2\psi(\Omega,T)}{\partial\Omega^2} + a_1\psi(\Omega,T) + a_2\psi^2(\Omega,T) + a_3\psi^3(\Omega,T) + ... + a_n\psi^n(\Omega,T) \quad (7)$$

subject to initial condition $\psi(\Omega,0) = \psi_o$.

Now using the Local Fractional complex transform method to switch the conventional differential equation into the local fractional differential equation.
To derive the fractional transform, we put

$$\Omega = \frac{y^\nu}{\Gamma(1+\nu)} \quad , \; T = \frac{t^\nu}{\Gamma(1+\nu)}$$

Then

$$\frac{\partial^\nu}{\partial t^\nu}\psi(y,t) = \frac{\partial\psi(\Omega,T)}{\partial\Omega}\frac{\partial^\nu\Omega}{\partial t^\nu} + \frac{\partial\psi(\Omega,T)}{\partial T}\frac{\partial^\nu T}{\partial t^\nu}$$
$$= 0 + \frac{1}{\Gamma(1+\nu)}\frac{\partial\psi(\Omega,T)}{\partial T}.$$

This implies

$$\frac{\partial\psi(\Omega,T)}{\partial T} = \Gamma(1+\nu)\frac{\partial^\nu\psi(y,t)}{\partial t^\nu}$$
$$\text{or } D_T\psi(\Omega,T) = \Gamma(1+\nu)D_t^\nu\psi(y,t)$$

Similarly,

$$\begin{aligned} D_y^{2\nu}\psi(y,t) &= D_\Omega^2\psi(\Omega,T)\frac{\partial^\nu\Omega}{\partial x^\nu} + D_T^2\psi(\Omega,T)\frac{\partial^\nu T}{\partial x^\nu} \\ &= \frac{1}{\Gamma(1+\nu)}D_\Omega^2\psi(\Omega,T) \\ \Rightarrow D_\Omega^2\psi(\Omega,T) &= \Gamma(1+\nu)D_y^{2\nu}\psi(y,t) \end{aligned}$$

and using local fractional derivatives under the constraints and characteristics of the fractional complex transform technique, we see that

$$\psi(y,t) = \frac{1}{\Gamma(1+\nu)}\psi(\Omega,T)$$
$$\psi^2(y,t) = \frac{1}{\Gamma(1+\nu)}\psi^2(\Omega,T)$$
$$\vdots =$$
$$\psi^n(y,t) = \frac{1}{\Gamma(1+\nu)}\psi^n(\Omega,T).$$

Thus, substituting the terms into 7 we obtain the local fractional KGE of n terms (Klein - Gordon equation) as

$$\frac{\partial\psi(y,t)}{\partial t^\nu} = \frac{\partial^{2\nu}\psi(y,t)}{\partial x^{2\nu}} + a_1\psi(y,t) + a_2\psi^2(y,t) + a_3\psi^3(y,t) + ... + a_n\psi^n(y,t), t > 0 \quad (8)$$

with initial conditions

$$\psi(y,0) = \psi_0$$

or we can write it as

$$D_t^\nu\psi = D_y^{2\nu}\psi\ + a_1\psi + a_2\psi^2 + a_3\psi^3 + \cdots\cdots + a_n\psi^n$$

with initial conditions

$$\psi(y,0) = \psi_o.$$

EXISTENCE AND UNIQUENESS OF SOLUTION OF N TERM LOCAL FRACTAL KLEIN- GORDON EQUATION

In this section, we apply the Banach fixed point theorem and contraction mapping theorem to ensure that the local fractional Klien Gordon equation with the initial condition has a unique solution.
Let us introduce a Banach space of real-valued functions by $C(\Omega\times[0,T])$ with the norm is given by

$$||\psi|| = \sup_{(y,t)\in\Omega\times[0,T]} ||\psi(y,t)||$$

Lemma 3. *Let $\psi(y,t)$ and its fractional partial derivatives are continuous on $\Omega\times[0,T]$ then $D_t^\nu\psi$ and $D_y^{2\nu}\psi$ are bounded.*

Proof. Let $A_1 = \sup_{0\le\tau\le t\le T} |t-\tau|^{-\nu}$. We will show that D_t^ν is bounded. Consider

$$|D_t^\nu\psi(y,t)| = |\frac{1}{\Gamma(1-\nu)}\int_0^t (t-\tau)^{-\nu}\psi(y,t)d\tau|$$
$$= |\frac{A_1}{\Gamma(1-\nu)}\int_0^t (t-\tau)^{-\nu}\psi(y,t)d\tau|$$
$$\le \frac{A_1}{\Gamma(1-\nu)}||\psi|| + max_{y\in\Omega}|\psi(y,0)|.$$

Let (L_1) be a positive constant such that $\max_{y\in\Omega}|\psi(y,0)| \le L_1||\psi||$,
this gives

$$|D_t^\nu\psi(y,t)| \le \frac{A_1}{\Gamma(1-\nu)}||\psi|| + L_1||\psi|| = L_2||\psi||$$

where $L_2 = \frac{A_1}{\Gamma(1-\nu)} + L_1$. Similarly, we can have $||D_y^{2\nu}\psi|| \le K||\psi||$ where K is some constant. Hence, the fractional derivatives are bounded. ☐

Now considering the subsequent fractional differential equation in the local fractional operator form as

$$L_\nu(\psi) - R_\nu(\psi) = 0 \quad (9)$$

where $\psi = \psi(y,t)$, $L_\nu(\psi) = \frac{\partial^\nu\psi}{\partial t^\nu}$ and $R_\nu(\psi) = \frac{\partial^{2\nu}\psi(y,t)}{\partial x^{2\nu}} + a_1\psi + a_2\psi^2 + a_3\psi^3 + \cdots\cdots + a_n\psi^n$.
We can rewrite this equation subject to initial conditions as

$$L_\nu\psi(y,t) = \Phi(\psi(y,t))$$

with initial condition

$$\psi(y,0) = \psi_0(y).$$

Here

$$\Phi(\psi(y,t)) = \frac{\partial^{2\nu}\psi(y,t)}{\partial y^{2\nu}} + a_1\psi + a_2\psi^2 + ... + a_n\psi^n = \frac{\partial^{2\nu}\psi(y,t)}{\partial y^{2\nu}} + f(\psi)$$

Theorem 3. *Assuming the function $\Phi(\psi(y,t))$ specified as*

$$\Phi(\psi(y,t)) = \frac{\partial^{2\nu}\psi(y,t)}{\partial x^{2\nu}} + f(\psi)$$

is a Locally fractional continuous function which satisfies Lipschitz continuity condition, that is,

$$|\Phi(\psi_1(y,t)) - \Phi(\psi_2(y,t))| \le \eta^\nu|\psi_1(y,t) - \psi_2(y,t)|, \nu\in(0,1]$$

where $0<\eta<1$.
then, the system

$$L_\nu\psi(y,t) = \Phi(\psi(y,t))$$

with initial condition

$$\psi(y,0) = \psi_0(y)$$

comprises a solution in $C_\nu[a,b]$ which is a unique solution.

Proof. Consider the function $V : C_\nu[a,b] \to C_\nu[a,b]$ be defined as

$$V(\psi(y,t)) = \psi_0(y) + \frac{1}{\Gamma(1+\nu)}\int_\nu^t [\Phi(\psi_1(y,s)) - \Phi(\psi_2(y,s))](ds)^\nu$$

we will use induction to show that

$$||V^n(\psi_1(y,t)) - V^n(\psi_2(y,t))||_\nu \le \frac{\eta^{n\nu}|b^\nu - a^\nu|^n}{\Gamma^n(1+\nu)}||\psi_1(y,t) - \psi_2(y,t)||_\nu.$$

For n=1, we have

$$|V(\psi_1(y,t)) - V(\psi_2(y,t))| = \left|\frac{1}{\Gamma(1+\nu)}\int_\nu^t \begin{bmatrix}\phi(\psi_1(y,s)) - \\ \phi(\psi_2(y,s))\end{bmatrix}(ds)^\nu\right|$$

$$\leqslant \left|\frac{1}{\Gamma(1+\nu)}\int_\nu^t \eta^\nu \begin{vmatrix}\psi_1(y,s) - \\ \psi_2(y,s)\end{vmatrix}(ds)^\nu\right|$$

$$\leqslant \frac{\eta^\nu}{\Gamma(1+\nu)}\left|\begin{array}{l}\frac{\partial^{2\nu}}{\partial y^{2\nu}}(\psi_1-\psi_2) + a_1(\psi_1-\psi_2) + a_2({\psi_1}^2 - {\psi_2}^2) + \\ .... + a_n({\psi_1}^n - {\psi_2}^n)\end{array}\right|$$

Now by the lemma 3 note that $D_y^{2\nu}\psi$ is bounded and since ψ is a bounded function therefore ψ^n is also bounded.
Also as the sum and difference of bounded functions are bounded, we can have

$$\begin{vmatrix}V(\psi_1(y,t)) - \\ V(\psi_2(y,t))\end{vmatrix} \leqslant \frac{\eta^\nu}{\Gamma(1+\nu)}|\psi_1 - \psi_2|\left[K + a_1m_1 + a_2m_2 + ... + a_nm_n\right]$$

where $|D_y^{2\nu}\psi| \leq K$ and $m_1, m_2, ..., m_n$ are the bounds for the other terms.

$$\leq \frac{\eta^\nu}{\Gamma(1+\nu)}|b^\nu - a^\nu|||\psi_1(y,t) - \psi_2(y,t)||_\nu,$$

where $|K + m_1 + m_2 + ... + m_n| \leq |b^\nu - a^\nu|$.
Hence for $n = 1$, the inequality holds.
Now let's assume it for $n = k$

$$||V^k(\psi_1(y,t)) - V^k(\psi_2(y,t))||_\nu \leq \frac{\eta^{k\nu}|b^\nu - a^\nu|^k}{\Gamma^k(1+\nu)}||\psi_1(y,t) - \psi_2(y,t)||_\nu \tag{10}$$

Now for $n = k+1$, we see that

$$\begin{aligned}
\left\| \begin{matrix} V^{k+1}(\psi_1(y,t)) - \\ V^{k+1}(\psi_2(y,t)) \end{matrix} \right\|_\nu &= \left| \frac{1}{\Gamma(1+\nu)} \int_\nu^t \begin{bmatrix} \phi(V^k(\psi_1(y,s))) - \\ \phi(V^k(\psi_2(y,s))) \end{bmatrix} (ds)^\nu \right| \\
\leqslant \left| \frac{1}{\Gamma(1+\nu)} \int_\nu^t \eta^\nu \begin{bmatrix} V^k(\psi_1(y,s)) \\ -V^k(\psi_2(y,s)) \end{bmatrix} (ds)^\nu \right| & \\
\leqslant \frac{\eta^{(k+1)\nu}|b^\nu - a^\nu|^{(k+1)}}{\Gamma^{(k+1)}(1+\nu)} \left\| \begin{matrix} \psi_1(y,t) - \\ \psi_2(y,t) \end{matrix} \right\|_\nu & \text{(using inequality 10)}
\end{aligned}$$

Thus for $n = k+1$, our assumption is proved, and we can say that

$$||V^n(\psi_1(y,t)) - V^n(\psi_2(y,t))||_\nu \leq \frac{\eta^{n\nu}|b^\nu - a^\nu|^n}{\Gamma^n(1+\nu)}||\psi_1(y,t) - \psi_2(y,t)||_\nu.$$

Now note that

$$\frac{\eta^{n\nu}|b^\nu - a^\nu|^n}{\Gamma^n(1+\nu)}||\psi_1(y,t) - \psi_2(y,t)||_\nu \to 0$$

as $n \to \infty$.
Therefore we can say that the map V^n is a contraction over $C_\nu[a,b]$ which conclusively says that the given system has a unique solution. □

APPROXIMATE ANALYTICAL SOLUTIONS OF N TERM LOCAL FRACTAL KLEIN- GORDON EQUATIONS

Theorem 4. *If we consider*

$$D_t^\nu \psi = D_y^{2\nu}\psi \ + a_1\psi + a_2\psi^2 + a_3\psi^3 + \cdots\cdots + a_n\psi^n \tag{11}$$

with initial condition

$$\psi(y,0) = \psi_o \tag{12}$$

where D_t^ν is Local fractional derivative operator with $\nu \in (0,1]$and $a_1, a_2, \cdots, a_n$ are real constants. Then the solution of (11) is given as

$$\psi(y,t) = \Psi_0 + \sum_{k=1}^{\infty} \Psi_k(y)t^{k\nu}. \tag{13}$$

Proof. We are going to apply the method of Local Fractional Reduced Differential Transform $LFRDTM$on (11)
For that, we recall that the reduced differential transform(Locally fractional) of $\psi(y,t)$ is $\Psi_k(y)$ or $\Psi(y,k)$ and is established as

$$\begin{aligned}
\Psi(y,k) \text{ or } \Psi_k(y) &= \frac{1}{\Gamma(1+k\nu)}\left[\frac{\partial^{k\nu}\psi(y,t)}{\partial t^{k\nu}}\right] \\
&= \frac{1}{\Gamma(1+k\nu)}[D_t^{k\nu}\psi(y,t)]
\end{aligned}$$

This implies

$$\Psi_{k+1}(y) = \frac{1}{\Gamma(1+(k+1)\nu)}[D_t^{(k+1)\nu}\psi(y,t)] \tag{14}$$

and since we are using local fractional derivative, therefore

$$\begin{aligned}
D_t^{(k+1)\nu}\psi(y,t) &= \frac{\partial^{(k+1)\nu}}{\partial t^{(k+1)\nu}}\psi(y,t) \\
&= \underbrace{\frac{\partial^\nu}{\partial t^\nu}\frac{\partial^\nu}{\partial t^\nu}\cdots\cdots\frac{\partial^\nu}{\partial t^\nu}}_{(k+1)times}\psi(y,t) \\
&= \frac{\partial^\nu}{\partial t^\nu}\left[\frac{\partial^{k\nu}}{\partial t^{k\nu}}\psi(y,t)\right] = \frac{\partial^\nu}{\partial t^\nu}[D_t^{k\nu}\psi(y,t)] \\
&= \frac{\partial^\nu}{\partial t^\nu}\left[\Gamma(1+k\nu)\Psi_k(y)\right]
\end{aligned} \tag{15}$$

Substituting (14) into (15), we get

$$\Gamma(1+(k+1)\nu)\Psi_{k+1}(y) = \frac{\partial^\nu}{\partial t^\nu}(\Gamma(1+k\nu)\Psi_k(y))$$

Hence we get the recurrence relation as,

$$\begin{aligned}
\Psi_{k+1}(y) &= \frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\frac{\partial^\nu\Psi_k(y)}{\partial t^\nu} \\
&= \frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}D_t^\nu\Psi_k(y)
\end{aligned} \tag{16}$$

thus using the properties of RTDM, and after applying LFRDTM to (11) we get

$$\Psi_{k+1}(y) = \frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\begin{bmatrix} D_y^{2\nu}a_1\Psi_k(y) + a_2\Psi_k^2 + a_3\Psi_k^3 + \cdots \\ \cdots + a_n\Psi_k^n(y) \end{bmatrix}$$

with (I.C)

$$\Psi_0(y) = \Psi_0$$

where $\Psi_k(y)$ is a Local fractional differential differential transform of $\psi(y,t)$
and similarly for $\psi^2, \psi^3, \cdots, \psi^n$ the transformed terms are

$$\begin{aligned}
\Psi_k^2(y) &= \sum_{r=0}^{k}\Psi_k(y)\Psi_{k-r}(y) \\
\Psi_k^3(y) &= \sum_{r=0}^{k}\sum_{s=0}^{r}\Psi_s(y)\Psi_{r-s}(y)\Psi_{k-r}(y) \\
\Psi_k^4(y) &= \sum_{r=0}^{k}\sum_{s=0}^{r}\sum_{t=0}^{s}\Psi_t(y)\Psi_{s-t}(y)\Psi_{r-s}(y)\Psi_{k-r}(y) \\
&\vdots \\
\Psi_k^n(y) &= \underbrace{\sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_{n-1}=0}^{r_{n-2}}}_{n-1\ times}\overbrace{\Psi_{r_{n-1}}(y)\Psi_{r_{n-2}}(y)\cdots\Psi_{k-r_1}(y)}^{n\ times}
\end{aligned}$$

Thus, the recurrence relation along with initial condition Ψ_0 is

$$\Psi_{k+1}(y)=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\Bigg[D_y^{2\nu}a_1\Psi_k(y)+a_2\sum_{r=0}^{k}\Psi_k(y)\Psi_{k-r}(y)+$$
$$a_3\sum_{r=0}^{k}\sum_{s=0}^{r}\Psi_s(y)\Psi_{r-s}(y)\Psi_{k-r}(y)+$$
$$\cdots\cdots+a_n\sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\cdots\sum_{r_{n-1}=0}^{r_{n-2}}\Psi_{r_{n-1}}(y)\Psi_{r_{n-2}}(y)\cdots\Psi_{k-r_1}(y)\Bigg] \tag{17}$$

and using this recurrence relation, we have

$$\Psi_1=\frac{\Gamma(1+0)}{\Gamma(1+\nu)}[D_y^{2\nu}\Psi_0+a_1\Psi_0+a_2\Psi_0^2+\cdots+a_n\Psi_0^n]$$
$$=\frac{1}{\Gamma(1+\nu)}[D_y^{2\nu}\Psi_0+\Psi_0[a_1+a_2\Psi_0+\cdots+a_n\Psi_0^{n-1}]]$$
$$\Psi_2=\frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}[D_y^{2\nu}\Psi_1+\Psi_1[a_1+a_2\Psi_1+\cdots+a_n\Psi_1^{n-1}]]$$
.
.
.
$$\Psi_k=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}[D_y^{2\nu}\Psi_{k-1}+\Psi_{k-1}[a_1+a_2\Psi_{k-1}+\cdots+a_n\Psi_{k-1}^{n-1}]] \tag{18}$$

Now using 6 we get the analytical solution to n term Klein Gordon equation as

$$\psi(y,t)=\sum_{k=0}^{\infty}\Psi_k(y)t^{k\nu}$$
$$=\Psi_0(y)+\Psi_1(y)t^{\nu}+\Psi_2(y)t^{2\nu}+\cdots$$

where $\Psi_0,\Psi_1,\Psi_2,\cdots$ are defined as above18. □

Now consider the following cases for particular solutions

Example 1. *consider the fractional differential equation*

$$\frac{\partial^{\nu}}{\partial t^{\nu}}\psi(y,t)=\frac{\partial^{2\nu}}{\partial x^{2\nu}}\psi(y,t) \tag{19}$$

with initial condition $\psi(y,0)=\psi_0$.
Note that this is a linear local fractional n term Klein-Gordon equation with $a_1=a_2=\cdots=a_n=0$ which is a special case of (11).
Let say $\psi_0=E_\nu(y^\nu)$
Taking local fractional reduced differential transform of (19),
We get the subsequent recurrence relation

$$\Psi_{k+1}(y)=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\left[D_y^{2\nu}\Psi_k(y)\right] \tag{20}$$

with (I.C) $\Psi_0(y)=E_\nu(y^\nu)$. (21)

On applying the recurrence relation and initial condition20, we attain

$$\Psi_1(y)=\frac{1}{\Gamma(1+\nu)}\left[D_y^{2\nu}\Psi_o(y)\right]$$
$$=\frac{1}{\Gamma(1+\nu)}E_\nu(y^\nu)$$
$$\Psi_2(y)=\frac{1}{\Gamma(1+2\nu)}\left[D_y^{2\nu}\Psi_1(y)\right]$$
$$=\frac{1}{\Gamma(1+2\nu)}E_\nu(y^\nu)$$
$$\vdots$$
$$\Psi_n(y)=\frac{1}{\Gamma(1+n\nu)}E_\nu(y^\nu).$$

Applying the inverse local fractional reduced differential transform, we attain the solution of 19.

$$\psi(y,t)=\sum_{k=0}^{\infty}\Psi_k(y)t^{k\nu}$$
$$=E_\nu(y^\nu)+E_\nu(y^\nu)\frac{t^\nu}{\Gamma(1+\nu)}+E_\nu(y^\nu)\frac{t^{2\nu}}{\Gamma(1+2\nu)}+\cdots$$
$$=\sum_{k=0}^{\infty}E_\nu(y^\nu)\frac{t^{k\nu}}{\Gamma(1+k\nu)}$$

This implies
$$\psi(y,t)=E_\nu(y^\nu)E_\nu(t^\nu).$$

The graphical illustration of the solution $\psi(y,t)$ is shown in [1] when $\nu=\frac{log(2)}{log(3)}$.

Example 2. *When $a_1=1,a_2=a_3=a_4=\cdots=a_n=0$, we get*

$$D_t^\nu\psi(y,t)=D_y^{2\nu}\psi(y,t)+\psi(y,t),\ t>0,\nu\in(0,1]$$

with initial condition
$$\psi(y,0)=E_\nu(y^\nu).$$

we know that Local fractional reduced differential transform of $\psi(y,t)$ is

$$\Psi_k(y)=\frac{1}{\Gamma(1+k\nu)}\left[\frac{\partial^{k\nu}\psi(y,t)}{\partial t^{k\nu}}\right]=\frac{1}{\Gamma(1+k\nu)}[D_t^{k\nu}\psi(y,t)]$$

This implies

$$\Psi_{k+1}(y)=\frac{1}{\Gamma(1+(k+1)\nu)}[D_t^{(k+1)\nu}\psi(y,t)]$$
$$\Rightarrow\Gamma(1+(k+1)\nu)\Psi_{k+1}(y)=\frac{\partial^\nu}{\partial t^\nu}(\Gamma(1+k\nu)\Psi_k(y)).$$

After applying local fractional partial derivative property, we get,

$$\Psi_{k+1}(y)=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\frac{\partial^\nu\Psi_k(y)}{\partial t^\nu}=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}D_t^\nu\Psi_k(y).$$

Thus, applying the properties of RTDM and apply LFRDTM, we get

$$\Psi_{k+1}(y)=\frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\left[D_y^{2\nu}\Psi_k(y)+\Psi_k(y)\right] \tag{22}$$

with (I.C)

$$\Psi_o(y) = E_\nu(y^\nu).$$

now applying Recurrence relation [22], we attain

$$\begin{aligned}
\Psi_1(y) &= \frac{1}{\Gamma(1+\nu)}\left[D_y^{2\nu}\Psi_o(y) + \Psi_o(y)\right]\\
&= \frac{2}{\Gamma(1+\nu)}E_\nu(y^\nu)\\
\Psi_2(y) &= \frac{1}{\Gamma(1+2\nu)}\left[D_y^{2\nu}\Psi_1(y) + \Psi_1(y)\right]\\
&= \frac{2^2}{\Gamma(1+2\nu)}E_\nu(y^\nu)\\
&\vdots\\
\Psi_k(y) &= \frac{2^k}{\Gamma(1+k\nu)}E_\nu(y^\nu)
\end{aligned}$$

Now using the inverse local fractional reduced differential transform

$$\begin{aligned}
\psi(y,t) &= \sum_{k=0}^{\infty}\Psi_k(y)t^{k\nu}\\
&= \Psi_0(y) + \Psi_1(y)t^\nu + \Psi_2(y)t^{2\nu} + \cdots\\
&= E_\nu(y^\nu)\sum_{k=0}^{\infty}\frac{(2t)^{k\nu}}{\Gamma(1+k\nu)}\\
&= E_\nu((2t-x)^\nu).
\end{aligned}$$

The graphical illustration of the solution $\psi(y,t)$ is shown in [2] when $\nu = \frac{log(2)}{log(3)}$.

Example 3. *Now considering the case $a_2 = -1, a_1 = a_3 = \cdots = a_n = 0$ along with initial condition $\psi(y,0) = 1 + sin_\nu(y^\nu)$.Then the non-linear KGE we have,*

$$D_t^\nu\psi(y,t) = D_y^{2\nu}\psi(y,t) - \psi^2(y,t),\ t>0, \nu\in(0,1]$$

,

with initial condition

$$\psi(y,0) = 1 + sin_\nu(y^\nu).$$

To get the next recurrence relation, we will use the local fractional reduced differential transform.

$$\Psi_{k+1}(y) = \frac{\Gamma(1+k\nu)}{\Gamma(1+(k+1)\nu)}\left[D_{y)}^{2\nu}\Psi_k(y) - \sum_{r=0}^{k}\Psi_r\Psi_{k-r}\right]$$

with (I.C)

$$\Psi_o(y) = 1 + sin_\nu(y^\nu).$$

thus we get

$$\begin{aligned}
\Psi_1(y) &= \frac{\Gamma(1)}{\Gamma(1+\nu)}[D_y^{2\nu}(\Psi_0) + [\Psi_0]^2]\\
&= \frac{1}{\Gamma(1+\nu)}[D_y^{2\nu}(1+sin_\nu(y^\nu)) - (1+sin_\nu(y^\nu))^2]\\
&= \frac{-1}{\Gamma(1+\nu)}[3sin_\nu(y^\nu) + sin_\nu^2(y^\nu) + 1]\\
&= \frac{1}{\Gamma(1+2\nu)\Gamma(1+\nu)}\begin{bmatrix}\Gamma(1+\nu)(3sin_\nu(y^\nu) - 2 + 4sin_\nu^2(y^\nu)) - 1\\ -11sin_\nu^2(y^\nu) + 6sin_\nu^3(y^\nu) +\\ 6sin_\nu(y^\nu) + sin_\nu^4(y^\nu)\end{bmatrix}\\
&.\\
&.\\
&.\\
&.
\end{aligned}$$

Now substituting using the inverse Local fractional reduced differential transform method (LFRDTM), we have

$$\begin{aligned}
\psi(y,t) &= \sum_{k=0}^{\infty}\Psi_k t^{k\nu}\\
&= \Psi_0 + \Psi_1 t^\nu + \Psi_2 t^{2\nu} + \cdots
\end{aligned}$$

$$\begin{aligned}
&= 1 + sin_\nu(y^\nu) - \frac{t^\nu}{\Gamma(1+\nu)}\left[3sin_\nu(y^\nu) + sin_\nu^2(y^\nu) + 1\right]\\
&+ \frac{t^{2\nu}}{\Gamma(1+\nu)\Gamma(1+2\nu)}\Big[(3sin_\nu(y^\nu) - 2 + 4sin_\nu^2(y^\nu))\Gamma(1+\nu)\\
&- 1 - 11sin_\nu^2(y^\nu) + 6sin_\nu^3(y^\nu) + 6sin_\nu(y^\nu) + sin_\nu^4(y^\nu)\Big] + \cdots\cdots
\end{aligned}$$

which is the series solution of this particular local fractional KGE. The graphical illustration of the solution $\psi(y,t)$ is shown in [3] when $\nu = \frac{log(2)}{log(3)}$.

Example 4. *when $a_1 = a_2 = a_3 = a_4 = a_5 = \cdots = a_n = 1$,
we get the non linear local fractional Klein Gordon equation*

$$D_t^\nu\psi(y,t) = D_y^{2\nu}\psi(y,t) + \psi + \psi^2 + \psi^3(y,t),\ t>0, \nu\in(0,1]$$

with initial condition

$$\psi(y,0) = 1 + sin_\nu(y^\nu)$$

on applying LFRDTM here, we get

$$\Psi_{k+1}(y) = \frac{\Gamma(1+k\nu)}{\Gamma(1+\nu+k\nu)}\left[D_y^{2\nu}\Psi_k + \Psi_k + \Psi_k^2 + \Psi_k^3 + \cdots\right]$$

where

$$\Psi_k^2 = \sum_{r=0}^{k}\Psi_r\Psi_{k-r}$$

$$\Psi_k^3 = \sum_{r=0}^{k}\sum_{s=0}^{r}\Psi_s\Psi_{r-s}\Psi_{k-r}$$

*and initial condition transforms into $\Psi_0(y) = 1 + sin_\nu(y^\nu)$
Now, using the recurrence relation along with the initial condition, we obtain*

$$\Psi_1(y) = \frac{1}{\Gamma(1+\nu)}[D_y^{2\nu}(\Psi_0) + \Psi_0 + (\Psi_0)^2 + (\Psi_0)^3 + \cdots]$$

$$\Psi_1(y) = \frac{1}{\Gamma(1+\nu)}\begin{bmatrix} D_y^{\nu}(D_y^{\nu}(1+sin_\nu(y^\nu))) + (1+sin_\nu(y^\nu)) + \\ (1+sin_\nu(y^\nu))^2 + (1+sin_\nu(y^\nu))^3 + \cdots \end{bmatrix}$$

$$= \frac{1}{\Gamma(1+\nu)}\begin{bmatrix} -sin_\nu(y^\nu) + 1 + sin_\nu(y^\nu) + 1 + sin_\nu^2(y^\nu) + 2sin_\nu(y^\nu) \\ +1 + sin_\nu^3(y^\nu) + 3sin_\nu(y^\nu) + 3sin_\nu^2(y^\nu) + \cdots \end{bmatrix}$$

$$= \frac{1}{\Gamma(1+\nu)}[sin_\nu^3(y^\nu) + 4sin_\nu^2(y^\nu) + 5sin_\nu(y^\nu) + n + \cdots]$$

$$\Psi_2(y) = \frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}[D_y^{2\nu}(\Psi_1) + \Psi_1 + (\Psi_1)^2 + (\Psi_1)^3 + \cdots]$$

$$\Psi_2(y) = \frac{1}{\Gamma(1+\nu)\Gamma(1+2\nu)}\begin{bmatrix} sin_\nu^9(y^\nu) + 4sin_\nu^8(y^\nu) + 9sin_\nu^7(y^\nu) \\ +105sin_\nu^6(y^\nu) + 123sin_\nu^5(y^\nu) + \\ 126sin_\nu^4(y^\nu) + 166sin_\nu^3(y^\nu) + \\ 29sin_\nu^2(y^\nu) - 3/4sin_\nu(y^\nu) + \cdots \end{bmatrix}$$

$$\Psi_3(y) = \frac{1}{\Gamma(1+\nu)\Gamma(1+2\nu)\Gamma(1+3\nu)}\begin{bmatrix} sin_\nu^{27}(y^\nu) + 10682sin_\nu^{26}(y^\nu) \\ +269874sin_\nu^{25}(y^\nu) + \cdots \end{bmatrix}$$

.

.

.

neglecting higher terms since with $0 < \nu \leq 1$ and as $|sin_\nu(y^\nu)| \leq 1$ the terms with higher power will eventually tend near zero.
Now, substituting all using the inverse local fractional reduced differential transform method (LFRDTM), we have

$$\psi(y,t) = \sum_{k=0}^{\infty} \Psi_k t^{k\nu}$$
$$= \Psi_0 + \Psi_1 t^\nu + \Psi_2 t^{2\nu} + \cdots$$

$$= 1 + sin_\nu(y^\nu) - \frac{t^\nu}{\Gamma(1+\nu)}\begin{bmatrix} sin_\nu^3(y^\nu) + 4sin_\nu^2(y^\nu) \\ +5sin_\nu(y^\nu) + n + \cdots \end{bmatrix}$$
$$+ \frac{t^{2\nu}}{\Gamma(1+\nu)\Gamma(1+2\nu)}\begin{bmatrix} sin_\nu^9(y^\nu) + 4sin_\nu^8(y^\nu) + 9sin_\nu^7(y^\nu) + \\ 105sin_\nu^6(y^\nu) + 123sin_\nu^5(y^\nu) + 126sin_\nu^4(y^\nu) \\ +166sin_\nu^3(y^\nu) + 29sin_\nu^2(y^\nu) - 3/4sin_\nu(y^\nu) + \cdots \end{bmatrix}$$
$$+ \frac{t^{3\nu}}{\Gamma(1+\nu)\Gamma(1+2\nu)\Gamma(1+3\nu)}\begin{bmatrix} sin_\nu^{27}(y^\nu) + 10682sin_\nu^{26}(y^\nu) \\ +269874sin_\nu^{25}(y^\nu) + \cdots \end{bmatrix} + \cdots$$

which is the series solution of the given local fractional Klein-Gordon equation. The figures 4,5, 6 and 7 shows the physical interpretation of $\psi(y,t)$ corresponding to $\nu = 0.25, 0.5, 0.6289$ and 1.

Figures here show the physical interpretation of $\psi(y,t)$ vs. t corresponding to a particular value of ν.

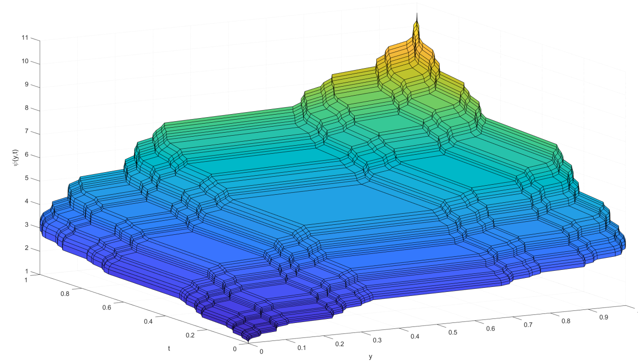

Figure 1 The figure illustrates Solution of Example 1 when $\nu = log(2)/log(3) = 0.6309$

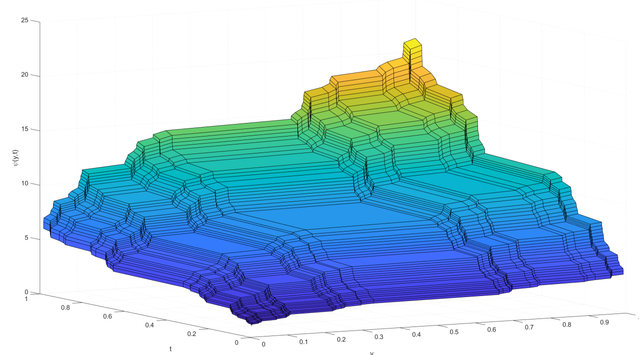

Figure 2 The figure illustrates Solution of Example 2 when $\nu = log(2)/log(3) = 0.6309$

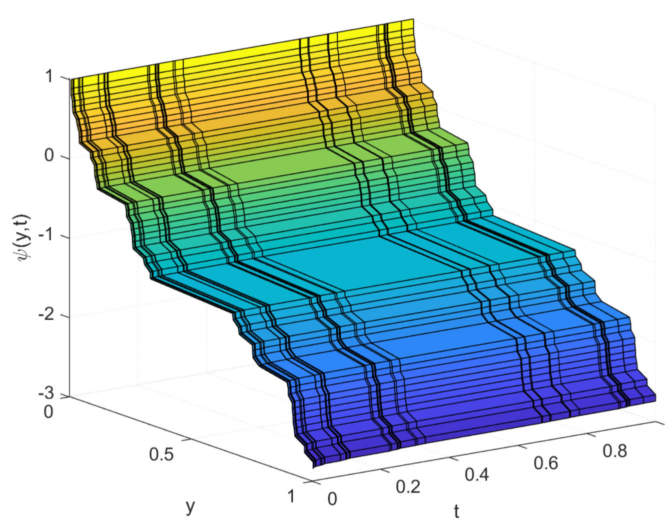

Figure 3 The figure illustrates Solution of Example 3 of $\psi(y,t)$ vs. time t when $\nu = log(2)/log(3) = 0.6309$

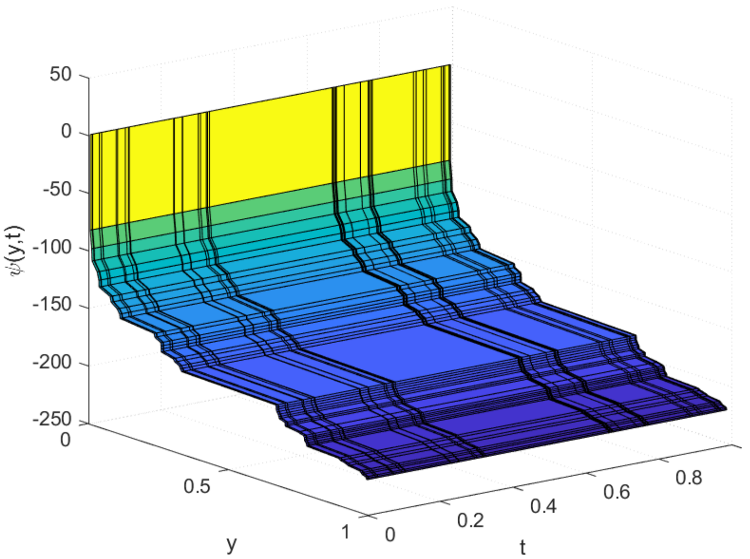

Figure 4 Example 4 when $\nu = 0.25$

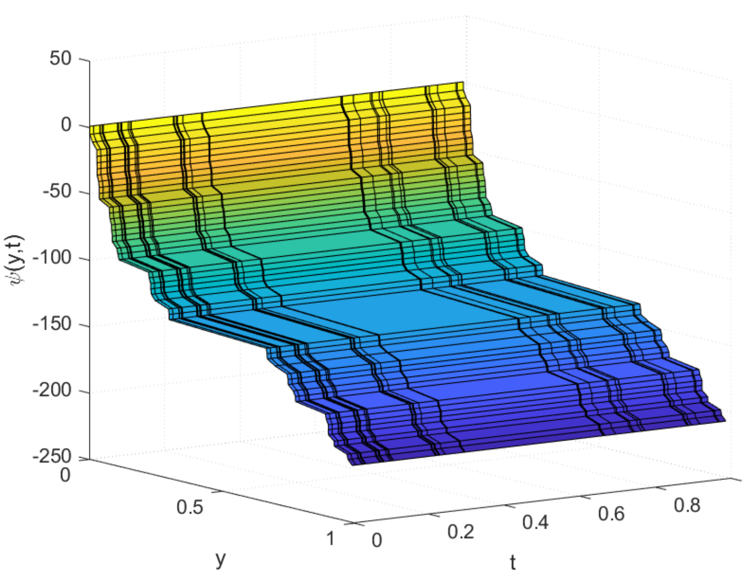

Figure 5 Example 4 when $\nu = 0.5$

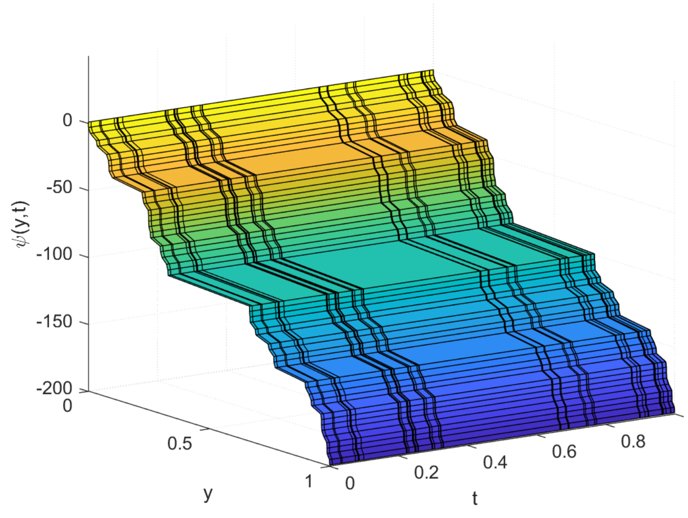

Figure 6 Example 4 when $\nu = 1$

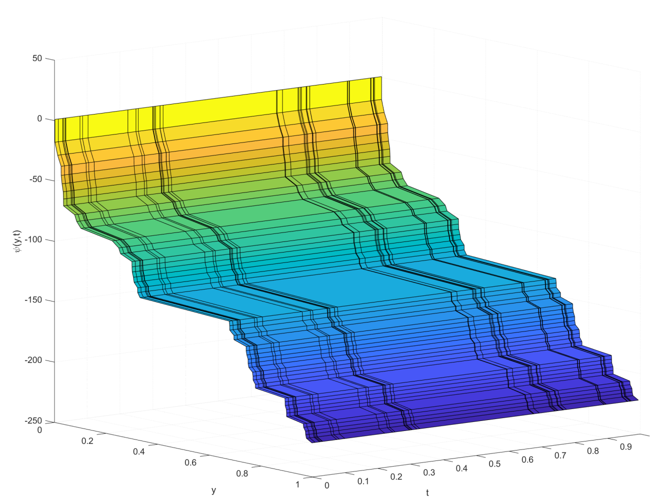

Figure 7 Example 4 when $\nu = log(2)/log(3)$

CONCLUSION

In this study, we have combined the fractional complex transform with the local fractional differential transform method to analyze the Klein-Gordon equations of n terms in cantor sets within the local fractional differential operator and have tried to approximate the solution of the same. Our results show that this method is an effective mathematical tool for solving local fractional linear differential equations. Furthermore, the versatility of this method makes it highly adaptable to solving a wide range of fractional differential equations. The examples are particular cases of the proposed n-term Klein-Gordon equation, and their corresponding corrected approximated solutions are presented along with their graphs. Hence, we can conclude that the fractional complex transform with the local fractional differential transform method is a powerful and flexible approach to obtain effective approximate solutions of local fractional partial differential equations.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Ethical standard

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