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Research Article

RELAXING MULTICURVES ON THE TWICE PUNCTURED MÖBIUS BAND

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Abstract: *An efficient way to describe multicurves (homotopy classes of finitely many essential simple closed curves) in an n-punctured non-orientable surface* $N_{g,n}$ $(g > 1, n \geq 1)$ *of genus g with one boundary component is achieved by the generalized Dynnikov coordinate system. In the case where* $g = 1$ *where the surface is, therefore, an* n*-punctured Möbius band, the generalized Dynnikov coordinate system gives a one-to-one map between the set of multicurves in* $N_{1,n}$ *and the set* $Z^{2n-1}\setminus\{0\}$*. In this paper, we describe an algorithm for relaxing an arbitrary multicurve in* $N_{1,2}$ *making use of generalized Dynnikov coordinates and the action of the mapping class group of* $N_{1,2}$ *in terms of generalized Dynnikov coordinates.*

Keywords: *Generalized Dynnikov coordinates, Möbius band, Multicurves*

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1. Introduction

Given a surface S, a simple closed curve in S is called essential if it is not parallel to a puncture or a boundary component. When S is non-orientable there are two types of essential simple closed curves in S. If a regular neighborhood of the curve is an annulus it is called 2-sided, and if it is a Möbius band it is called 1-sided. A multicurve in S is a disjoint union of finitely many essential simple closed curves in S up to homotopy.

A beautiful way to describe multicurves in $N_{g,n}$ $(g > 1, n \geq 1)$ is achieved by the generalized Dynnikov coordinate system [9, 10]. In the case where $g = 1$ where the surface is an n-punctured Möbius band, the generalized Dynnikov coordinate system gives a one-to-one map between the set of multicurves in $N_{1,n}$, and $Z^{2n-1}\setminus\{0\}$. Let $\mathcal{L}_{1,n}$ be the set of multicurves in $N_{1,n}$. In this paper we focus on a combinatorial problem regarding multicurves in $N_{1,2}$. The problem is to improve a relaxation algorithm for a multicurve $\mathcal{L} \in \mathcal{L}_{1,2}$. More precisely, we compute a mapping class f (isotopy class of a homeomorphism) of $N_{1,2}$ such that $f(\mathcal{L})$ is relaxed, a particular multicurve where each component is either the core curve of the Möbius band or a curve intersecting the horizontal diameter of $N_{1,2}$ exactly twice.

Section 2 serves as a background to the paper which contains a short introduction to the generalized Dynnikov coordinates and a description of the mapping class group MCG($N_{1,2}$). Section 3 provides the necessary tools for Section 4 where an algorithm to relax a multicurve $\mathcal{L} \in \mathcal{L}_{1,2}$ is given.

2. Preliminaries

2.1. Generalized Dynnikov Coordinates

Consider the standard model of the n–punctured Möbius band $N_{1,n}$ as illustrated in Figure 1 where the punctures and the crosscap are aligned in the horizontal diameter of $N_{1,n}$. Here the disk with a cross drawn within it represents the crosscap which is obtained by removing the interior of the disk and then identifying the antipodal points in the resulting boundary component.

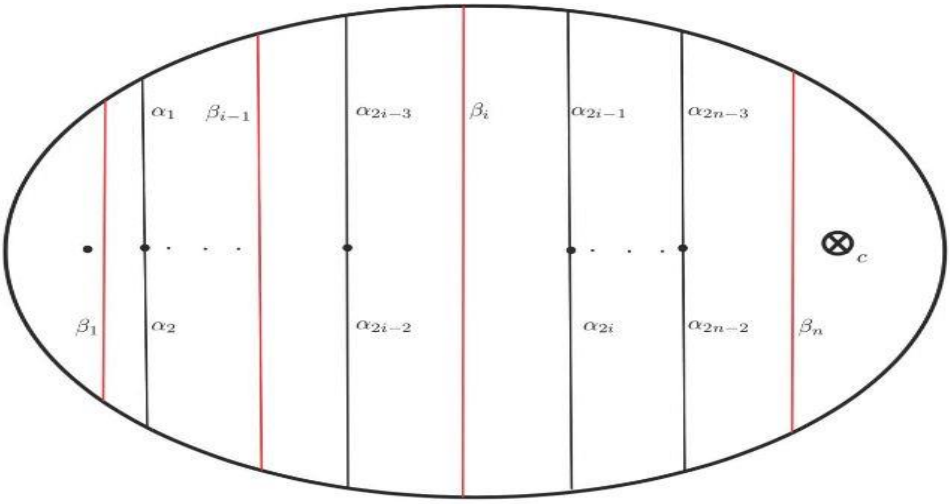

Figure 1. The core curve c and the arcs α_i, β_i on $N_{1,n}$

Consider the core curve c of the crosscap together with the arcs α_i $(1 \leq i \leq 2n-2)$ and β_i $(1 \leq i \leq n)$ depicted in Figure 1. Take a representative L of $\mathcal{L}$ intersecting the core curve and each of these arcs minimally. We call L a minimal representative of $\mathcal{L}$. For convenience, the number of intersections of L with the core curve c and each of the arcs α_i and β_i will also be denoted by the same symbols. We define the generalized Dynnikov coordinate function [9,10] ρ: $\mathcal{L}_{1,n} \rightarrow \mathbb{Z}^{2n-1} \setminus \{0\}$ by

$$\rho(\mathcal{L}) \coloneqq (a; b; c) = (a_1, a_2, \dots a_{n-1}; b_1, b_2, \dots, b_{n-1}; c)$$

where

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2}, \quad 1 \leq i \leq n-1,$$

(1)

$$b_i = \frac{\beta_i - \beta_{i+1}}{2}, \quad 1 \leq i \leq n-1$$

The vector $(\alpha; \beta; c) \in \mathbb{Z}^{3n-1} \setminus \{0\}$ therefore the multicurve $\mathcal{L}$) can be obtained from the generalized Dynnikov coordinates $(a; b; c) \in \mathbb{Z}^{2n-1} \setminus \{0\}$. Let

$$\beta_i^* = 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_{r,0}) + \sum_{j=1}^{r-1} b_j \right\} - 2 \sum_{j=1}^{i-1} b_j$$

$$R = \max(0, c - \beta_n^* /2)$$

Then,

(2) $$\beta_i = \beta_i^* + 2R$$

(3)
$$\alpha_i = \begin{cases} (-1)^i a_{[i/2]} + \dfrac{\beta_{[i/2]}}{2} & \text{if } b_{[i/2]} \geq 0, \\ (-1)^i a_{[i/2]} + \dfrac{\beta_{1+[i/2]}}{2} & \text{if } b_{[i/2]} \leq 0, \end{cases}$$

where $[x]$ denotes the smallest integer which is not smaller than the value x.

This paper focuses on the case where $n = 2$. Therefore, the generalized Dynnikov coordinates of a multicurve $\mathcal{L}$ are given by the vector $(a; b; c) \in \mathbb{Z}^3 \setminus \{0\}$ where $(\alpha; \beta; c) \in \mathbb{Z}^5 \setminus \{0\}$ can be obtained by the above formula. Remark 2.1 points out the geometric interpretation of the parameter R.

Remark 2.1. Observe that for any essential simple closed curve except for such as those depicted in Figure 2, we have either $\alpha_1 - b \neq 0$ or $\alpha_2 - b \neq 0$ otherwise, there would be curves parallel to the boundary component. The parameter R counts the number of arcs of such curves which intersect α_1, α_2, and c exactly once.

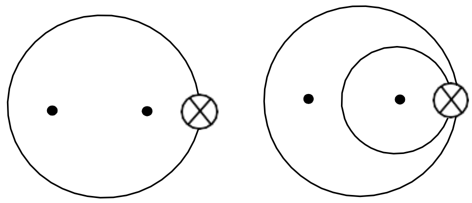

Figure 2. Geometric interpretation of the parameter R

2.2. Generators of MCG($N_{1,n}$)

The mapping class group MCG(S) of a given surface S is the group of isotopy classes of homeomorphisms of S (homeomorphisms are orientation preserving when S is orientable). The elements of MCG(S) are called mapping classes of S. In the case when $S = N_{1,n}$ the mapping class group MCG($N_{1,n}$) is generated by the braid generators σ_i $(1 \leq i \leq n-1)$ and the puncture slides v_i $(1 \leq i \leq n)$ [7]. The braid generator σ_i is a mapping class supported in a disk with two punctures exchanging puncture i and $i+1$ in an anticlockwise manner and leaves the exterior of the disk invariant. Now consider a Möbius band M with a puncture p. The puncture slide v pushes p once along the core curve of M fixing a neighborhood of the boundary of M. The effect of σ_i and the n-th puncture slide v_n on the arcs β_i and β_n are shown in Figure 3. Due to the well-known relation [7] $v_i = \sigma_i v_{i+1} \sigma_i^{-1}$ each mapping class of MCG($N_{1,n}$) can be written as a sequence of braid generators σ_i, the n-th puncture slide v_n, and their inverses. Therefore, each mapping class considered in this paper corresponds to a word where each letter belongs to the set $\{\sigma_1, \sigma_1^{-1}, v_2, v_2^{-1}\}$.

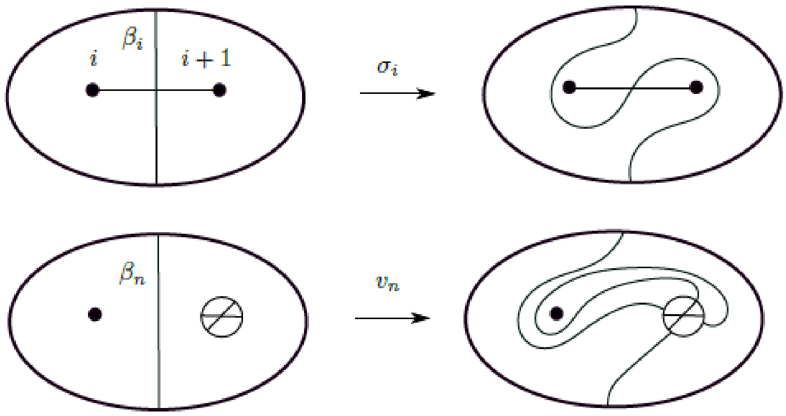

Figure 3. The action of σ_i on β_i and v_n *on* β_n

Definition 2.2. An essential simple closed curve $C \in \mathcal{L}_{1,2}$ is called relaxed if it satisfies one of the following: $\rho(C) = (0;1;0)$ or $\rho(C) = (0;-1;0)$ or C is the core curve of the crosscap as shown in Figure 4.

It is always possible to transform an arbitrary curve in $\mathcal{L}_{1,2}$ into one of the relaxed curves depicted in Figure 4. This process is known as *the relaxation algorithm* for multicurves. Before we give a relaxation algorithm for multicurves in $N_{1,2}$ we present some necessary tools.

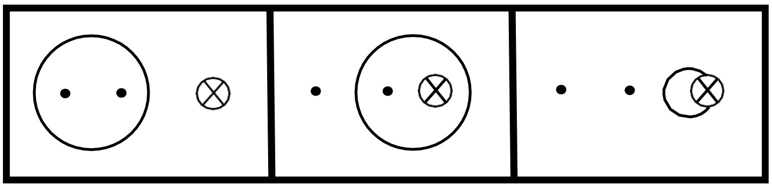

Figure 4. Relaxed curves in $N_{1,2}$

3. Tools for the Algorithm

In this section, we provide some necessary tools to improve the relaxation algorithm in Section 4.

3.1. Dynnikov Coordinates and the action of MCG(D_3)

Removing the crosscap and the arcs $\alpha_{2n-2}, \alpha_{2n-3}$, and β_n from the standard model for the generalized Dynnikov coordinate system depicted in Figure 1, we can construct the Dynnikov coordinate system for the n-punctured disk D_n [3, 5]. For $n \geq 3$, Dynnikov coordinate system [3] provides a bijection ρ: $\mathcal{L}_n \rightarrow \mathbb{Z}^{2n-4}\backslash\{0\}$ from the set of multicurves $\mathcal{L}_n$ in D_n to $\mathbb{Z}^{2n-4}\backslash\{0\}$ given by

$$\rho(\mathcal{L}) \coloneqq (a;b) = (a_1, a_2, \dots a_{n-2}; b_1, b_2, \dots, b_{n-2})$$

where a_i and b_i are as described in equations (1). Since MCG(D_n) is isomorphic to Artin's braid group B_n modulo its center [1] each mapping class of D_3 can be written as a sequence of Artin's braid generators σ_1, σ_2, and their inverses. B_n acts on $\mathcal{L}_n$ piecewise linearly and is described by the *update rules* in terms of the Dynnikov coordinates [3, 4, 5, 6, 8]. Theorem 3.1 gives *updated rules* for B_3.

Theorem 3.1 (Update rules [6]**).** Let $\mathcal{L}$ be a multicurve with $\rho(\mathcal{L}) = (a;b) \in \mathbb{Z}^2\backslash\{0\}$. Let $\rho(\sigma_i(\mathcal{L})) = (a';b')$ *and* $\rho(\sigma_i^{-1}(\mathcal{L})) = (a';b')$. *Let* $x^+ = \max(0,x)$. *Then*

- *If* $i = 1$ *then*

$$a' = a + b - max\,(0, a, b)b'$$
$$= b^{+} - a$$

- $If\ i = 2\ \ then$
- $$a' = \max\,(a + b^{+}, b)$$
- $$b' = b - a - b^{+}$$
- $If\ i = 1\ \ then$

$$a'' = \max\,(0, a + b^{+})\text{-}b$$
$$b'' = a + b^{+}$$

- $If\ i = 2\ \ then$
- $$a'' = \min\,(a - b^{+}, -b)$$

$$b'' = a + b - b^{+}$$

Next, we define a move of the algorithm that is only used when $c = 0$.

3.2. Blowdown Move

This move blows down the crosscap to a point labeled p and transforms a multicurve $\mathcal{L}$ in $N_{1,2}$ with generalized Dynnikov coordinates $(a; b; 0) \in Z^{3}\backslash\{0\}$ into the same multicurve in D_3 which has hence Dynnikov coordinates $(a; b) \in Z^{2}\{0\}$. Suppose that the punctures are labeled 1, 2, and 3 from left to right. Then we say that p is in position 3 or in crosscap position.

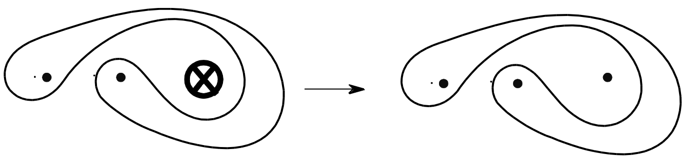

Figure 5. Blowdown move in $N_{1,2}$

4. Relaxation Algorithm

Let $(a; b) \in Z^{2}\backslash\{0\}$ (respectively $(a; b; c) \in Z^{3}\backslash\{0\}$) be the Dynnikov coordinates (respectively generalized Dynnikov coordinates) of a multicurve $\mathcal{L}$ *in* D_3 (respectively $N_{1,2}$). We write $(a'; b') \in Z^{2}\backslash\{0\}$ (respectively $(a'; b'; c') \in Z^{3}\backslash\{0\}$ to denote the Dynnikov (respectively generalized Dynnikov coordinates) of $\phi(\mathcal{L})$ where ϕ is a generator of the mapping class group MCG(D_3) (respectively MCG($N_{1,2}$)). Given a multicurve $\mathcal{L} \in \mathcal{L}_{1,2}$ with generalized Dynnikov coordinates $\rho(\mathcal{L}) = (a; b; c) \in Z^{3}\backslash\{0\}$ the following algorithm finds a mapping class f such that $f(\mathcal{L})$ is relaxed.

Main Algorithm. **If** $c = 0$ apply Algorithm 1 **otherwise** apply Algorithm 2.

The following algorithm works for the case $c = 0$. The algorithm works with the pair $((a; b), p)$ where p is the position of the labeled puncture p.

Algorithm 1. Given $\mathcal{L} \in \mathcal{L}_{1,2}$ let $\rho(\mathcal{L}) = (a; b; 0) \in Z^{3}\backslash\{0\}$.

Step 1: Apply *Blow down* move and replace $(a; b; 0) \in Z^{3}\backslash\{0\}$ with $(a; b) \in Z^{2}\backslash\{0\}$ and input $((a; b), p)$ to Step 2.

Step 2: If $b \geq 0$ let $(a'; b') = \rho(\sigma_1(\mathcal{L}))$ if $a > 0$ and $(a'; b') = \rho(\sigma_1^{-1}(\mathcal{L}))$ if $a < 0$. If $a' = 0$ input the pair $((a'; b'), p)$ to Step 4: If $a' \neq 0$ and $b' \geq 0$ then input $((a'; b'), p)$ to Step 2.
Otherwise, input $((a'; b'), p)$ to Step 3.

Step 3: If $b \leq 0$ let $(a'; b') = \rho(\sigma_2(\mathcal{L}))$ if $a < 0$ and $(a'; b') = \rho(\sigma_2^{-1}(\mathcal{L}))$ if $a > 0$. If $a' = 0$ input $((a'; b'), p)$ to Step 4: If $a' \neq 0$ and $b' \leq 0$ then input $\big((a'; b'), p\big)$ to Step 3.
Otherwise, input $((a'; b'), p)$ to Step 2.

Step 4: Since $a = 0$ then $\mathcal{L}$ is relaxed. **If** $p \neq 3$ let $\rho(\sigma_1\sigma_2\sigma_1)(\mathcal{L}) = (a'; b')$ if $p = 1$, and let $\rho(\sigma_2)(\mathcal{L}) = (a'; b')$ if $p = 2$. Then input $((a'; b'), p)$ to Step 5.

Step 5: Since $a = 0$ and $p = 3$ then $\mathcal{L}$ is a relaxed curve in D_3 with p being in the crosscap position. Blow up p to the crosscap to obtain a relaxed curve in $N_{1,2}$.

The following algorithm works for case $c \neq 0$.

Algorithm 2. Given $\mathcal{L} \in \mathcal{L}_{1,2}$ let $\rho(\mathcal{L}) = (a; b; c) \in \mathbb{Z}^3 \backslash \{0\}$ such that $c \neq 0$.

Step 1: If $b \geq 0$ let $(a'; b') = \rho(\sigma_1(\mathcal{L}))$ if $a > 0$ and $(a'; b') = \rho(\sigma_1^{-1}(\mathcal{L}))$ if $a < 0$.
If $a' = 0$ input $(a'; b')$ to Step 3: If $a' \neq 0$ and $b' \geq 0$ then input $(a'; b')$ to Step 1.
Otherwise, input $(a'; b')$ to Step 2.

Step 2: If $b \leq 0$ let $(a'; b') = \rho\big(v_2(\mathcal{L})\big)$ if $a \leq 0$ and $(a'; b') = \rho(v_2^{-1}(\mathcal{L}))$ if $a \geq 0$.
If $a' = 0$ and $b' = 0$ input $(a'; b')$ to Step 3: If $b' \leq 0$ then input $(a'; b')$ to Step 2.
Otherwise, input $(a'; b')$ to Step 1.

Step 3: Since $a = 0$, $\mathcal{L}$ is relaxed. Write the generators used in Step 1 and Step 3 in order to express the mapping class f relaxing $\mathcal{L}$.

Remark 4.1. We note that while Algorithm 2 has the advantage of computing the mapping class f relaxing an arbitrary multicurve $\mathcal{L}$, we do not have a tool to describe the action of a puncture slide in terms of generalized Dynnikov coordinates yet.

Conflict of Interest:
The authors declare no conflict of interest.

Publication Ethics:
The authors declare that this document does not require ethics committee approval or any special permission. Our study does not cause any harm to the environment and does not involve the use of animal or human subjects.

Authors' Contributions:

Methodology: F.A, O.Y.

Writing: F.A., A.B., O.Y.

Program Coding: A.B.

All authors read and approved the final manuscript.

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