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Geometry of Twisted Products and Applications on Static Perfect Fluid Spacetimes

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ABSTRACT

In this paper, first we study the harmonicity of the functions and forms on the twisted products, and then we determine its sectional curvature. We explore some characteristics of static perfect fluid and static vacuum spacetimes on twisted product manifolds by proving the existence and obstructions on Ricci curvature. Finally, we study the problem of the existence static perfect fluid spacetime associated with the twisted generalized Robertson-Walker and standard static spacetime metrics.

Keywords: *Twisted product, gradient Ricci soliton, static perfect fluid, vacuum static space, generalized Robertson-Walker spacetime, standard static spacetime metric.*

AMS Subject Classification (2020): *Primary: 53C25; Secondary: 53C40.*

1. Introduction

In recent years, various properties of different kind of product manifolds have been studied in both Riemannian and Lorentzian settings. While warped product manifolds are the most ideal example that provides Einstein's field equations and characterizes the universe, twisted products offer a much more realistic characterization in this regard [3, 9, 19]. The most important reason for this, twisted products are introduced as a generalization of warped products and their twisting functions are defined on the points of both base and fiber. It was proved [20] that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two canonical foliations of the product manifold $(M = M_1 \times M_2, g)$, whose leaves intersect perpendicularly, the leaves of $\mathcal{D}_1$ are totally geodesic and the leaves of $\mathcal{D}_2$ are totally umbilic, then (M, g) is isometric to a twisted product $M_1 \times_f M_2$. In [14], it was proved that a mixed Ricci-flat twisted product semi-Riemannian manifold (i.e., the product manifold satisfying $\mathrm{Ric}(X, V) = 0$, for all $X \in \chi(M_1)$ and $V \in \chi(M_2)$) can be expressed as a warped product. Very recently, Chen proved that the necessary and sufficient condition for the existence of the torqued vector field on an m-dimensional semi-Riemannian manifold M is that the manifold M can be expressed as a twisted product $I \times_f M^*$, where M^* is an $(m-1)$-dimensional manifold, [8]. Moreover, in [18], the Ricci tensor of the twisted product Robertson-Walker and generalized Robertson-Walker spacetimes were characterized in terms of the stress-energy tensor of an imperfect fluid. For more, see [7, 17, 16] In the present paper, we study the harmonicity of the functions and arbitrary p-forms on the twisted products by proving the existence and obstructions on curvature. The next purpose of this paper is to study and explore some characteristics of static perfect fluid spacetimes on twisted product manifolds. Static perfect fluids spacetimes are special global solutions of Einstein's equations that show the relationship between matter content and spacetime in general relativity, [10, 11, 21].

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2. Preliminaries

Throughout the paper, all geometric objects of this paper are assumed to be smooth and connected. In this section, we give the notation and basic formulas for the Levi-Civita connection, the Riemannian curvature tensor and the Ricci tensor of the twisted products that will be used in the proofs of our main results.

Let (M_1, g_1) and (M_2, g_2) be two semi-Riemannian manifolds of dimensions m_i, $i = 1, 2$. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the canonical projections. Then twisted product manifold $(M_1 \times_f M_2, g)$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric

$$g = \pi^* g_1 + f^2 \sigma^* g_2 \tag{2.1}$$

where $f : M_1 \times M_2 \to (0, \infty)$ is called the twisting function. If f only depends on the points of M_1, then $M_1 \times_f M_2$ reduces to a warped product.

Notation 2.1. For the sake of simplicity, from now on, all relations will be written, without involving the projection maps from $M_1 \times M_2$ to each component M_1 and M_2 as in $g = g_1 \oplus f^2 g_2$. Moreover, all objects such as Levi-Civita connection ${}^i\nabla$, Riemannian curvature tensor R_i, Ricci tensor Ric_i etc. having the indices or powers i denote the objects of the manifold (M_i, g_i), where $i = 1, 2$. Also, all non-index and non-power objects are considered to belong to the twisted product manifold.

Let $\mathfrak{L}(M_1)$ and $\mathfrak{L}(M_2)$ be the set of lifts of vector fields on M_1 and M_2 to $M_1 \times M_2$ respectively, and we denote the same symbol for the vector fields and their lifts. Also, let $k = \ln f$. By using the similar computations of [19, p. 206–211], we find the components of the Levi-Civita connection, Riemannian tensor, Ricci tensor and the scalar curvature of the twisted product $(M_1 \times_f M_2, g)$. We skip the proofs that are long but straightforward, as is the case of warped product manifolds.

Lemma 2.1. *[15] Let $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Then:*

(1) $\nabla_X Y = {}^1\nabla_X Y$,

(2) $\nabla_X V = \nabla_V X = X(k)V$,

(3) $\nabla_U V = {}^2\nabla_U V + U(k)V + V(k)U - g(U, V)\nabla k$.

Now, for a smooth function φ on a twisted product $(M = M_1 \times_{f_1} M_2, g)$, we define, $h_1^\varphi(X, Y) = XY(\varphi) - ({}^1\nabla_X Y)(\varphi)$ for all $X, Y \in \mathfrak{L}(M_1)$ and $h_2^\varphi(U, V) = UV(\varphi) - ({}^2\nabla_U V)(\varphi)$ for all $U, V \in \mathfrak{L}(M_2)$. Then using Lemma 1, the Hessian tensor h^φ of φ on a twisted product $(M = M_1 \times_{f_1} M_2, g)$ satisfies [15]

$$h^\varphi(X, Y) = h_1^\varphi(X, Y), \tag{2.2}$$

$$h^\varphi(U, V) = h_2^\varphi(U, V) - U(k)V(\varphi) - V(k)U(\varphi) + g(U, V)g(\nabla k, \nabla\varphi) \tag{2.3}$$

$$h^\varphi(X, U) = -X(k)U(\varphi). \tag{2.4}$$

The following formula is also useful.

$$h^k = h^{\ln f} = \frac{1}{f} h^f - \frac{1}{f^2} df \otimes df. \tag{2.5}$$

Let R_1 and R_2 be the lifts of curvature tensors of (M_1, g_1) and (M_2, g_2), respectively and R be the curvature tensor of the twisted product $(M_1 \times_f M_2, g)$. Then by Lemma 2.1 and equations (2.2) – (2.4), we have the following relations:

Lemma 2.2. *[15] Let $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$. Then:*

$$R(X, Y)Z = R_1(X, Y)Z, \tag{2.6}$$

$$R(X, Y)U = 0, \tag{2.7}$$

$$R(X, V)Y = [h_1^k(X, Y) + X(k)Y(k)]V, \tag{2.8}$$

$$R(V, W)X = VX(k)W - WX(k)V, \tag{2.9}$$

$$R(X, V)W = XW(k)V - X(k)g(V, W)\nabla k - g(V, W)\mathrm{H}^k(X), \tag{2.10}$$

$$\begin{aligned} R(U, V)W = {} & R_2(V, W)U + \mathrm{h}_2^k(V, U)W - \mathrm{h}_2^k(W, U)V \\ & + W(k)U(k)V - V(k)U(k)W - g(U, W)V(k)\nabla k \\ & + g(U, V)W(k)\nabla k + g(U, V)\mathrm{H}^k(W) - g(U, W)\mathrm{H}^k(V), \end{aligned} \tag{2.11}$$

where $\mathrm{h}^k(\cdot, \cdot) = g(\mathrm{H}^k(\cdot), \cdot)$.

Now, let Ric_1 and Ric_2 be the lifts of Ricci tensors of (M_1, g_1) and (M_2, g_2), respectively and Ric be the Ricci tensor of the twisted product $(M = M_1 \times_f M_2, g)$. Then using the equations (2.2) – (2.11), we have the following relations:

Lemma 2.3. *[15] Let $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Then the components of the Ricci tensor of the twisted product $(M = M_1 \times_f M_2, g)$ are:*

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_1(X, Y) - \frac{m_2}{f} h_1^f(X, Y)\,, \tag{2.12}$$

$$\mathrm{Ric}(X, V) = (1 - m_2) XV(k) = 0\,, \tag{2.13}$$

$$\begin{aligned}\mathrm{Ric}(U, V) = &\ \mathrm{Ric}_2(U, V) + \frac{(2 - m_2)}{f} h_2^f(U, V) \\ &+ \frac{2(m_2 - 2)}{f^2}(df \otimes df)(U, V) - \left(\frac{1}{f}\Delta f - \frac{1}{f^2} g(\nabla f, \nabla f)\right) g(U, V)\,,\end{aligned} \tag{2.14}$$

where Δ denotes the Laplacian on M and $m_i = \dim(M_i)$.

3. Harmonicity of Functions on Twisted Products

In this section, we deal with harmonicity on twisted product manifolds. We recall that any smooth function φ on a Riemannian manifold (M, g) is harmonic if its Laplacian $\Delta\varphi$ vanishes identically. If ∇ denotes the Levi-Civita connection of g and $\{E_j\}_j$ is an orthonormal frame on M, then

$$\Delta\varphi = trace(Hess\varphi) = trace(\nabla d\varphi) = E_j E_j(\varphi) - (\nabla_{E_j} E_j)\varphi, \tag{3.1}$$

where $Hess\varphi(X, Y) = g(\nabla_X \nabla\varphi, Y)$ denotes the Hessian of φ, for all $X, Y \in \chi(M)$. Equivalently, $\Delta\varphi = div(grad\varphi)$. The divergence of a vector field $X \in \chi(M)$ is given by $divX = g(\bar{\nabla}_{E_j} X, E_j)$ and the gradient of φ is defined by $grad\varphi = \nabla\varphi = (d\varphi)^\sharp$, where $\sharp$ denotes the musical isomorphism with respect to g.

Also, note that for any function φ, the following relation holds:

$$\frac{m}{\varphi}\mathrm{H}^\varphi = \mathrm{H}^{m\ln\varphi} + \frac{1}{m}\mathrm{d}(m\ln\varphi) \otimes \mathrm{d}(m\ln\varphi), \quad m \in \mathbb{R}. \tag{3.2}$$

For the next statement, we need the following:

Remark 3.1. If $\{e_i\}_{i=1}^{m_1}$ and $\{u_l\}_{l_1}^{m_2}$ denote the orthonormal frames on the manifolds M_i with respect to g_i $(i = 1, 2)$ respectively, then $\{e_i\}_{i=1}^{m_1}$ and $\{\frac{u_l}{f}\}_{l_1}^{m_2}$ are respectively orthonormal frames on M_i $(i = 1, 2)$ all with respect to g. Therefore, $\{E_j\}_{j_1}^{n} = \{e_i\}_{i=1}^{m_1} \bigcup \{\frac{u_l}{f}\}_{l_1}^{m_2}$ is an orthonormal frame on the twisted product manifold $M = M_1 \times_f M_2$ with respect to g.

Now, we deal with the Laplacian, which is one of the most used operator in both PDE and Differential Geometry.

Proposition 3.1. *Let $(M = M_1 \times_f M_2, g)$ be a twisted product manifold. Then for any smooth function φ on M, one has:*

$$\Delta\varphi = \Delta_1\varphi + \frac{1}{f^2}\Delta_2\varphi + \left(m_2 - \frac{2}{f^2}\right) g(\nabla k, \nabla\varphi). \tag{3.3}$$

Proof. Remark 3.1 and (3.1) yield

$$\begin{aligned}\Delta\varphi = &E_j E_j(\varphi) - (\nabla_{E_j} E_j)\varphi \\ &= [e_i e_i(\varphi) - (\nabla_{e_i} e_i)\varphi] + \frac{1}{f^2}[u_l u_l(\varphi) - (\nabla_{u_l} u_l)\varphi].\end{aligned} \tag{3.4}$$

From Lemma 2.1, we obtain

$$\Delta\varphi = \Delta_1\varphi + \frac{1}{f^2}[u_l u_l(\varphi) - (\nabla^2_{u_l} u_l)\varphi] - \frac{2}{f^2} u_l(k) u_l(\varphi) + m_2 \nabla k(\varphi).$$

which gives (3.3) by straightforward computation, that complete the proof. □

As consequences of (3.3), we obtain the following:

Proposition 3.2. *If $(M = M_1 \times_f M_2, g)$ is a twisted product manifold, then for any smooth function φ_1 on M_1, one has:*

(i) φ_1 is harmonic on (M, g) if and only if $\Delta_1 \varphi_1 = \left(m_2 - \frac{2}{f^2}\right) g(\nabla k, \nabla \varphi)$;
(ii) Any two of the following assertions imply the third one:
(a) φ_1 is harmonic on (M, g); (b) φ_1 is harmonic on (M_1, g_1);
(c) ∇k is orthogonal to $\nabla \varphi_1$.

Remark 3.2. Note that if for any smooth function φ_1 on M_1, ∇k is orthogonal to $\nabla \varphi_1$, then by Lemma 2.1, $k \in C^\infty(M_2)$. Thus the metric g becomes $g = g_1 \oplus \tilde{g}_2$, where $\tilde{g}_2$ is a conformal metric $f^2 g_2$ on M_2. Therefore the twisted product M reduces to the direct product of (M_1, g_1) and $(M_2, \tilde{g}_2)$.

Proposition 3.3. *If $(M = M_1 \times_f M_2, g)$ is a twisted product manifold, then for any smooth function φ_2 on M_2, one has:*

(i) φ_2 is harmonic on (M, g) if and only if $\Delta_2 \varphi_2 = (2 - m_2 f^2) g(\nabla k, \nabla \varphi_2)$.
(ii) Any two of the following assertions imply the third one:
(a) φ_2 is harmonic on (M, g); (b) φ_2 is harmonic on (M_2, g_2);
(c) ∇k is orthogonal to $\nabla \varphi_2$.

Remark 3.3. Note that if for any smooth function φ_2 on M_2, ∇k is orthogonal to $\nabla \varphi_1$, then by Lemma 2.1, $k \in C^\infty(M_1)$. Therefore the twisted product M reduces to the warped product of (M_1, g_1) and (M_2, g_2).

Now, we study harmonic forms on a twisted product manifold $(M = M_1 \times_f M_2, g)$ for which we recall the following:

Definition 3.1. On a Riemannian manifold (M, g), we say that a p-form $\omega \in \mathcal{A}_p(M)$ is co-closed if its co-differential $\delta^g \omega$ given by

$$\delta^g \omega(X_1, \ldots, X_{p-1}) = (\nabla . \omega)(\cdot, X_1, \ldots, X_{p-1}), \quad \forall X_1, \ldots, X_{p-1} \in \chi(M) \tag{3.5}$$

vanishes identically. Moreover, a p-form on (M, g) is harmonic if it is both closed and co-closed.

Proposition 3.4. *The co-differential operator δ^g of the twisted product manifold $(M = M_1 \times_f M_2, g)$ is related to the co-differential operators δ^{g_j} of (M_j, g_j), $j = 1, 2$, by:*

$$\delta^g \omega = \delta^{g_1} \omega + \frac{1}{f^2} \delta^{g_2} \omega + \left(m_2 - \frac{2}{f^2}\right) w(\nabla k); \quad \forall \omega \in \mathcal{A}_1(M), \tag{3.6}$$

Proof. With the notations of Remark 3.1, we express the co-differential operator δ^g by

$$\begin{aligned} \delta^g \omega =& (\nabla_{E_j} \omega)(E_j) = E_j(\omega(E_j)) - \omega(\nabla_{E_j} E_j) \\ =& e_i(\omega(e_i)) - \omega(\nabla_{e_i} e_i) + \frac{u_l}{f}(\omega(\frac{u_l}{f})) - \omega(\nabla_{\frac{u_l}{f}} \frac{u_l}{f}). \end{aligned} \tag{3.7}$$

By using Lemma 2.1 in (3.7), we complete the proof. □

As a consequence, we obtain the following results:

Theorem 3.1. *Let $(M = M_1 \times_f M_2, g)$ be a twisted product manifold. For each $\omega \in \mathcal{A}_1(M)$, any of the following three assertions imply the fourth one:*

(i) ω is co-closed with respect to g;
(ii) ω restricted to M_1 is co-closed with respect to g_1;
(iii) ω restricted to M_2 is co-closed with respect to g_2;
(iv) $\omega(\nabla k) = 0$ holds.

Corollary 3.1. *Let $(M = M_1 \times_f M_2, g)$ be a twisted product manifold. For each $\omega \in \mathcal{A}_1(M)$, any of the following three assertions imply the fourth one:*

(i) ω is harmonic with respect to g;
(ii) ω restricted to M_1 is harmonic with respect to g_1;
(iii) ω restricted to M_2 is harmonic with respect to g_2;
(iv) ω is closed and $\omega(\nabla k) = 0$ holds.

Remark 3.4. The above Proposition 3.4, Theorem 3.1 and Corollary 3.1 can be generalized to arbitrary p-forms.

4. Sectional Curvature of Twisted Products

By using Remark 3.1, we express the sectional curvature $K(\sigma) = K(E,F) = \bar{g}(\bar{R}(E,F)F,E)$ of any 2-plane σ spanned by a basis $\{E,F\}$, orthonormal with respect to g. From a long calculation, we obtain:

Theorem 4.1. *For any 2-plane σ, tangent to the twisted product manifold $(M = M_1 \times_f M_2, g)$, the sectional curvature K of the metric g can be calculated from the following three cases, involving the sectional curvatures K_j of g_j, $j = 1,2$:*

(i) If σ is tangent to M_1, then

$$K(\sigma) = K_1(\sigma); \tag{4.1}$$

(ii) If σ is tangent to M_2 spanned by the vector fields $U,V \in \chi(M_2)$, then

$$K(\sigma) = \frac{1}{f^2}[K_2(\sigma) - h_2^k(U,U) - h_2^k(V,V) + U(k)^2 + V(k)^2 - ||\nabla f||^2]; \tag{4.2}$$

(iii) If σ is spanned by arbitrary vector fields $A \in \chi(M_1)$ and $U \in \chi(M_2)$, then

$$K(\sigma) = -\frac{1}{f^2}[A(k)^2 + h_1^f(A,A)]; \tag{4.3}$$

Proof. By Lemma 2.2 and Remark 3.1, we have the following three cases:

(i) For any 2-plane spanned by arbitrary unit vector fields $X,Y \in \chi(M_1)$, $K(X,Y) = g(R(X,Y)Y,X) = K_1(X,Y)$.

(ii) For any 2-plane spanned by arbitrary unit vector fields $\mathcal{U},\mathcal{V} \in \chi(M_2)$,

$$\begin{aligned} K(\mathcal{U},\mathcal{V}) =& g(R(\mathcal{U},\mathcal{V})\mathcal{V},\mathcal{U}) = \frac{1}{f^4}g(R(U,V)V,U) \\ =& \frac{1}{f^2}[R_2(U,V,V,U) - h_2^k(V,V) + V(k)^{-}U(k)^2 - h^k(U,U)], \end{aligned} \tag{4.4}$$

where $\mathcal{U} = \frac{U}{f}$ and $\mathcal{V} = \frac{V}{f}$. Using (2.1) and Lemma 2.1, the last equation yields (4.2).

(iii) At last, for any unit vector field $X \in \chi(M_1)$ and $\mathcal{U} \in \chi(M_2)$, we have $K(X,\mathcal{U}) = g(R(X,\mathcal{U})\mathcal{U},X) = \frac{1}{f^2}g(R(X,U)U,X)$ yields (4.3).

□

As a consequence of Theorem 4.1, we obtain the following:

Corollary 4.1. *For a twisted product manifold $(M = M_1 \times_f M_2, g)$:*

(i) If the sectional curvature of M is of constant sign, then K_1 is of constant sign on M_1;
(ii) If the sectional curvature of M is positive, then $Hess_1 f(A,A)$ is always negative, for all unit vector field $A \in \chi(M_1)$.

5. Static Perfect Fluid Spacetimes on Twisted Products

Let (M,g) be an n-dimensional Riemannian manifold. Then (M,g) is said to be a static perfect fluid space if it admits a nontrivial solution φ of the static equation

$$\varphi Ric - h^{\varphi} = \frac{1}{n}(\tau\varphi - \Delta\varphi)g, \tag{5.1}$$

where Ric is the Ricci tensor, τ is the scalar curvature, h is the Hessian tensor and Δ is the Laplacian operator [11, 21]. The concept of static perfect fluid space plays an important role in both general relativity and differential geometry.

If for some smooth function φ

$$\mathrm{Ric} + \mathrm{h}^{\varphi} = \lambda g \tag{5.2}$$

holds, then the triple (M,g,φ,λ) satisfying (5.2) is called a gradient Ricci soliton. Here λ is usually a real constant. But, sometimes λ can be smooth function on M and in this case, (M,g,φ,λ) is called a gradient almost Ricci soliton, [6] There are also several related notions, such as almost η-Ricci and almost η-Yamabe solitons [4, 5]: The manifold satisfying the condition

$$\mathrm{Ric} + \mathrm{h}^{\varphi} = \gamma g + \mu \eta \otimes \eta, \tag{5.3}$$

for some smooth functions φ, γ and μ and the corresponding non-zero 1-form η, is called the gradient almost η-Ricci soliton.

Lemma 5.1. *Let $(M = M_1 \times_f M_2, g)$ be a non-trivial twisted product manifold. If (M, g) is a static perfect fluid spacetime with the potential function φ, then we have*

$$d\varphi(V) = 0, \tag{5.4}$$

for any $V \in \mathfrak{L}(M_2)$.

Proof. For the twisted product $(M_1 \times_f M_2, g)$, we have $XV(k) = 0$. From (2.4), (2.13) and (5.1), for any $X \in \mathfrak{L}(M_1)$ and $V \in \mathfrak{L}(M_2)$, $X(k)V(\varphi) = 0$. Hence, either $X(k) = 0$ or $V(\varphi) = 0$. In the first case, the function f only depends on the points of M_2. Hence, we can write $g = g_1 \oplus \tilde{g}_2$, where $\tilde{g}_2 = f^2 g_2$. Namely, $M_1 \times_f M_2$ can be expressed as a usual product $M_1 \times M_2$, where the metric tensor of M_2 is $\tilde{g}_2$ given above. In the second case, we immediately get (5.4). □

Theorem 5.1. *Let $(M = M_1 \times_f M_2, g)$ be a non-trivial twisted product manifold. Then (M, g) is a static perfect fluid spacetime with the potential function φ if and only if the following conditions hold:*

1. *the relation given below holds on (M_1, g_1):*

$$\mathrm{Ric}_1 + \mathrm{h}_1^{-m_2 \ln f - \ln \varphi} - m_2 d(\ln f) \otimes d(\ln f) - d(\ln \varphi) \otimes d(\ln \varphi) = \frac{1}{n}(\tau - \frac{\Delta\varphi}{\varphi}) g_1. \tag{5.5}$$

2. *(M_2, g_2) is the gradient almost η-Ricci soliton with the potential function is $\tilde{\varphi} = (2 - m_2)k - \ln \varphi$, associated 1-form $\eta = dk$ and the associated soliton functions $\mu = m_2 - 2$ and $\lambda = f\Delta f - ||\nabla f||^2 + f^2 g(\nabla k, \nabla \ln \varphi) + \frac{f^2}{n}(\tau - \frac{\Delta\varphi}{\varphi})$.*

Proof. If $(M = M_1 \times_f M_2, g)$ is a non-trivial twisted product manifold admitting a static perfect fluid spacetime structure with potential function φ, then by using Lemma 2.1-(1), (2.5) and (2.12) into the fundamental equation (5.1) of static perfect fluid spacetime, we get the relation (5.9) on (M_1, g_1). This gives the assertion (1).

Similarly, using Lemma 2.1-(2) and (2.13) into the fundamental equation (5.1) of static perfect fluid spacetime, we get

$$\begin{aligned} \varphi\Big[\mathrm{Ric}_2(U,V) + \frac{(2-m_2)}{f} h_2^f(U,V) + \frac{2(m_2-2)}{f^2}(df \otimes df)(U,V) \\ -\Big(\frac{1}{f}\Delta f - \frac{1}{f^2} g(\nabla f, \nabla f)\Big) g(U,V)\Big] - h_2^{\varphi}(U,V) + U(k)V(\varphi) + V(k)U(\varphi) \\ -g(\nabla k, \nabla \varphi) g(U,V) = \frac{f^2}{n}(\tau\varphi - \Delta\varphi) g_2(U,V). \end{aligned} \tag{5.6}$$

Using (5.4) and (2.5) into (5.6) and dividing both sides by $\varphi \neq 0$, we get

$$\mathrm{Ric}_2 + h_2^{\tilde{\varphi}} + \mu dk \otimes dk = \lambda g_2, \tag{5.7}$$

where $\tilde{\varphi} = (2 - m_2)k - \ln \varphi$, $\eta = dk$ and $\mu = m_2 - 2$ and $\lambda = f\Delta f - ||\nabla f||^2 + f^2 g(\nabla k, \nabla \ln \varphi) + \frac{f^2}{n}(\tau - \frac{\Delta\varphi}{\varphi})$. □

If $\Delta\varphi = -\frac{\tau}{n-1}\varphi$ holds in (5.1), then it turns into

$$\varphi Ric - h^{\varphi} = -(\Delta\varphi) g, \tag{5.8}$$

which is the equation of vacuum static space. Therefore, static perfect fluid spacetimes behave like a generalization of static vacuum spaces which are an important subject of study in both differential geometry and general relativity, [13]. From Theorem 5.1, we finally have:

Theorem 5.2. *Let $(M = M_1 \times_f M_2, g)$ be a non-trivial twisted product manifold. Then (M, g) is a vacuum static spacetime with the potential function φ if and only if the following conditions hold:*

1. *the relation given below holds on (M_1, g_1):*

$$\mathrm{Ric}_1 + \mathrm{h}_1^{-m_2 \ln f - \ln \varphi} - m_2 d(\ln f) \otimes d(\ln f) - d(\ln \varphi) \otimes d(\ln \varphi) = \frac{1}{n-1} g. \tag{5.9}$$

2. *(M_2, g_2) is the gradient almost η-Ricci soliton with the potential function is $\tilde{\varphi} = (2 - m_2)k - \ln \varphi$, associated 1-form $\eta = dk$ and the associated soliton functions $\mu = m_2 - 2$ and $\lambda = f\Delta f - ||\nabla f||^2 + f^2 g(\nabla k, \nabla \ln \varphi) + \frac{f^2 \tau}{n-1}$.*

5.1. Static Spacetimes with Twisted GRW Metric

In this section, we give some applications of twisted product manifolds.
An m-dimensional product manifold $M = I \times_f N$ equipped with the metric tensor

$$g = -\mathrm{d}t^2 \oplus f^2 g_N$$

is called a *generalized Robertson-Walker spacetime* (briely GRW), where I is an open interval in $\mathbb{R}$, $\mathrm{d}t^2$ is the usual Euclidean metric tensor on I and (N, g_N) be a Riemannian manifold and f is a positive smooth function on I. This notion has been studied by many authors, such as [7, 17, 16, 22].

In [15], the authors generalized this notion by defining the relevant function f on the whole manifold $M = I \times_f N$ and then give basic geometric formulas of this new spacetime, namely we call it as *twisted generalized Robertson-Walker spacetime* (briefly say TGRW).

Let's consider the Lorentzian manifold $M = I \times_f M_2$ endowed with the Lorentzian metric

$$g = -\mathrm{d}t^2 \oplus f^2 g_2, \tag{5.10}$$

where I is a real open interval and f is a positive smooth function on M. Then $(M = I \times_f M_2, g)$ is called the twisted generalized Robertson-Walker spacetime (TGRW). Also, let denote the standard vector field on I by ∂_t. Then we can directly obtain the following lemmas, which are the direct applications of Lemma 2.1 and Lemma 2.2.

Lemma 5.2. *[15] Let $U, V \in \mathfrak{L}(M_2)$. Then the components of the Levi-Civita connection of TGRW $(M = I \times_f M_2, g)$ are:*

(1) $\nabla_{\partial_t} \partial_t = 0$,

(2) $\nabla_{\partial_t} V = \nabla_V \partial_t = k' V$,

(3) $\nabla_U V = {}^2\nabla_U V + U(k)V + V(k)U - g(U, V)\nabla k$.

Lemma 5.3. *[15] Let $U, V, W \in \mathfrak{L}(M_2)$. Then, the non-zero components of the Riemannian curvature tensor of TGRW $(M = I \times_f M_2, g)$ are given by:*

$$R(V, \partial_t)\partial_t = -[k'' + (k')^2]V, \tag{5.11}$$

$$R(V, W)\partial_t = V(k')W - W(k')V, \tag{5.12}$$

$$R(\partial_t, V)W = \partial_t(W(k))V - k' g(V, W)\nabla k - g(V, W)\mathrm{H}^k(\partial_t), \tag{5.13}$$

$$\begin{aligned} R(V, W)U &= R_2(V, W)U + \mathrm{h}_2^k(V, U)W - \mathrm{h}_2^k(W, U)V \\ &+ W(k)U(k)V - V(k)U(k)W - g(U, W)V(k)\nabla k + g(U, V)W(k)\nabla k \\ &+ g(U, V)\mathrm{H}^k(W) - g(U, W)\mathrm{H}^k(V), \end{aligned} \tag{5.14}$$

where $\mathrm{h}^k(\cdot, \cdot) = g(\mathrm{H}^k(\cdot), \cdot)$.

Then by simple calculations, we have the following:

Lemma 5.4. *Let $U, V, W \in \mathfrak{L}(M_2)$. Then, the non-zero components of the Ricci tensor of TGRW $(M = I \times_f M_2, g)$ are given by:*

$$\begin{aligned} &\mathrm{Ric}(\partial_t, \partial_t) = -m_2[k'' + (k')^2], \\ &\mathrm{Ric}(V, \partial_t) = (1 - m_2)V(k') = 0, \\ &\mathrm{Ric}(U, V) = \mathrm{Ric}_2(U, V) + (2 - m_2)h_2^k(U, V) - (2 - m_2)(dk \otimes dk)(U, V) - \Delta_2 k g(U, V), \end{aligned} \tag{5.15}$$

where Δ_2 denotes the Laplacian on M_2 and $m_i = \dim(M_i)$.

Then using Lemma 5.2, the Hessian tensor h^φ of φ on a TGRW $(M = I \times_f M_2, g)$ satisfies

$$h^\varphi(\partial_t, \partial_t) = \varphi'', \tag{5.16}$$

$$\begin{aligned} h^\varphi(U, V) &= h_2^\varphi(U, V) - U(k)V(\varphi) - V(k)U(\varphi) \\ &+ g(U, V)g(\nabla k, \nabla \varphi) \end{aligned} \tag{5.17}$$

$$h^\varphi(\partial_t, V) = -k'V(\varphi). \tag{5.18}$$

Theorem 5.3. *Let $(M = I \times_f M_2, g)$ be a non-trivial TGRW. Then (M, g) is a static perfect fluid spacetime with the potential function φ if and only if the following conditions hold:*

1. *the following second order ordinary differential equation between the potential function φ and the warping function k holds:*
$$\varphi'' + m_2[k'' + (k')^2] = \frac{1}{n}(\tau - \frac{\Delta\varphi}{\varphi}). \tag{5.19}$$
2. $d\varphi(V) = 0$*, for any $V \in \chi(M_2)$.*
3. (M_2, g_2) *is the gradient almost η-Ricci soliton with the potential function is $\tilde{\varphi} = (2 - m_2)k - \ln\varphi$, associated 1-form $\eta = dk$ and the associated soliton functions $\mu = m_2 - 2$ and $\lambda = f\Delta f - ||\nabla f||^2 + f^2 g(\nabla k, \nabla \ln\varphi) + \frac{f^2}{n}(\tau - \frac{\Delta\varphi}{\varphi})$.*

The proof of the above theorem is completely based on the proof of the Lemma (5.1) and Theorem (5.1). Thus we may skip.

5.2. Static Spacetimes on Twisted SSST Metric

Now, we recall the definition of standard static spacetimes. Let (F, g_F) be an s-dimensional Riemannian manifold and $f : F \to (0, \infty)$ be a smooth function. The $(s + 1)$-dimensional product manifold ${}_fI \times F$ endowed with the metric tensor

$$g = -f^2 \mathrm{d}t^2 \oplus g_F$$

is called a standard static spacetime (briefly SSS-T) and is denoted by $M =_f I \times F$ where I is an open, connected subinterval of $\mathbb{R}$ and $\mathrm{d}t^2$ is the Euclidean metric tensor on I.

Standard static spacetime metrics play very important roles to find the solutions of the Einstein's field equations so that they have been studied intensively for many years. Some famous examples of standard static spacetimes are the Minkowski spacetime, the Einstein's static universe, the universal covering space of anti-de Sitter spacetime and the Exterior Schwarzschild spacetime (for more details see [2, 1, 12]).

As a second application of twisted product, in [15], the auhors extend this notion by redefining the relevant function f on the whole manifold $M =_f I \times F$ and then give basic geometric formulas on this new product and we call it as *twisted standard static spacetime* (briefly say TSSS-T).

We consider a semi-Riemannian manifold $M =_f I \times M_2$ endowed with the Lorentzian metric

$$g = -f^2 \mathrm{d}t^2 \oplus g_2, \tag{5.20}$$

where I is a real open interval and f is a positive smooth function on M. Then $(M =_f I \times M_2, g)$ is called the twisted standard static spacetime (TSSS-T). Again by taking the standard vector field on I by ∂_t, we can directly obtain the following lemmas, which are the direct applications of the previous section.

Lemma 5.5. *[15] Let $U, V \in \mathfrak{L}(M_2)$. Then the components of the Levi-Civita connection of TSSS-T $(M =_f I \times M_2, g)$ are:*

(1) $\nabla_{\partial_t}\partial_t = 2k'\partial_t + f\nabla f$,

(2) $\nabla_{\partial_t}V = \nabla_V\partial_t = V(k)\partial_t$,

(3) $\nabla_U V = {}^2\nabla_U V$.

Lemma 5.6. *[15] Let $U, V, W \in \mathfrak{L}(M_2)$. Then, the non-zero components of the Riemannian curvature tensor of TSSS-T $(M =_f I \times M_2, g)$ are given by:*

$$R(V, \partial_t)\partial_t = V(k')\partial_t + V(f)\nabla f + f^2\nabla_V\nabla k, \tag{5.21}$$

$$R(V, \partial_t)W = [\mathrm{h}_2^k(V, W) + V(k)W(k)]\partial_t, \tag{5.22}$$

$$R(V, W)U = R_2(V, W)U. \tag{5.23}$$

Then again by simple calculations, we have the following:

Lemma 5.7. *Let $U, V, W \in \mathfrak{L}(M_2)$. Then, the non-zero components of the Ricci tensor of TSSS-T $(M =_f I \times M_2, g)$ are given by:*

$$\begin{aligned} &\mathrm{Ric}(\partial_t, \partial_t) = f\Delta_2 f, \\ &\mathrm{Ric}(V, \partial_t) = 0, \\ &\mathrm{Ric}(U, V) = \mathrm{Ric}_2(U, V) - \frac{\mathrm{h}_2^f(U, V)}{f}, \end{aligned} \tag{5.24}$$

where Δ_2 denotes the Laplacian on M_2.

Then using Lemma 5.5, the Hessian tensor h^φ of φ on a TSSS-T $(M = I \times_f M_2, g)$ satisfies

$$h^\varphi(\partial_t, \partial_t) = \varphi'' - 2k'\varphi' - fg(\nabla f, \nabla\varphi), \tag{5.25}$$
$$h^\varphi(U, V) = h_2^\varphi(U, V) \tag{5.26}$$
$$h^\varphi(\partial_t, V) = -V(k)\varphi'. \tag{5.27}$$

Theorem 5.4. *Let $(M =_f I \times M_2, g)$ be a non-trivial TSSS-T. Then (M, g) is a static perfect fluid spacetime with the potential function φ if and only if the following conditions hold:*

1. *the potential function is of the form $\varphi = \alpha + \tilde{\varphi}$, where $\alpha \in \mathbb{R}$ and $\tilde{\varphi} \in C^\infty(M_2)$.*
2. *(M_2, g_2) is also a static perfect fluid spacetime with the same potential function φ.*
3. *The twisted function f and the potential function φ is related by:*

$$\varphi f\Delta_2 + fg(\nabla f, \nabla\varphi) = -\frac{f^2}{n}(\tau\varphi - \Delta\varphi). \tag{5.28}$$

Proof. To prove the necessary condition, first, we use (5.24) and (5.27) for any $V \in \mathfrak{L}(M_2)$ and get $V(k)\varphi' = 0$. Since TSSS-T is non-trivial, we may assume that $V(k) \neq 0$. Otherwise, the metric reduces to the direct product. Thus, $\varphi' = 0$ that implies the assertion (1). By the TSSS-T metric, (5.24) and (5.26), the second assertion is obvious. Finally, by using (5.24) and (5.25) into the fundamental equation of the static perfect fluid, the relation (5.28) is obtained. The sufficient condition can be directly verified. □

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