http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 410–419 (2024)
DOI:10.31801/cfsuasmas.1286719
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:April 25, 2023; Accepted: February 13, 2024

INTUITIONISTIC FINE SPACE

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Abstract. In the exploration of intuitionistic fine spaces, this article introduces a novel concept known as intuitionistic fine open sets (I_fOS). Delving into the properties of these sets, the study analyzes both intuitionistic fine open and closed sets within the context of intuitionistic fine spaces. The article establishes fundamental definitions, accompanied by illustrative real time example, to provide a comprehensive understanding of the newly introduced sets. Furthermore, the exploration extends to defining and examining key concepts such as intuitionistic fine continuity, intuitionistic fine irresoluteness, and intuitionistic fine irresolute homeomorphism. This progression aims to contribute to the broader comprehension and application of intuitionistic fine spaces in topological contexts.

1. Introduction

Intuitionistic topology (IT), fuzzy and intuitinistic fuzzy topology [2, 3] plays a vital role in applied sciences such as pattern recognition, optimization technique, medical diagnosis, decision-making etc., [1, 8, 12, 17, 18] creates interest in introducing the new set, intuitionistic fine open set (I_fOS) in this article. The classical version of intuitionistic sets, as proposed by Coker [2, 3], serves as a foundational framework for understanding certain topological structures. Additionally, the concept of fine sets, pioneered by P.L. Powar and K. Rajak [14–16], adds further depth to the study of these sets. The linkage between fine sets and the newly introduced concept in this article, namely intuitionistic fine open sets, lies in the amalgamation of these two theoretical underpinnings.

2020 *Mathematics Subject Classification.* 54A99.

Keywords. I_fOS, intuitionistic fine continuous, intuitionistic fine irresolute, intuitionistic fine irresolute homeomorphism.

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The introduction of intuitionistic fine open sets can be seen as an evolution or extension of these ideas. In essence, the concept of intuitionistic fine open sets builds upon the foundational notions of intuitionistic sets and fine sets, tailoring them to address and explore more specialized properties or relations. The objective of this article is to introduce a new set of concepts termed intuitionistic fine open sets (I_fOS) in intuitionistic fine spaces. As the discourse advances, an in-depth examination of the characteristics of intuitionistic fine open and closed sets in such spaces is undertaken. The foundational definitions are meticulously laid out, accompanied by essential real time example to elucidate the nuances of these sets.

Moreover, the article delves into the exploration and definition of intuitionistic fine continuity, intuitionistic fine irresoluteness, and intuitionistic fine irresolute homeomorphism, thus adding layers of understanding to the intricacies of intuitionistic fine spaces. In particular, there exists applications in image processing predominantly leverage fuzzy topology [12, 17] and intuitionistic fuzzy topology [1, 8, 19], this article motivates us to anticipate and forecast the potential applications of intuitionistic fine space within the same domain. The focus lies on extrapolating the applications based on the unique characteristics and features inherent to intuitionistic fine space. The aim is to project how the distinct attributes of intuitionistic fine space can contribute to and enhance various aspects, thereby expanding the scope and utility of this topological framework in practical applications.

2. Preliminaries

Definition 1. *[1] Suppose X be a non-empty set, an intuitionistic set (IS) C is an element of form $C = \langle X, C_1, C_2\rangle$, C_1 and C_2 are subsets of X holding $C_1 \cap C_2 = \phi$. C_1 is known as the set of members of C, and C_2 is known as the set of non-members of C.*

Definition 2. *[1] An IT on a non-empty set X is a family τ of ISs in X holding:*
(i) $\underset{\sim}{X}, \underset{\sim}{\phi} \in \tau$.
(ii) $C_1 \cap C_2 \in \tau$ *for any* $C_1, C_2 \in \tau$.
(iii) $\cup C_i \in \tau$ *for arbitrary family* $\{C_i : i \in L\} \subseteq \tau$
(X,τ) is called intuitionistic topological space (ITS) and IS in τ is called an intuitionistic open set (IOS) in X, the complement of it is said to be intuitionistic closed set (ICS).

Definition 3. *[3] Suppose X be a non empty set, $p \in X$ an element in X. IS $\underset{\sim}{p} = \langle X, \{p\}, \{p\}^c\rangle$ is an intuitionistic point (IP) in X and the IS $(\underset{\approx}{p}) = \langle X, \phi, \{p\}^c\rangle$ is said to be a vanishing intuitionistic point (VIP) in X.*

Definition 4. *[12] Suppose (X, τ) be a TS, we define $\tau(C_\alpha) = \tau_\alpha = \{K_\alpha(\neq X) : K_\alpha \cap C_\alpha \neq \phi$, for $C_\alpha \in \tau$ and $C_\alpha \neq \phi$, X for some $\alpha \in I$, I an index set$\}$. We define $\tau_f = \{\phi, X\} \bigcup_\alpha \{\tau_\alpha\}$. τ_f of subsets of X is said to be fine collection of subsets of X, (X, τ, τ_f) is known as fine space X generated by topology τ on X.*

Definition 5. *[12] A subset O of (X, τ, τ_f) is called fine open in X, if $O \in$ to τ_f. Its complement is fine-closed set.*

Example 1. *[12] Suppose $X = \{p, q, r\}$ &*
$\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}\}$. *Let* $A_1 = \{p\}, A_2 = \{q\}, A_3 = \{p, q\}$ *then*
$\tau_1 = \tau(A_1) = \tau\{p\} = \{\{p\}, \{p, q\}, \{p, r\}\}$,
$\tau_2 = \tau(A_2) = \tau\{q\} = \{\{q\}, \{p, q\}, \{q, r\}\}$,
$\tau_3 = \tau(A_3) = \tau\{p, q\} = \{\{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$
$\tau_f = \{\phi, X\} \cup_\alpha \{\tau_\alpha\}$
$\tau_f = \{\phi, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$.

Definition 6. *[12]*
(i) The largest fine open set $\subseteq C$ is fine interior of C denoted as $fint(C)$.
(ii) The smallest fine closed set $\supseteq C$ is fine closure of C denoted as $fcl(C)$.

Definition 7. *[12] Suppose (X, τ, τ_f) and $x \in X$, then a fine open set O of $X \in x$ is said to be a fine neighborhood of X.*

Definition 8. *[4, 8] A subset C of (X, τ) is:*
(i) intuitionistic α-open if $C \subseteq Iint(Icl(Iint(C)))$.
(ii) intuitionistic semi-open set if $C \subseteq Icl(Iint(C))$.
(iii) intuitionistic pre-open if $C \subseteq Iint(Icl(C))$.
(iv) intuitionistic β-open if $C \subseteq Icl(Iint(Icl(C))$.
(v) intuitionistic regular-open if $C = Iint(Icl(C))$.

Definition 9. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is known as fine continuous if $g^{-1}(V)$ is open in X for every fine open set V of Y.*

Definition 10. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is known as fine irresolute (or f-irresolute) if $g^{-1}(V)$ is fine-open in X for every fine-open set V of Y.*

Definition 11. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is f-irresolute homeomorphism if*
(i) g is 1-1 and onto.
(ii) Both maps g and inverse map $g^{-1} : (Y, \tau', \tau'_f) \to (X, \tau, \tau_f)$ are f-irresolute.

3. Intuitionistic Fine Open Sets

Definition 12. *Suppose (X, τ) be an ITS, we define*
$\tau(C_\alpha) = \widehat{\tau}_\alpha = \{K_\alpha (\neq \underset{\sim}{X}) : K_\alpha \cap C_\alpha \neq \underset{\sim}{\phi}$, *for* $C_\alpha \in \tau$ *and* $C_\alpha \neq \underset{\sim}{\phi}, \underset{\sim}{X}$ *for some* $\alpha \in I$, *I an indexed set*$\}$. *We define* $\widehat{\tau_f} = \left\{\underset{\sim}{\phi}, \underset{\sim}{X}\right\} \cup_\alpha \{\widehat{\tau}_\alpha\}$. $\widehat{\tau}_f$ *of subsets of X is known as intuitionistic fine collection of subsets of X &* $(X, \tau, \widehat{\tau}_f)$ *is known as an intuitionistic fine space* $(I_f S)$ *X generated by* τ *on X.*

Definition 13. *A subset O of $I_f S$ X is known as intuitionistic fine open sets $(I_f OS)$ if $O \in \widehat{\tau}_f$. Complement of $(I_f OS)$ is intuitionistic fine closed set $(I_f CS)$.*

Example 2. *Consider* $X = \{p, q, r\}$ *and* $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ *where*
$A_1 = \langle X, \{r\}, \{p, q\}\rangle$ *and* $A_2 = \langle X, \{r\}, \{p\}\rangle$.
Let $A_\alpha = A_1$ *and* A_2.
$\tau(A_\alpha) = \widehat{\tau}_\alpha = \{\langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle,$
$\langle X, \{q, r\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle,$
$\langle X, \{q\}, \{p, r\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle,$
$\langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
$\widehat{\tau_f} = \{\underset{\sim}{X}, \underset{\sim}{\phi}\} \cup \{\widehat{\tau}_\alpha\}$.
$\because \widehat{\tau_f} = I_f OS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle,$
$\langle X, \{\phi\}, \{p\}\rangle, \langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle,$
$\langle X, \{q\}, \{r\}\rangle, \langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle,$
$\langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{\phi\}\rangle,$
$\langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle, \langle X, \{r\}, \{p, q\}\rangle,$
$\langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle$.

3.1. **Real-Time example: Advantage of Intuitionistic Fine open sets.**

Suppose $X = \{a, b, c\}$ represents the set of three participants involving in various activities. All combination of subsets (i.e) the power set (intuitionistic) P(X) consist of 27 intuitionistic subsets like $A_1, A_2, ..., A_{27}$ involving membership and non-membership values, in which ϕ represents a set with no participants, X represents a set with all the participants, $\{a\}$ represents a set with one participant and so on. For example, if the following table (Table 1) illustrates the sets to which team they belong to:

TABLE 1. Intuitionistic subsets and corresponding teams

$X = \langle X, \{p, q, r\}, \{\phi\}\rangle$	Intuitionistic set representing social activity team
$\phi = \langle X, \{\phi\}, \{p, q, r\}\rangle$	Intuitionistic set representing music team
$A_1 = \langle X, \{p, r\}, \{\phi\}\rangle$	Intuitionistic set representing Project group
$A_2 = \langle X, \{p, q\}, \{r\}\rangle$	Intuitionistic set representing Study group
$A_3 = \langle X, \{p, r\}, \{q\}\rangle$	Intuitionistic set representing Sports team
$A_4 = \langle X, \{p\}, \{q\}\rangle$	Intuitionistic set representing individual activity-1 participant
$A_5 = \langle X, \{r\}, \{p\}\rangle$	Intuitionistic set representing individual activity-2 participant

and so on. Also if $\tau = \{\{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{r\}, \{p, q\}\rangle,$ $\langle X, \{r\}, \{p\}\}\rangle\}$ (Ref.Example 2) associated with X, represents possibilities of collection of intuitionistic sets involving in political activity, then we get the collection of intuitionistic fine open sets, $\widehat{\tau_f}$ (Ref.Example 2) gives a clear picture of the various combinations (intuitionistic sets) of participants engaged in different activities(teams) also involve in political activity so that their intersection is not empty, with the union of possibilities of no and all participants ($\{\underset{\sim}{X}$ and $\underset{\sim}{\phi}\}$).

Definition 14. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS, suppose $p \in X$, then an intuitionistic fine open set O of X containing $\underset{\sim}{p}$ is known as an intuitionistic fine neighborhood.*

Example 3. *Consider $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ where*
$A_1 = \langle X, \{\phi\}, \{p, q\}\rangle$ *and* $A_2 = \langle X, \{\phi\}, \{p\}\rangle$.
$\widehat{\tau_f} = I_fOS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle, \langle X, \{r\}, \{p\}\rangle,$
$\langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle,$
$\langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
Consider $r \in X$ then the intuitionistic fine neighborhoods of the intuitionistic point $\underset{\sim}{r} = \langle X, \{r\}, \{r\}^c\rangle$ ($\underset{\sim}{r} = \langle X, \{r\}, \{p, q\}\rangle$) in X are
$\{\langle X, X, \phi\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle\}$.

Definition 15. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS and suppose $C = \langle X, C_1, C_2\rangle$ be an IS in X then:*
$Icl_f(C) = \bigcap\{J : J$ *is an* I_fCS *in* X & $C \subseteq J\}$
$Iint_f(C) = \bigcup\{J : J$ *is an* I_fOS *in* X & $C \supseteq J\}$.

Example 4. *Suppose $X = \{p, q\}$ and $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ where*
$A_1 = \langle X, \{q\}, \{p\}\rangle$ *and* $A_2 = \langle X, \{\phi\}, \{p\}\rangle$.
$\widehat{\tau_f} = I_fOS = \{\underset{\sim}{\phi}, \underset{\sim}{X}, \langle X, \{\phi\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{p\}, \{\phi\}\rangle,$
$\langle X, \{q\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
$I_fCS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle$.
$Icl(\langle X, \{p\}, \{q\}\rangle) = \underset{\sim}{X}$.
$I_fcl(\langle X, \{p\}, \{q\}\rangle) = \langle X, \{p\}, \{q\}\rangle$.

Theorem 1. *Suppose $(X, \tau, \widehat{\tau}_f)$ be I_fS, then the (arbitrary) union of I_fOS in X is I_fOS in X.*

Proof. Suppose $\{K_\alpha\}_{\alpha \in I}$ be set of I_fOSs of X. Implies $K_\alpha \cap C_\alpha \neq \underset{\sim}{\phi}, \forall \alpha \in I$ and $C_\alpha(\neq \underset{\sim}{\phi}, \underset{\sim}{X}) \in \tau$. We need T.P that $\cup_{\alpha \in I} K_\alpha = K$ is I_fOS. It's enough to S.T $K \cap C_\beta \neq \underset{\sim}{\phi}$ for $C_\beta(\neq \underset{\sim}{\phi}, \underset{\sim}{X}) \in \tau$. Here $(\cup_{\alpha \in I} K_\alpha \cap C_\beta) = (K_\alpha \cap C_\beta) \cup (K_\beta \cap C_\beta)$..... $\Rightarrow$ $\exists$ an index $\beta \in I$ s.t $K_\beta \cap C_\beta \neq \underset{\sim}{\phi}$ ($\because K_\beta \in \widehat{\tau}_f$). Therefore $(\cup K_\alpha) \cap C_\beta \neq \phi \Rightarrow K$ is an I_fOS. □

Remark 1. (1) *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fTS then the union of two I_fCS in X need not be I_fCS in X.*
(2) *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fTS then $\cap$ of two I_fOS in X need not be I_fOS in X.*

Theorem 2. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS, then the arbitrary intersection of I_fCSs in X is I_fCS in X.*

Proof. Assuming $\{F_\alpha\}_{\alpha \in I}$ to be set of intuitionistic fine-closed sets of X.
T.P: $\bigcap F_\alpha = F$ is intuitionistic fine closed. It is sufficient T.P F^c is intuitionistic fine-open. Using De Morgan's law to get $F^c = \cup F^c_\alpha$. Using the above remark the union of intuitionistic fine open set implies that $F^c = \cup F^c_\alpha$ is I_fOS. Therefore F is I_fCS. □

Example 5. *Consider Example 3, Suppose $A = \langle X, \{r\}, \{\phi\}\rangle$ and*
$B = \langle X, \{p\}, \{r\}\rangle$ be two I_fOS then $A \cup B = \langle X, \{p, r\}, \{\phi\}\rangle$ which is an I_fOS.
Now $C = \langle X, \{q\}, \{r\}\rangle$ and $D = \langle X, \{r\}, \{q\}\rangle$ be I_fOS
$C \cap D = \langle X, \{\phi\}, \{q, r\}\rangle$ which is not an I_fOS.
Here $I_fCS = \{\underset{\sim}{X}, \phi, \langle X, \{\phi\}, \{p\}\rangle, \langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{\phi\}\rangle,$
$\langle X, \{q\}, \{\phi\}\rangle, \langle X, \tilde{\{r\}}, \{\phi\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle,$
$\langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle, \langle X, \{r\}, \{p\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{q, r\}\rangle,$
$\langle X, \{\phi\}, \{p, r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{q\}\rangle,$
$\langle X, \{p, q\}, \{r\}\rangle, \langle X, \{p\}, \{q, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
Let $E = \langle X, \{q\}, \{p, r\}\rangle$ and $F = \langle X, \{r\}, \{p, q\}\rangle$ be I_fCS
$E \cup F = \langle X, \{q, r\}, \{p\}\rangle$ which is not an I_fCS.
$E \cap F = \langle X, \{\phi\}, \{X\}\rangle$ which is an I_fCS.

Definition 16. *An intuitionistic fine subset C of $(X, \tau, \widehat{\tau}_f)$ is:*
(i) an $I_f\alpha OS$ if $C \subseteq I_fint(I_fcl(I_fint(C)))$. (ii) an I_fSOS if $C \subseteq I_fcl(I_fint(C))$. (iii) an I_fPOS if $C \subseteq I_fint(I_fcl(C))$. (iv) an $I_f\beta OS$ if $C \subseteq I_fcl(I_fint(I_fcl(C)))$. (v) an I_fROS if $C = I_fint(I_fcl(C))$.

Remark 2. *An I_fOS C of $(X, \tau, \widehat{\tau}f)$ is:*
(i) an intuitionistic fine α-open set ($If\alpha OS$) if C is an intuitionistic α open subset of (X, τ). (ii) an intuitionistic fine semi-open set (I_fSOS) if C is an intuitionistic semi open subset of (X, τ). (iii) an intuitionistic fine pre-open set (I_fPOS) if C is an intuitionistic pre open subset of (X, τ). (iv) an intuitionistic fine β-open set ($I_f\beta OS$) if C is an intuitionistic β open subset of (X, τ). (v) an intuitionistic fine regular-open (I_fROS) if C is an intuitionistic regular open subset of (X, τ).

Theorem 3. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS w.r.t the TS (X, τ), then $\widehat{\tau}_f \subset$ every $ISOS$ and $I\alpha OS$.*

Proof. Assume, $E \subset \underset{\sim}{X}$ and $E \notin \widehat{\tau}_f$.
T.P: E is not $ISOS$ in $\underset{\sim}{X}$.
$\because E \notin \widehat{\tau}_f \Rightarrow C_\alpha \cap E = \underset{\sim}{\phi}$.
$\forall \alpha \in J \Rightarrow I_fint(E) = \underset{\sim}{\phi} \Rightarrow I_fcl(I_fint(E)) = \underset{\sim}{\phi} \Rightarrow E$ not contained in $I_fcl(I_fint(E))$.
Hence E is not intuitionistic semi open and hence E is not contained in
$I_fint(I_fcl(I_fint(E)))$ and therefore E is not an $I\alpha OS$. □

Theorem 4. *Suppose $(X, \tau, \widehat{\tau}_f)$ be the I_fS w.r.t the TS (X, τ), then $\widehat{\tau}_f \subset$ every $IPOS$ and $I\beta OS$.*

Proof. Assume, $E \subset \underset{\sim}{X}$ and $E \notin (X, \tau)$.
T.P: E is not $IPOS$ and not $I\beta OS$ in $\underset{\sim}{X}$
$\because$ it is known that $E \notin \widehat{\tau}_f \Rightarrow C_\alpha \cap E = \phi \forall \alpha \in J \Rightarrow E \subseteq (C_\alpha)^c$ and $I_f cl(E) \subseteq (C_\alpha)^c$ and $(C_\alpha)^c$ is an $I_f CS$ containing E. $\because C_\alpha \cap (C_\alpha)^c = \phi$ and $I_f cl(E) \subseteq (C_\alpha)^c \Rightarrow C_\alpha \cap I_f cl(E) = \underset{\sim}{\phi} \Rightarrow I_f int(I_f cl(E)) = \underset{\sim}{\phi}$ and hence E not contained in $I_f int(I_f cl(E))$. Therefore E is not intuitionistic pre open and thus E is not contained in $I_f cl(I_f int(I_f cl(E)))$. Hence E is not $I\beta OS$. □

Example 6. *Suppose* $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1\}$ *where*
$A_1 = \langle X, \{r\}, \{p, q\}\rangle$
$I_f OS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{\phi\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle, \langle X, \{\phi\}, \{q\}\rangle, \langle X, \{p\}, \{\phi\}\rangle,$
$\langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p\}, \{q\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle.$
It is found that $\langle X, \{\phi\}, \{r\}\rangle, \langle X, \{\phi\}, \{q, r\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle,$
$\langle X, \{p\}, \{q, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{q\}, \{r\}\rangle, \langle X, \{p\}, \{r\}\rangle$
are not members of $\widehat{\tau_f}$, *they are not* $I\alpha OS, I\beta OS, ISOS$ *and* $IPOS$ *but they are* $I_f CS$.

Theorem 5. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* $I_f S$ *and C any arbitrary subset of X. Then:*
(i)$Iint(C) \subseteq I_f int(C)$
(ii)$I_f cl(C) \subseteq Icl(C)$.

Proof. By the definitions of interior, closure, intuitionistic fine interior and intuitionistic fine closure $Iint(C) \subseteq I_f int(C)$ and $I_f cl(C) \subseteq Icl(C)$. □

Theorem 6. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* $I_f S$ *and C be any arbitrary subset of X. Then* $x \in I_f cl(C)$ *iff every* $I_f OS$ *'O' containing x intersects C.*

Proof. Assume $(X, \tau, \widehat{\tau}_f)$ to be an $I_f S$ and C be any arbitrary subset of X. Let $x \in I_f cl(C)$, consider every $IfOS$ 'O' containing x by the def. of intuitionistic fine closure we find that every $I_f OS$ 'O' containing x intersects C.
Conversely assume that every intuitionistic fine open set O containing x intersects C then using the definition of intuionistic fine open set we find that $x \in I_f cl(C)$. □

4. Intuitionistic Fine Maps

Definition 17. *Map* $g : (X, \tau, \widehat{\tau}_f) \rightarrow (Y, \tau^{'}, \widehat{\tau}^{'}_f)$ *is known as intuitionistic fine-continuous if* $g^{-1}(V)$ *is intuitionistic open set (IOS) in* $X \,\forall\, I_f OS$ V *of Y.*

Definition 18. *Map* $g : (X, \tau, \widehat{\tau}_f) \rightarrow (Y, \tau^{'}, \widehat{\tau}^{'}_f)$ *is known as intuitionistic fine-irresolute if* $g^{-1}(V)$ *is intuitionistic fine open in* $X \,\forall\, I_f OS$ V *of Y.*

Definition 19. *A map* $g : (X, \tau, \widehat{\tau}_f) \rightarrow (Y, \tau^{'}, \widehat{\tau}^{'}_f)$ *is known as intuitionistic fine-irresolute homeomorphism if*
(i) g is 1-1 and onto. (ii) Both maps g and inverse map of g are intuitionistic fine irresolute.

Example 7. *Suppose* $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A\}$ *s.t* $A = \langle X, \{p\}, \{q\}\rangle$ *and suppose* $Y = \{1, 2, 3\}$ *and* $\tau' = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B\}$ *where* $B = \langle X, \{3\}, \{1, 2\}\rangle$. *Suppose* $g : (X, \tau, \widehat{\tau}_f) \to (Y, \tau', \widehat{\tau}'_f)$ *where g(p)=1, g(q)=2 and g(r)=3.*
Here I_fOS *in X are* $\{\langle X, \phi, X\rangle, \langle X, X, \phi\rangle, \langle X, \phi, \{p\},\rangle, \langle X, \phi, \{q\}\rangle, \langle X, \phi, \{r\}\rangle,$
$\langle X, \{p\}, \phi\rangle, \langle X, \{q\}, \phi\rangle, \langle X, \{r\}, \phi\rangle, \langle X, \phi, \{p, q\}\rangle, \langle X, \phi, \{q, r\}\rangle,$
$\langle X\{p, q\}, \phi\rangle, \langle X, \{q, r\}, \phi\rangle, \langle X, \{p, r\}, \phi\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p\}, \{q, r\}\rangle,$
$\langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \phi, \phi\rangle\}.$
I_fOS *in Y are* $\{\langle Y, \phi, Y\rangle, \langle Y, Y, \phi\rangle, \langle Y, \phi, \{1\},\rangle, \langle Y, \phi, \{2\}\rangle, \langle Y, \phi, \{3\}\rangle,$
$\langle Y, \{1\}, \phi\rangle, \langle Y, \{2\}, \phi\rangle, \langle Y, \{3\}, \phi\rangle, \langle Y, \phi, \{1, 2\}\rangle, \langle Y, \phi, \{2, 3\}\rangle,$
$\langle Y\{1, 2\}, \phi\rangle, \langle Y, \{2, 3\}, \phi\rangle, \langle Y, \{1, 3\}, \phi\rangle, \langle Y, \{1\}, \{2\}\rangle, \langle Y, \{2\}, \{3\}\rangle,$
$\langle Y, \{3\}, \{1\}\rangle, \langle Y, \{2\}, \{1\}\rangle, \langle Y, \{3\}, \{2\}\rangle, \langle Y, \{1\}, \{3\}\rangle, \langle Y, \{1\}, \{2, 3\}\rangle,$
$\langle Y, \{3\}, \{1, 2\}\rangle, \langle Y, \{2, 3\}, \{1\}\rangle, \langle Y, \{1, 3\}, \{2\}\rangle, \langle Y, \{1, 2\}, \{3\}\rangle, \langle Y, \phi, \phi\rangle\}.$
Here the given function is not intuitionistic fine continuous but it is intuitionistic fine irresolute and intuitionistic fine irresolute homeomorphism.

Theorem 7. *Suppose X & Y be* I_fSs *and suppose* $g : X \to Y$. *The following are equivalent :*
(i) g is intuitionistic fine irresolute.
(ii) For every subset C of X, $g(I_fcl(C)) \subseteq I_fclg(C)$.
(iii) For every I_fCS *D in Y,* $g^{-1}(D)$ *is* I_fCS *in X.*

Proof. $(i) \Rightarrow (ii)$: Suppose that g is intuitionistic fine irresolute. Suppose C be a subset of X. We S.T if $x \in I_fcl(C)$, then $g(x) \in g(I_fcl(C)) \Rightarrow g(x) \in I_fcl(g(C))$. Assume V to be an intuitionistic fine neighbourhood of g(x). Hence $g^{-1}(V)$ is I_fOS of X containing x ; this intersects C in some point y. Then V intersects g(C) in g(y) s.t $g(x) \in I_fcl(g(C))$, which is required.
$(ii) \Rightarrow (iii)$: Suppose D be I_fCS in Y and suppose that $C = g^{-1}(D)$. We have to P.T C is $I_fCS$in X; We P.T $I_fcl(C) = C$. By basic set theory, $g(C) = g(g^{-1}(D)) \subset D$. Hence if $x \in I_fcl(C), g(x) \in g(I_f(C)) \subset I_fcl(C)) \subset I_fcl(D) = D$, $\because$ D is an I_fCS, s.t $x \in g^{-1}(D) = C$. Hence $I_f(C) \subset C$, s.t $I_fcl(C) = C$ as required.
$(iii) \Rightarrow (i)$: Suppose V be an I_fOS of Y. Set $D = Y - V$. Hence $g^{-1}(D) = g^{-1}(Y) - g^{-1}(V) = X - g^{-1}(V)$. Here D is an I_fCS of Y. Hence $g^{-1}(D)$ is I_fCS in X by hypothesis s.t $g^{-1}(V)$ is I_fOS, as required. □

5. Envisaging Applications in Image Processing: The Probable Impacts of Intuitionistic Fine Space

Given that intuitionistic fuzzy sets inherently handle both membership and non-membership values, their significance in image processing is well-established. The similarity between intuitionistic fine space and intuitionistic topological space with fuzzy measures suggests a promising avenue for the application of intuitionistic fine space in the realm of image processing. In the digital plane, each pixel functions as an open set [21]. Notably, intuitionistic sets encompass both membership and non-membership values, with measures for these values defined in intuitionistic fuzzy sets. This framework forms the foundation for various applications such as image extraction, image segmentation, optimization, and more within the domain of image processing. This article posits that the same rationale can be extended to intuitionistic fine open sets, potentially yielding intriguing and valuable results in the context of image processing.

6. Conclusion

The continued advancement of the intuitionistic fine topology, as introduced in this article, into fuzzy, binary space holds the potential to unlock further applications in diverse areas. These areas span from digital lines to computer networking, image processing, and data analysis, among others. Consequently, this article lays the groundwork for future applications of I_fS in both intuitionistic topology and intuitionistic fuzzy topology, paving the way for broader and more varied practical implementations.

Author Contribution Statements All the three authors equally contributed to this article and all the three reviewed the final article.

Declaration of Competing Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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