Universal Journal of Mathematics and Applications, 6 (2) (2023) 65-75
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1287524

Exact Traveling Wave Solutions of the Schamel-KdV Equation with Two Different Methods

Seydi Battal Gazi Karakoç[1*], Khalid K. Ali[2] and Mona Samir Mehanna[3]

[1]*Department of Mathematics, Faculty of Science and Art, Nevşehir Hacı Bektaş Veli University, Nevşehir, 50300, Türkiye*
[2]*Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt*
[3]*Faculty of Engineering, MTI University, Cairo, Egypt*
[*]*Corresponding author*

Article Info

Keywords: *Auxiliary equation method, Bernoulli Sub-ODE method, Modified, Schamel–Korteweg–de Vries equation, Travelling wave solutions*
2010 AMS: *35C07, 35Q53, 35E05, 35L05*
Received: *25 April 2023*
Accepted: *9 June 2023*
Available online: *30 June 2023*

Abstract

The Schamel-Korteweg-de Vries (S-KdV) equation including a square root nonlinearity is very important pattern for the research of ion-acoustic waves in plasma and dusty plasma. As known, it is significant to discover the traveling wave solutions of such equations. Therefore, in this paper, some new traveling wave solutions of the S-KdV equation, which arises in plasma physics in the study of ion acoustic solitons when electron trapping is present and also it governs the electrostatic potential for a certain electron distribution in velocity space, are constructed. For this purpose, the Bernoulli Sub-ODE and modified auxiliary equation methods are used. It has been shown that the suggested methods are effective and give different types of function solutions as: hyperbolic, trigonometric, power, exponential, and rational functions. The applied computational strategies are direct, efficient, concise and can be implemented in more complex phenomena with the assistant of symbolic computations. The results found in the paper are of great interest and may also be used to discover the wave sorts and specialities in several plasma systems.

1. Introduction

Nonlinear partial differential equations (NPDEs) are used to describe complex problems with numerous phenomena in different fields, including engineering, chemical kinematics, biology, wave theory, optics, physics, fluid mechanics, biomedical science, and others [1]- [4]. S-KdV equation, based upon both usual KdV equation (when $\alpha = 0$) [5]- [10],

$$u_t + \beta u u_x + \delta u_{xxx} = 0,$$

and Schamel equation (when $\beta = 0$)

$$u_t + \alpha u^{1/2} u_x + \delta u_{xxx} = 0,$$

which was derived a German scientist Hans Schamel in 1973 has the form [11]- [13]

$$u_t + (\alpha u^{1/2} + \beta u)_x + \delta u_{xxx} = 0, \quad \alpha\beta \neq 0 \tag{1.1}$$

where α, β and δ are constants which they are refer to the activation trapping, the convection and the dispersion coefficients, respectively. The advantage of implementing the nonlinear S-KdV equation to analyze dynamics of modulated waves in dispersive media lies in the diversity of its solutions [14]- [17]. Here we should point out that the S-KdV equation has a stronger nonlinearity than the usual KdV

***Email addresses and ORCID numbers*: sbgkarakoc@nevsehir.edu.tr, 0000-0002-2348-4170 (S.B.G. Karakoç), khalidkaram2012@azhar.edu.eg, 0000-0002-7801-2760 (K.K. Ali), monamehanna@gmail.com, 0000-0003-4349-1113 (M.S. Mehanna)**
Cite as: S. B. G. Karakoç, K. K. Ali, M. S. Mehanna, *Exact traveling wave solutions of the Schamel-KdV equation with two different methods*, Univers. J. Math. Appl., 6(2) (2023), 65-75.

equation in that the single soliton solution possesses a smaller width and higher velocity [18]. This equation is contained in many physical phenomena involving electromagnetic theory, physical chemistry, geophysics and other fields are examples [19, 20]. The square root in the nonlinear term then translates to lowest order some of the kinetic effects, associated with electron trapping [21]. Schamel [22] stated that when uu_x is replaced by $(| u |^{3/2})_x$, compared to the classical KdV equation, the Schamel equation possesses a stronger nonlinearity, which reveals that the wave has a smaller width and higher velocity and exact traveling wave solutions for the regularized Schamel equation [23].To create different exact solutions and to notice their properties, various significant methods have been developed [19]- [21], [24]- [29].
The implementations of the Sub-ODE and modified auxiliary equation methods in this paper highlight our main motivation and indicates its capacity to handle nonlinear equations, permitting it to be utilized to solve many types of nonlinearity models. The body of our paper is structured as follows: Methodologies of the Sub-ODE and modified auxiliary equation methods and their detailed structures are given in Section 2. In Section 3, we apply the different methods, introduced in Section 2, to the Schamel-KdV equation to find the exact solutions. Different forms of the exact solutions are derived from these methods. Section 4 is devoted to graphical illustrations of the methods. A discussion section is presented in Section 5. Finally, Section 6 provides conclusions stemming from the results of our work.

2. The Bernoulli Sub-ODE Method [30]- [33]

Herein, we introduce the steps of the Bernoulli Sub-ODE method. Suppose, nonlinear partial differential equation is given by

$$F(v, v_t, v_x, v_{xx}, v_{xxx}, ...) = 0, \tag{2.1}$$

where $v = v(x,t)$ is wave function to be calculated.

Step 1: Apply the traveling wave transformation,

$$v(x,t) = v(\eta), \ \ \eta = kx - ct \tag{2.2}$$

where k is constant and c is speed of the traveling wave. Substituting (2.2) into (2.1), then (2.1) converted to an ordinary differential equation:

$$F(v', v'', v''', ...) = 0 \tag{2.3}$$

where F is a polynomial in $v(\eta)$ and its derivatives.

Step 2: Presume solutions of (2.3) presented by a series in G:

$$v(\eta) = \sum_{i=0}^{N} b_i (G(\eta))^i \tag{2.4}$$

where $b_i (0 \leq i \leq N)$ are constants to be calculated, $b_N \neq 0$, and $G(\eta)$ satisfies the next ODE,

$$G'(\eta) + \lambda G(\eta) = \mu G(\eta)^2 \tag{2.5}$$

which has the following solution:

$$G(\eta) = \frac{1}{\frac{\mu}{\lambda} + de^{\lambda \eta}}$$

where $\lambda, \mu \neq 0$ are arbitrary constants.

Step 3: The positive integer N determined by balancing the highest order derivative term with the highest power nonlinear term in (2.3).

Step 4: Replacing (2.4) into (2.3), we acquire a polynomial in $G(\eta)$. Gathering all terms with the same power and equating each one to zero. We get a system of equations which can be solved by using Mathematica program.

2.1. The modified auxiliary equation method (MAE) [34]- [37]

Main steps of the modified auxiliary equation method are explained as follows:

Step 1: Solution of (2.3) is given by:

$$v(\eta) = \sum_{i=0}^{N} (b_i a^{if(\eta)} \tag{2.6}$$

where $f(\eta)$ satisfies the following ODE:

$$f' = \frac{1}{\ln a} \left(\mu a^{-f(\eta)} + \sigma + \lambda a^{f(\eta)} \right) \tag{2.7}$$

where $b_i \, (i = 0, 1, 2, \ldots, N)$, $b_N \neq 0$, λ, σ and μ, are constants to be calculated.

Step 2: In (2.3), N is a positive integer determined via the homogeneous balance principle as illustrated before.

Step 3: Substituting (2.6) and (2.7) in (2.3), and gathering the terms which had like powers of $(a^{f(\eta)})$ and putting their coffecients equal to zero, we obtain a set of algebraic equations, which can be solved by the aid of Mathematica program.

Step 4: There is various sets of solutions of (2.7):

Set 1: $\sigma^2 - 4\lambda\mu < 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda} \tan\left(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda} \cot\left(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2}\right).$$

Set 2: $\sigma^2 - 4\lambda\mu > 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda} \tanh\left(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda} \coth\left(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2}\right).$$

Set 3: $\sigma^2 + 4\mu^2 < 0, \lambda \neq 0 and \lambda = -\mu$,

$$a^{f(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu} \tan\left(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu} \cot\left(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2}\right).$$

Set 4: $\sigma^2 + 4\mu^2 > 0, \lambda \neq 0$ and $\lambda = -\mu$,

$$a^{f(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu} \tanh\left(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu} \coth\left(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2}\right).$$

Set 5:$\sigma^2 - 4\mu^2 < 0$ and $\lambda = \mu$,

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu} \tan\left(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu} \cot\left(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2}\right).$$

Set 6: $\sigma^2 - 4\mu^2 > 0$ and $\lambda = \mu$,

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \tanh\left(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \coth\left(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}\right).$$

Set 7: $\sigma^2 = 4\lambda\mu$ and $\lambda = \mu$,

$$a^{f(\eta)} = -\frac{2 + \sigma\eta}{2\lambda\eta}.$$

Set 8: $\lambda\mu < 0, \sigma = 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = -\sqrt{\frac{-\mu}{\lambda}}\tanh\left(\sqrt{-\mu\lambda}\eta\right),$$

or

$$a^{f(\eta)} = -\sqrt{\frac{-\mu}{\lambda}}\coth\left(\sqrt{-\mu\lambda}\eta\right).$$

Set 9:$\sigma = 0$ and $\mu = -\lambda$,

$$a^{f(\eta)} = \frac{1+e^{-2\lambda\eta}}{-1+e^{-2\lambda\eta}}.$$

Set 10: $\mu = \lambda = 0$,

$$a^{f(\eta)} = \cosh(\sigma\eta) + \sinh(\sigma\eta).$$

Set 11: $\mu = \sigma = h$ and $\lambda = 0$,

$$a^{f(\eta)} = e^{h\eta} - 1.$$

Set 12: $\lambda = \sigma = h$ and $\mu = 0$,

$$a^{f(\eta)} = \frac{e^{h\eta}}{1-e^{h\eta}}.$$

Set 13: $\sigma = \lambda + \mu$,

$$a^{f(\eta)} = -\frac{1-\mu e^{(\mu-\lambda)\eta}}{1-\lambda e^{(\mu-\lambda)\eta}}.$$

Set 14: $\sigma = -(\lambda + \mu)$,

$$a^{f(\eta)} = \frac{\mu - e^{(\mu-\lambda)\eta}}{\lambda - e^{(\mu-\lambda)\eta}}.$$

Set 15: $\mu = 0$,

$$a^{f(\eta)} = \frac{\sigma e^{\sigma\eta}}{1-\lambda e^{\sigma\eta}}.$$

Set 16: $\lambda = \mu = \sigma \neq 0$,

$$a^{f(\eta)} = \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\mu\eta\right) - 1.$$

Set 17: $\lambda = \sigma = 0$,

$$a^{f(\eta)} = \mu\eta.$$

Set 18: $\mu = \sigma = 0$,

$$a^{f(\eta)} = \frac{-1}{\lambda\eta}.$$

Set 19: $\lambda = \mu$ and $\sigma = 0$,

$$a^{f(\eta)} = \tan(\mu\eta).$$

Set 20: $\lambda = 0$,

$$a^{f(\eta)} = e^{\sigma\eta} - \frac{\mu}{\sigma}.$$

3. Applications of the Methods

Begin with the following transformation:

$$v(x,t)=u(x,t)^2,$$

with wave transformation (2.2) into (1.1), we obtain following ordinary differential equation:

$$2k\beta u^3u'+6k^3\delta u'u''+2u((-c+k\alpha u)u'+k^3\delta u''')=0.$$

Integrating once with respect to η, we get

$$-cu^2+\frac{2}{3}k\alpha u^3+\frac{1}{2}k\beta u^4+2k^3\delta(u')^2+2k^3\delta uu''=0. \tag{3.1}$$

Balancing uu'' with u^4 in (3.1), we get $4N=2N+2$, then $N=1$.

3.1. The Bernoulli sub-ODE method

Using (2.4), solution of (3.1) is given by

$$u(\eta)=b_0+b_1G(\eta). \tag{3.2}$$

Substituting (3.2) in (3.1), then collecting terms of the same powers and putting their coefficients equal to zero, next system of equations are acquired :

$$\begin{aligned}
&-cb_0^2+\frac{2}{3}k\alpha b_0^3+\frac{1}{2}k\beta b_0^4=0,\\
&-2cb_0b_1+2k^3\delta\lambda^2b_0b_1+2k\alpha b_0^2b_1+2k\beta b_0^3b_1=0,\\
&-6k^3\delta\lambda\mu b_0b_1-cb_1^2+4k^3\delta\lambda^2b_1^2+2k\alpha b_0b_1^2+3k\beta b_0^2b_1^2=0,\\
&4k^3\delta\mu^2b_0b_1-10k^3\delta\lambda\mu b_1^2+\frac{2}{3}k\alpha b_1^3+2k\beta b_0b_1^3=0,\\
&6k^3\delta\mu^2b_1^2+\frac{1}{2}k\beta b_1^4=0.
\end{aligned}$$

In what follows, we present the two sets of solution:
Set 1:

$$\delta=-\frac{4\alpha^2}{75k^2\beta\lambda^2},\quad b_0=-\frac{4\alpha}{5\beta},\quad b_1=\frac{4\alpha\mu}{5\beta\lambda},\quad c=-\frac{16k\alpha^2}{75\beta},\quad v(x,t)=\left(-\frac{4\alpha}{5\beta}+\frac{4\alpha\mu}{5\beta\lambda}\left(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\left(kx+\left(\frac{16k\alpha^2}{75\beta}\right)t\right)}}\right)\right)^2. \tag{3.3}$$

Set 2:

$$\delta=-\frac{4\alpha^2}{75k^2\beta\lambda^2},\quad b_0=0,\quad b_1=-\frac{4\alpha\mu}{5\beta\lambda},\quad c=-\frac{16k\alpha^2}{75\beta},\quad v(x,t)=\left(-\frac{4\alpha\mu}{5\beta\lambda}\left(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\left(kx+\left(\frac{16k\alpha^2}{75\beta}\right)t\right)}}\right)\right)^2. \tag{3.4}$$

3.2. The modified auxiliary equation method (MAE)

(2.6) presents the solution in the form:

$$u(\eta)=b_0+b_1a^{f(\eta)}. \tag{3.5}$$

Substituting (3.5)in (3.1), then summing terms of like powers and setting their coefficients equal to zero, the next system of equations are obtained:

$$\begin{aligned}
&-cb_0^2+\frac{2}{3}k\alpha b_0^3+\frac{1}{2}k\beta b_0^4+2k^3\delta\mu\sigma b_0b_1+2k^3\delta\mu^2b_1^2=0,\\
&-2cb_0b_1+4k^3\delta\lambda\mu b_0b_1+2k^3\delta\sigma^2b_0b_1+2k\alpha b_0^2b_1+2k\beta b_0^3b_1+6k^3\delta\mu\sigma b_1^2=0,\\
&6k^3\delta\lambda\sigma b_0b_1-cb_1^2+8k^3\delta\lambda\mu b_1^2+4k^3\delta\sigma^2b_1^2+2k\alpha b_0b_1^2+3k\beta b_0^2b_1^2=0,\\
&4k^3\delta\lambda^2b_0b_1+10k^3\delta\lambda\sigma b_1^2+\frac{2}{3}k\alpha b_1^3+2k\beta b_0b_1^3=0,\\
&6k^3\delta\lambda^2b_1^2+\frac{1}{2}k\beta b_1^4=0.
\end{aligned}$$

Solving the previous system yields two sets of solutions:

$$c=4k^3\delta(-4\lambda\mu+\sigma^2),\quad \alpha=\pm\frac{15k^2\delta\lambda\sqrt{(-4\lambda\mu+\sigma^2)b_1^2}}{b_1^2},\quad \beta=-\frac{12k^2\delta\lambda^2}{b_1^2},\quad b_0=\frac{\sigma b_1\pm\sqrt{(-4\lambda\mu+\sigma^2)b_1^2}}{2\lambda}.$$

Therefore, using the above sets gives the solitary wave solutions to (2.5) in the following formulas:

$(kx-(4k^3\delta(-4\lambda\mu+\sigma^2))t)$

Set 1: $\sigma^2-4\lambda\mu<0$ and $\lambda\neq 0$,

$$v_{1,2}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}+\frac{\sqrt{4\mu\lambda-\sigma^2}}{2\lambda}\tan\left(\frac{\sqrt{4\mu\lambda-\sigma^2}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{3,4}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}+\frac{\sqrt{4\mu\lambda-\sigma^2}}{2\lambda}\cot\left(\frac{\sqrt{4\mu\lambda-\sigma^2}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 2: $\sigma^2-4\lambda\mu>0$ and $\lambda\neq 0$,

$$v_{5,6}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}-\frac{\sqrt{\sigma^2-4\mu\lambda}}{2\lambda}\tanh\left(\frac{\sqrt{\sigma^2-4\mu\lambda}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{7,8}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}-\frac{\sqrt{\sigma^2-4\mu\lambda}}{2\lambda}\coth\left(\frac{\sqrt{\sigma^2-4\mu\lambda}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2. \tag{3.6}$$

Set 3: $\sigma^2+4\mu^2<0, \lambda\neq 0$ and $\lambda=-\mu$,

$$v_{9,10}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}-\frac{\sqrt{-\sigma^2-4\mu^2}}{2\mu}\tan\left(\frac{\sqrt{-\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{11,12}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}-\frac{\sqrt{-\sigma^2-4\mu^2}}{2\mu}\cot\left(\frac{\sqrt{-\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 4: $\sigma^2+4\mu^2>0$, $\lambda\neq 0$ and $\lambda=-\mu$,

$$v_{13,14}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}+\frac{\sqrt{\sigma^2+4\mu^2}}{2\mu}\tanh\left(\frac{\sqrt{\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{15,16}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}+\frac{\sqrt{\sigma^2+4\mu^2}}{2\mu}\coth\left(\frac{\sqrt{\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 5: $\sigma^2-4\mu^2<0$ and $\lambda=\mu$,

$$v_{17,18}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}+\frac{\sqrt{-\sigma^2+4\mu^2}}{2\mu}\tan\left(\frac{\sqrt{-\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{19,20}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}+\frac{\sqrt{-\sigma^2+4\mu^2}}{2\mu}\cot\left(\frac{\sqrt{-\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2. \tag{3.7}$$

Set 6: $\sigma^2-4\mu^2>0$ and $\lambda=\mu$,

$$v_{21,22}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}-\frac{\sqrt{\sigma^2-4\mu^2}}{2\mu}\tanh\left(\frac{\sqrt{\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{23,24}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}-\frac{\sqrt{\sigma^2-4\mu^2}}{2\mu}\coth\left(\frac{\sqrt{\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 7: $\sigma^2=4\lambda\mu$ and $\lambda=\mu$,

$$v_{25,26}(x,t)=\left(\frac{\sqrt{\lambda\mu}b_1}{\lambda}-\frac{\left(2+2kx\sqrt{\lambda\mu}\right)b_1}{2kx\lambda}\right)^2.$$

Set 8: $\lambda\mu<0$, $\sigma=0$ and $\lambda\neq0$,

$$v_{27,28}(x,t)=\left(\pm\frac{\sqrt{-\lambda\mu b_1^2}}{\lambda}-\sqrt{-\frac{\mu}{\lambda}}b_1\tanh\left(\sqrt{-\lambda\mu}\left(kx+16k^3t\delta\lambda\mu\right)\right)\right)^2,$$

or (3.8)

$$v_{29,30}(x,t)=\left(\pm\frac{\sqrt{-\lambda\mu b_1^2}}{\lambda}-\sqrt{-\frac{\mu}{\lambda}}b_1\coth\left(\sqrt{-\lambda\mu}\left(kx+16k^3t\delta\lambda\mu\right)\right)\right)^2.$$

Set 9: $\sigma=0$ and $\mu=-\lambda$,

$$v_{31,32}(x,t)=\left(\pm\frac{\sqrt{\lambda^2b_1^2}}{\lambda}+b_1\left(\frac{1+e^{-2\lambda\left(kx-16k^3\delta\lambda^2t\right)}}{-1+e^{-2\lambda\left(kx-16k^3\delta\lambda^2t\right)}}\right)\right)^2.$$

Set 10: $\lambda=\sigma=h$ and $\mu=0$,

$$v_{33,34}(x,t)=\left(\frac{hb_1\pm\sqrt{h^2b_1^2}}{2h}+b_1\left(\frac{e^{h\left(kx-4h^2k^3\delta t\right)}}{1-e^{h\left(kx-4h^2k^3\delta t\right)}}\right)\right)^2.$$

Set 11: $\sigma=\lambda+\mu$,

$$v_{35,36}(x,t)=\left(\frac{b_1\left(\lambda+\mu\right)\pm b_1\left(\lambda-\mu\right)}{2\lambda}-b_1\left(\frac{1-\mu e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}{1-\lambda e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}\right)\right)^2.$$

Set 12: $\sigma=-(\lambda+\mu)$,

$$v_{37,38}(x,t)=\left(\frac{b_1\left(-\lambda-\mu\right)\pm b_1\left(\lambda-\mu\right)}{2\lambda}+b_1\left(\frac{\mu-e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}{\lambda-e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}\right)\right)^2$$

Set 13: $\mu=0$,

$$v_{39,40}(x,t)=\left(\frac{\sigma b_1\pm\sigma b_1}{2\lambda}+b_1\left(\frac{\sigma e^{\sigma\left(kx-4k^3\delta\sigma^2t\right)}}{1-\lambda e^{\sigma\left(kx-4k^3\delta\sigma^2t\right)}}\right)\right)^2.$$

Set 14: $\lambda=\mu=\sigma\neq0$,

$$v_{41,42}(x,t)=\left(\frac{b_1\pm\sqrt{-3b_1^2}}{2}+\frac{1}{2}b_1\left(-1+\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\sigma\left(kx+12k^{3\delta}\sigma^2t\right)\right)\right)\right)^2.$$

Set 15: $\mu = \sigma = 0$,

$$v_{43,44}(x,t) = \frac{b_1^2}{k^2 \lambda^2 x^2}.$$

Set 16: $\lambda = \mu$ and $\sigma = 0$,

$$v_{45,46}(x,t) = \left(\pm\sqrt{-b_1^2} + b_1 \tan\left(\mu \left(kx + 16k^3 \delta \mu^2 t \right) \right) \right)^2.$$

4. Graphical Illustrations

The majority of our solutions are presented in the following graphs to illustrate solutions.
In Figure 4.1, we present graph of (3.3) using the Bernoulli Sub-ODE method at $k = 2, \alpha = 0.5, \beta = 0.3, \mu = 0.3, \lambda = 0.3, d = 2$. Figure 4.2 shows graph of (3.4) using the Bernoulli Sub-ODE method at $k = 2, \alpha = 0.5, \beta = 0.3, \mu = 0.3, \lambda = 0.3, d = 2$. Graph of (3.6) using the modified auxiliary equation method at $k = 2, b_1 = 0.3, \mu = 0.02, \lambda = 0.1, \delta = 0.1, \sigma = 0.3$ is presented in Figure 4.3. Graph of (3.7) using the modified auxiliary equation method at $k = 0.6, b_1 = 0.1, \mu = 0.03, \delta = 1.6, \sigma = 0.04$ is given in Figure 4.4. Lastly, Figure 4.5 presents graph of (3.8) using the modified auxiliary equation method at $k = 0.7, b_1 = 0.1, \mu = 0.3, \delta = 0.3, \sigma = 0, \lambda = -0.5$.

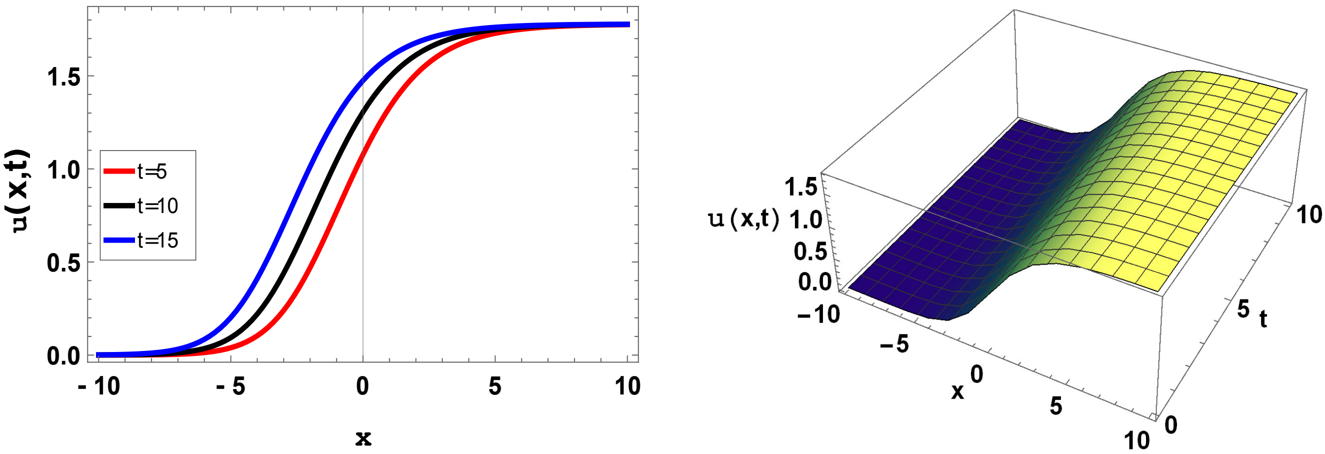

Figure 4.1: Graph of Eq. (3.3) using the Bernoulli Sub-ODE method at $k = 2, \alpha = 0.5, \beta = 0.3, \mu = 0.3, \lambda = 0.3, d = 2$.

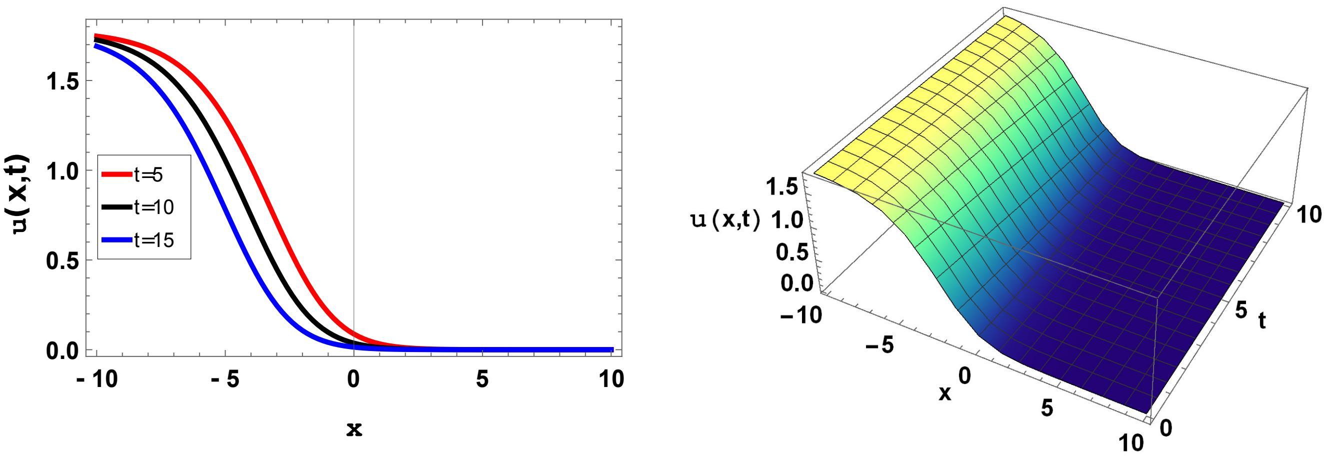

Figure 4.2: Graph of Eq. (3.4) using the Bernoulli Sub-ODE method at $k = 2, \alpha = 0.5, \beta = 0.3, \mu = 0.3, \lambda = 0.3, d = 2$.

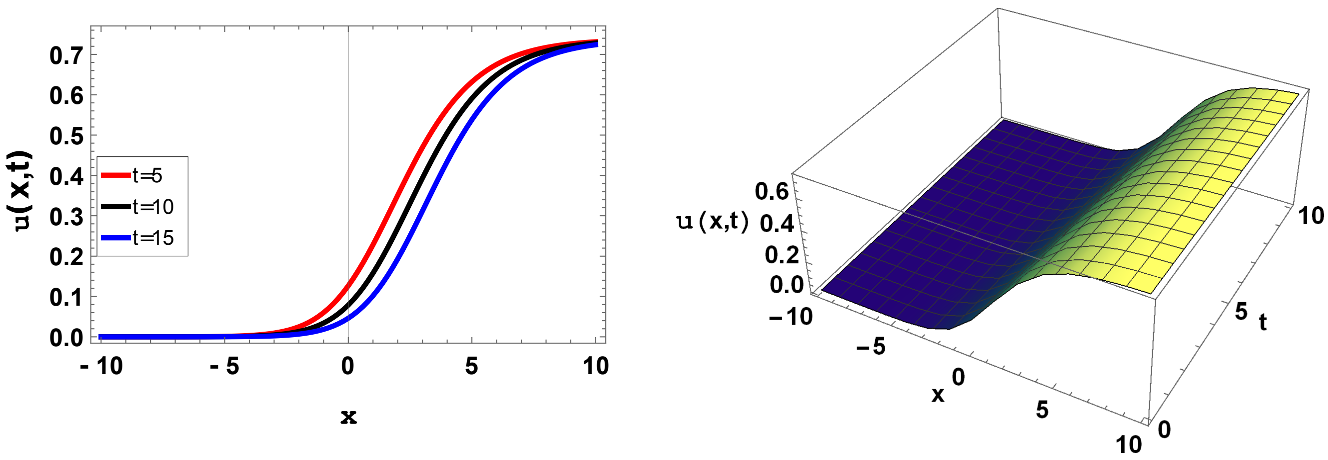

Figure 4.3: Graph of Eq. (3.6) using the modified auxiliary equation method at $k = 2, b_1 = 0.3, \mu = 0.02, \lambda = 0.1, \delta = 0.1, \sigma = 0.3$.

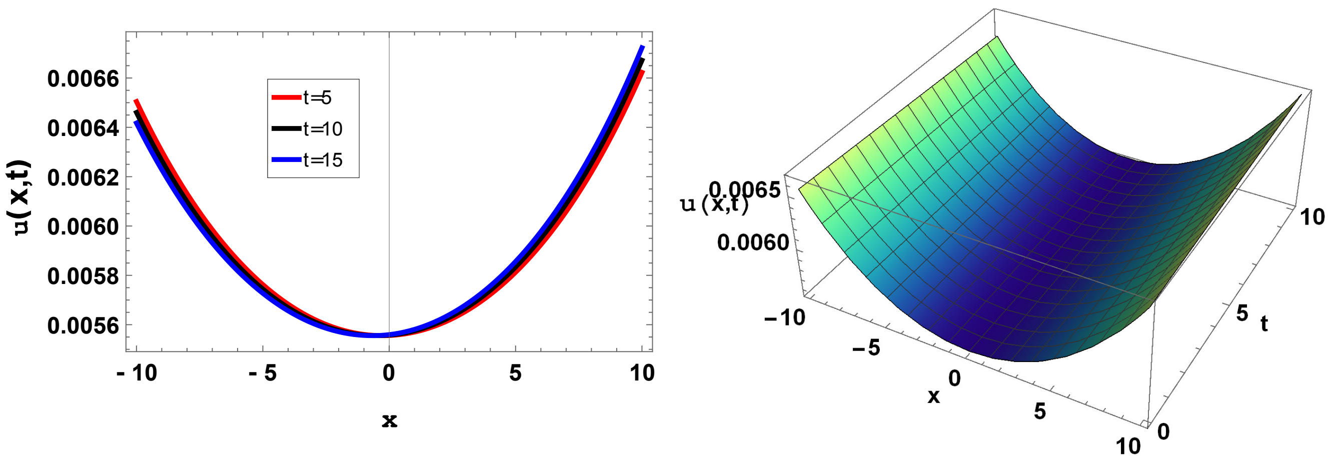

Figure 4.4: Graph of Eq. (3.7) using the modified auxiliary equation method at $k = 0.6, b_1 = 0.1, \mu = 0.03, \delta = 1.6, \sigma = 0.04$.

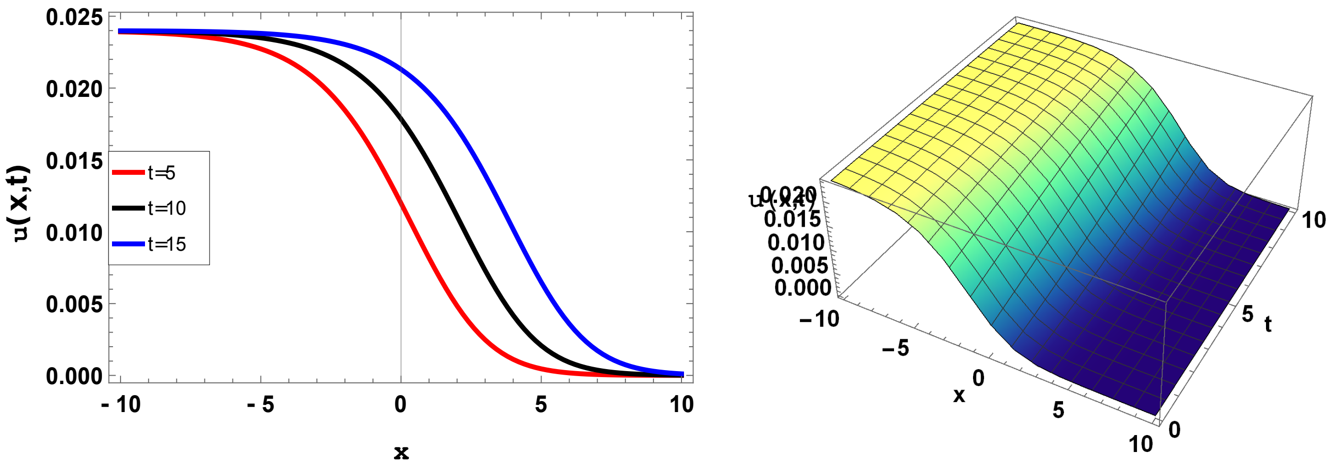

Figure 4.5: Graph of Eq. (3.8) using the modified auxiliary equation method at $k = 0.7, b_1 = 0.1, \mu = 0.3, \delta = 0.3, \sigma = 0, \lambda = -0.5$.

5. Discussion

The graph is one of the best tools for describing and presenting solutions. In the following, we review the behavior of the wave in the solutions presented: In Figures 4.1-4.2 the wave travels to the left with increasing time$t = 0, 5, 10$. Contrarily, in Figures 4.3-4.5 the wave moves towards left as time passes $t = 0, 5, 10$. The flipped wave is presented in Figure 4.4 as time goes on.

6. Conclusion

In this work, a class of some new travelling wave solutions of the Schamel–Korteweg-de Vries equation are successfully found out by using the Bernoulli Sub-ODE and modified auxiliary equation methods. The Bernoulli Sub-ODE is a simple and straightforward method and is applicable to a wide range of problems in science and engineering, but it can be time-consuming to apply the method if the equation involves complex functions. The modified auxiliary equation method can be used to solve a wide range of differential equations, also it can provide closed-form solutions, but it can be difficult to determine the appropriate auxiliary equation to use for a given differential equation. The presented exact solutions provided here may describe various new characteristics of waves and then may be useful in the theoretical and numerical studies of the considered equation. A graphical representation of newly discovered solutions are also shown to explain the dynamics of soliton profiles. The found new soliton solutions of the S-KdV equation are of significant importance and can be used in other areas of physics such as plasma physics.

Article Information

Acknowledgements: The authors thank the referees for valuable comments and suggestions which improved the presentation of this paper.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

Availability of Data and Materials: Not applicable.

References

[1] B. Karaagac, A. Esen, Y. Ucar, N. M. Yagmurlu, *A new outlook for analysis of Noyes-Field model for the nonlinear Belousov-Zhabotinsky reaction using operator splitting method*, Computers Math. Appl. **136** (2023), 127-135.
[2] S. Kutluay, S. Ozer, N. M. Yagmurlu, *A new highly accurate numerical scheme for Benjamin-Bona-Mahony-Burgers equation describing small amplitude long wave propagation*, Mediterr. J. Math. **20**(3) (2023), 1-24.
[3] S. Kutluay, N. M. Yagmurlu, A. S. Karakas, *Operator time-splitting techniques combined with quintic B-spline collocation method for the generalized Rosenau–KdV equation*, Numerical Meth. Partial Differential Equ., **35** (2019), 2221-2235.
[4] S. Kutluay, M. Karta, N. M. Yagmurlu, *An Effective Numerical approach based on cubic Hermite B-spline collocation method for solving the 1D Heat conduction equation*, New Trends Math. Sci., **10**(4) (2022), 20-31.
[5] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and a new type of long stationary wave*, Philosophical Magazine, **39** (1895) 422-443.
[6] N. J. Zabusky, *A synergetic approach to problem of nonlinear dispersive wave propagation and interaction, in: W. Ames (Ed.).* Proc. Symp. Nonlinear Partial Diff. Equ., Academic Press (1967) 223-258.
[7] S. B. G. Karakoc, *A quartic subdomain finite element method for the modified KdV equation*, Statistics, Optimization Information Comp. **6** (2018), 609-618.
[8] A. Bashan, A. Esen, *Single soliton and double soliton solutions of the quadratic-nonlinear Korteweg-de Vries equation for small and long-times*, Numerical Meth. Partial Differential Equ., **37**(2) (2021), 1561-1582.
[9] S. B. G. Karakoc, *Numerical solutions of the modified KdV equation with collocation method* Malaya J. Mat., **6**(4) (2018), 835-842.
[10] S. B. G. Karakoc, A. Saha, D. Sucu, *A novel implementation of Petrov-Galerkin method to shallow water solitary wave pattern and superperiodic traveling wave and its multistability generalized Korteweg-de Vries equation*, Chinese J. Phys., **68** (2020), 605-617.
[11] A. I. Zemlyanukhin, I. V. Andrianov, A. V. Bochkarev, L. I. Mogilevich, *The generalized Schamel equation in nonlinear wave dynamics of cylindrical shells*, Nonlinear Dyn. **98** (2019), 185-194.
[12] M. G.Kuzyk, *Polymer Fiber Optics: Materials, Materials, Physics, and Applications*, CRC Press, 2018.
[13] K. Shimoda, *Introduction to Laser Physics*. Springer-Verlag, Heidelberg, 1986.
[14] M. W. Coffey, *On the integrability of schamel's modified Korteweg-de Vries equation*, J. Phys. A: Mathematical and General **24**(23) (1991).
[15] Y. Wu, Z. Liu, *New types of nonlinear waves and bifurcation phenomena in Schamel-Korteweg-de Vries equation*, Abstr. Appl. Anal., **2013** (2013), 1-18.
[16] J. Yang, S. Q. Tand, *Exact traveling wave solutions of the Schamel-Korteweg-de Vries equation*, J. Math. Sci. Adv. Appl., **31**(25) (2015).
[17] E. Kengne, A. Lakhssassi, W. M. Liu, *Nonlinear Schamel–Korteweg deVries equation for a modified Noguchi nonlinear electric transmission network: Analytical circuit modeling*, Chaos, Solitons, Fractals **140** (2020), 110229.
[18] H. Schamel, *Stationary solitary, snoidal and sinusoidal ion acoustic waves*, Plasma Phys. **14**(10) (1972), 905-924.
[19] K. U. Tariq, H. Rezazadeh, M. Zubair, M. S. Osman, L. Akinyemi, *New exact and solitary wave solutions of nonlinear Schamel–KdV equation*, Int. J. Appl. Comput. Math., **8**(114) (2022), 1-16.
[20] Z. Pinar, A. Yildirim, S. T. Mohyud-Din, K. F. Oğuz, S. Djabrailov, A. Biswas, *New exact solutions for Schamel-Korteweg-de-Vries equation*, Studies in Nonlinear Sci. **3**(3) (2012), 102-106.
[21] F. Kangalgil, *Travelling wave solutions of the Schamel–Korteweg–de Vries and the Schamel equations*, J. Egyptian Math. Soc., **24** (2016), 526-531.
[22] H. Schamel, *A modified Korteweg-de Vries equation for ion acoustic waves due to resonant electrons*, J. Plasma Phys. **9**(3) (1973), 377-387.
[23] Q. Cai, K. Tan, J. Li, *Bifurcations and exact traveling wave solutions for the regularized Schamel equation*,Open Math., **19** (2021), 1699-1712.
[24] O. Dönmez, D. Dağhan, *Analytic solutions of the Schamel-KdV equation by using different methods: Application to a dusty space plasma*, Süleyman Demirel Univ. J. Natural and Appl. Sci. **21**(1) (2017), 208-215.

[25] I. B. Giresunlu, Y. S. Özkan, E. Yaşar, *On the exact solutions, lie symmetry analysis, and conservation laws of Schamel–Korteweg–de Vries equation*, Math. Methods Appl. Sci. **40**(11) (2017), 3927-3936.

[26] K. U. Tariq, M. Inc, H. Y. Martinez, M. M. A. Khater, *Explicit, periodic and dispersive soliton solutions to the Schamel-KdV equation with constant coefficients*, Journal of Ocean Engineering and Science, https://doi.org/10.1016/j.joes.2022.05.025 (In press).

[27] S. K. Mohanty, A. N. Dev, *Recent trends in applied mathematics*, Lect. Notes Mech. Engrg., **174** (2021), 109-136.

[28] J. Lee, R. Sakthivel, *Exact travelling wave solutions of the Schamel-Korteweg- de Vries equation*, Rep. Mathematical Phys., **68**(2) (2011), 153-161.

[29] H. I. Abdel-Gawad, M. Tantawy, *Exact solutions of the Shamel-Korteweg-de Vries equation with time dependent coefficients*, Inf. Sci. Lett. **3** (2014), 103-109.

[30] N. Taghizadeh, M. Akbari, P. Esmaeelnejhad, *Parirokh Esmaeelnejhad, application of Bernoulli Sub-ODE method for finding travelling wave solutions of Schrödinger Equation Power Law Nonlinearity*, Appl. Appl. Math., **12**(1) (2017), 596-603.

[31] Q. Feng, *Traveling wave solution of (3+1) dimensional potential-YTSF equation by Bernoulli Sub-ODE method*, Adv. Mater. Res., **403** (2012), 212-216.

[32] B. Zheng, *Application of a Generalized Bernoulli Sub-ODE Method For Finding Traveling Solutions Of Some Nonlinear Equations*, Wseas Trans. Math., **7**(11) (2012), 618-626.

[33] B. Zheng, *New analytical solutions for two equations by a proposed Sub-ODE method*, International Conference on Computer Technology and Science (ICCTS 2012) **47** (2012), 360-364.

[34] M. A. Akbar, N. H. Mohd Ali, T. Tanjim, *Outset of multiple soliton solutions to the nonlinear Schrödinger equation and the coupled Burgers equation*, J. Phys. Commun., **3** (2019), 1-17.

[35] S. M. R.Islam, S. Khan, S. M. Y. Arafat, M. A. Akbar, *Diverse analytical wave solutions of plasma physics and water wave equations*, Results in Physics, **40** (2022), 1-8.

[36] M. A. Khani, M. A. Akbar, N. HJ. M. Ali, M. Abbas, *The new auxiliary method in the solution of the generalized Burgers-Huxley equation*, J. Prime Res. Math., **16**(2) (2020), 16-26.

[37] M. M. A. Khater, A. R. Seadawy, D. Lu, *Dispersive optical soliton solutions for higher order nonlinear Sasa-Satsuma equation in mono mode fibers via new auxiliary equation method*, Superlattices and Microstructures **113** (2018), 346-358.