Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (2) (2024), 471 – 487
DOI : 10.15672/hujms.1287530

Research Article

Suzuki type $\mathcal{Z}_c$-contraction mappings and the fixed-figure problem

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Abstract

Geometric approaches are important for the study of some real-life problems. In metric fixed point theory, a recent problem called "*fixed-figure problem*" is the investigation of the existence of self-mapping which remain invariant at each points of a certain geometric figure (e.g. a circle, an ellipse and a Cassini curve) in the space. This problem is well studied in the domain of the extension of this line of research in the context of fixed circle, fixed disc, fixed ellipse, fixed Cassini curve and so on. In this paper, we introduce the concept of a Suzuki type $\mathcal{Z}_c$-contraction. We deal with the fixed-figure problem by means of the notions of a $\mathcal{Z}_c$-contraction and a Suzuki type $\mathcal{Z}_c$-contraction. We derive new fixed-figure results for the fixed ellipse and fixed Cassini curve cases by means of these notions. Also fixed disc and fixed circle results given for Suzuki type $\mathcal{Z}_c$-contraction. There are couple of illustration related to the obtained theoretical results.

Mathematics Subject Classification (2020). 54H25, 47H09, 47H10, 54C30, 46T99

Keywords. Suzuki type $\mathcal{Z}_c$-contraction, fixed disc, fixed circle, fixed ellipse, fixed Cassini curve

1. Introduction

Topological and geometrical properties of the fixed point set have been widely studied for number of manner in the non-linear analysis. In recent times geometrical aspects of the fixed point set of an operator have been considered with different form of the fixed-figure as, fixed circle, fixed disc, fixed ellipse. In [12, 13, 15], Özgür and Taş examined the fixed circle and fixed disc problem in metric space. From that these topics are generated much interest recently in fixed figure problem (see, for example, [5], [15] and the references therein).

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Received: 26.04.2023; Accepted: 04.06.2023

Recently, the notions of a $\mathcal{Z}_c$-contraction and of a Suzuki type $\mathcal{Z}$-contraction have been defined by means of the set of simulation functions defined in [10] (see [12] and [11] for more details). In present article, we consider the fixed-figure problem by means of the notions of $\mathcal{Z}_c$-contraction and Suzuki type $\mathcal{Z}_c$-contraction. We derive new fixed-figure results for the fixed ellipse and fixed Cassini curve cases. Let $T : X \to X$ be a map, where X is a metric space with the metric d. The set of fixed points of T is denoted by $Fix(T)$ and can be defined as follows:

$$Fix(T) = \{u \in X : Tu = u\}.$$

A geometric figure (e.g. a circle, an Apollonius circle, an ellipse, a Cassini curve) contained within $Fix(T)$ is known as a fixed figure of T. For example, if $Tu = u$ for any $u \in E_r(u_1, u_2) = \{u \in X : d(u, u_1) + d(u, u_2) = r\}$ then the ellipse is a fixed ellipse of T. The interested reader can refer to [15] and the references therein. For our purpose, we use some properties of simulation functions defined in [10].

Now, onwards in this paper consider $\mathbb{R}_0^+ = [0, \infty)$. The function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is said to be a simulation function, if it possesses the stipulations given below :

(ζ_1) $\zeta(0, 0) = 0$,
(ζ_2) $\zeta(t, s) < s - t$ for all $s, t > 0$,
(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Collection of simulation functions is noted by $\mathcal{Z}$ [10]. In [11], the notion of a Suzuki type $\mathcal{Z}$-contraction was defined as follows:

Definition 1.1 ([11])**.** Let the self-mapping T on the set X, where X is a metric space with the metric d and $\zeta \in \mathcal{Z}$. Then T is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the condition given below is satisfied for all $u, v \in X$ with $u \neq v$:

$$\frac{1}{2} d(u, Tu) < d(u, v) \Rightarrow \zeta\left(d(Tu, Tv), d(u, v)\right) \geq 0. \tag{1.1}$$

From Definition 1.1, we have

$$\frac{1}{2} d(u, Tu) < d(u, v) \Rightarrow d(Tu, Tv) < d(u, v),$$

for all distinct $u, v \in X$ for a Suzuki type $\mathcal{Z}$-contraction T. Hence, every Suzuki type $\mathcal{Z}$-contraction mapping is a Suzuki type contraction (see [11] for more details). In [11], Kumam et al. used the notion of a Suzuki type $\mathcal{Z}$-contraction mapping to generalize the Banach contraction. It was proved that the fixed point of a Suzuki type $\mathcal{Z}$-contraction is unique, provided it exists.

Present article is divided into following sections. In Section 2, we examine the geometric properties of the fixed point set of a $\mathcal{Z}_c$-contraction mapping. In Section 3, we define the notion of a Suzuki type $\mathcal{Z}_c$-contraction mapping and then discuss about the fixed figure problem in particular fixed disc, fixed ellipse and fixed Cassini curve. In Section 4, as an application, we propose a new type of activation function for complex valued neural networks. We construct a one-parameter generalization of the well-known zReLU function.

Author noted the concept of a simulation function has many impressive applications (see [9, 10] and the references therein). Also, the fixed ellipse and the fixed Cassini curve cases have special interests for some possible applications. For example, the differential geometry of a normal red blood cell was investigated using the Cassinian oval for modelling its profile [1]. In [17], by using of a special activation function whose fixed point set is an ellipse, an application to a complex-valued Hopfield neural network (CVHNN) was given.

Fixed point results are also important for theoretical studies (see, for instance, [7], [19], [22]).

2. Fixed-figure problem for $\mathcal{Z}_c$-contractions

In this section, we examine the fixed-figure problem for $\mathcal{Z}_c$-contractive mappings. First, we recall the definition and basic properties of a $\mathcal{Z}_c$-contractive mapping. For more details, one can see [12].

Definition 2.1 ([12]). Let $\zeta \in \mathcal{Z}$ be any simulation function. T is said to be a $\mathcal{Z}_c$-contraction with respect to ζ if there exists an $u_0 \in X$ such that the following condition holds for all $u \in X$:

$$d(Tu,u) > 0 \Rightarrow \zeta\left(d(Tu,u), d(Tu,u_0)\right) \geq 0. \tag{2.1}$$

If T is a $\mathcal{Z}_c$-contraction with respect to ζ, then

$$d(Tu,u) < d(Tu,u_0), \tag{2.2}$$

for each $u \in X$ with $Tu \neq u_0$.

Throughout the paper, to establish a fixed figure result, we use the number $\rho \in \mathbb{R}^+ \cup \{0\}$ defined by

$$\rho = \inf\{d(u,Tu) \mid Tu \neq u, u \in X\}. \tag{2.3}$$

Let T be a $\mathcal{Z}_c$-contraction with respect to ζ with $u_0 \in X$. From [12], we know that the fixed point set $Fix(T)$ of a $\mathcal{Z}_c$-contraction contains the disc $D_{u_0,\rho}$ if the condition $0 < d(Tu,u_0) \leq \rho$ holds for all $u \in D_{u_0,\rho} - \{u_0\}$.

Theorem 2.2 ([12]). *Let T be a $\mathcal{Z}_c$-contraction map with the simulation function ζ and $u_0 \in X$. If the condition*

$$0 < d(Tu,u_0) \leq \rho \tag{2.4}$$

holds for each $u \in D_{u_0,\rho} - \{u_0\}$ then $D_{u_0,\rho}$ is a fixed disc of T, that is, $D_{u_0,\rho} \subset Fix(T)$.

The proof of the following corollary can be easily deduced similar to the proof of Theorem 2.2.

Corollary 2.3. *Let T be a $\mathcal{Z}_c$-contraction map with the simulation function ζ and $u_0 \in X$. If the condition*

$$0 < d(Tu,u_0) \leq \mu \ (0 < \mu \leq \rho)$$

holds for each $u \in D_{u_0,\mu} - \{u_0\}$ then $D_{u_0,\mu}$ is a fixed disc of T, that is, $D_{u_0,\mu} \subset Fix(T)$.

2.1. $\mathcal{Z}_c$-contractions and the fixed-ellipse problem

First, we consider the fixed-ellipse problem. Let us consider an ellipse $E_r\left(u_1,u_2\right)$ defined by

$$E_r\left(u_1,u_2\right) = \{u \in X : d(u,u_1) + d(u,u_2) = r\}$$

and the set

$$\overline{E}_r\left(u_1,u_2\right) = \{u \in X : d(u,u_1) + d(u,u_2) \leq r\}.$$

The points u_1 and u_2 are called the foci of the ellipse $E_r\left(u_1,u_2\right)$. Now, we will see the result for fixed ellipse in sense of a $\mathcal{Z}_c$-contraction.

Proposition 2.4. *Let (X,d) be a metric space. Consider a set $\overline{E}_r\left(u_0,u_1\right)$ with any distinct $u_0,u_1 \in X$. If $\mu = d\left(u_1,u_0\right) \leq r$ then we have*

$$\overline{E}_r\left(u_0,u_1\right) \subset D_{u_0,r} \text{ and } \overline{E}_r\left(u_0,u_1\right) \subset D_{u_1,r}.$$

That is, we have

$$\overline{E}_r\left(u_0,u_1\right) \subset D_{u_0,r} \cap D_{u_1,r}.$$

Proof. Let $\mu = d(u_1, u_0) \leq r$. For any point $u \in D_{u_1,r-\mu}$, using the triangle inequality, we obtain

$$d(u, u_0) \leq d(u, u_1) + d(u_1, u_0) \leq (r - \mu) + \mu = r.$$

This shows that $u \in D_{u_0,r}$, and so $D_{u_0,\mu} \cup D_{u_1,r-\mu} \subset D_{u_0,r}$. For any $u \in \overline{E}_r(u_0, u_1)$, if $u \notin D_{u_0,\mu} \cup D_{u_1,r-\mu}$ then we get

$$d(u, u_0) + d(u, u_1) > r,$$

which is a contradiction. Therefore, we find $u \in D_{u_0,\mu} \cup D_{u_1,r-\mu}$ and so $u \in D_{u_0,r}$. Consequently, we deduce that

$$\overline{E}_r(u_0, u_1) \subset D_{u_0,r}.$$

On the other hand, since $u_1 \in D_{u_0,r}$ then we have also $u_0 \in D_{u_1,r}$. Again similarly, we obtain

$$\overline{E}_r(u_0, u_1) \subset D_{u_1,r}.$$

Combining these last inclusions, we get

$$\overline{E}_r(u_0, u_1) \subset D_{u_0,r} \cap D_{u_1,r}.$$

□

Corollary 2.5. *Let (X, d) be a metric space. Consider a set $\overline{E}_r(u_0, u_1)$ with any distinct $u_0, u_1 \in X$. If $u_1 \in D_{u_0,\gamma}$ $(0 < \gamma \leq r)$ then we have*

$$\overline{E}_r(u_0, u_1) \subset D_{u_0,\gamma} \cup D_{u_1,r-\gamma} \subset D_{u_0,r}.$$

Proof. Let $u \in D_{u_1,r-\gamma}$ be any point. Since $u_1 \in D_{u_0,\gamma}$ $(0 < \gamma \leq r)$, using the triangle inequality, we find

$$d(u, u_0) \leq d(u, u_1) + d(u_1, u_0) \leq (r - \gamma) + \gamma = r.$$

This shows that $u \in D_{u_0,r}$, and so $D_{u_0,\gamma} \cup D_{u_1,r-\gamma} \subset D_{u_0,r}$. Then the rest of the proof follows similarly. □

Let us consider the number ρ defined in (2.3).

Theorem 2.6. *Let (X, d) be a metric space, $T : X \to X$ be a $\mathcal{Z}_c$-contraction with respect to ζ with $u_0 \in X$. For a point $u_1 \in X$, if the condition*

$$0 < d(Tu, u_0) + d(Tu, u_1) \leq \rho \tag{2.5}$$

holds for all $u \in \overline{E}_\rho(u_0, u_1) - \{u_0\}$, then $Fix(T)$ contains the ellipse $E_\rho(u_0, u_1)$, that is, $E_\rho(u_0, u_1)$ is a fixed ellipse of T. Furthermore, we have $\overline{E}_\rho(u_0, u_1) \subset Fix(T)$.

Proof. If $u_0 = u_1$, then the ellipse $E_\rho(u_0, u_1)$ becomes the circle $C_{u_0,\frac{\rho}{2}}$ and the set $\overline{E}_\rho(u_0, u_1)$ becomes the disc $D_{u_0,\frac{\rho}{2}}$. By the assumption (2.5), we have

$$0 < d(Tu, u_0) \leq \frac{\rho}{2}$$

for all $u \in D_{u_0,\frac{\rho}{2}} - \{u_0\}$. Then by Corollary 2.3, we have $D_{u_0,\frac{\rho}{2}} \subset Fix(T)$.

Let $u_0 \neq u_1$ and $u \in E_\rho(u_0, u_1)$ be arbitrary but fixed. Assume that $Tu \neq u$. Using the condition (ζ_2) and (2.5) together with the definition of the number ρ, we obtain

$$\begin{aligned} \zeta(d(Tu, u), d(Tu, u_0)) &< d(Tu, u_0) - d(Tu, u) \\ &\leq d(Tu, u_0) + d(Tu, u_1) - d(Tu, u) \\ &\leq \rho - d(Tu, u) \\ &\leq \rho - \rho = 0 \end{aligned}$$

and so

$$\zeta(d(Tu, u), d(Tu, u_0)) < 0,$$

which conflicts the hypothesis (2.1). Hence we have $Tu = u$. Since $u \in E_\rho(u_0, u_1)$ is an arbitrary point then we deduce that $E_\rho(u_0, u_1)$ is a fixed ellipse of T. Similar arguments are valid for all $u \in \overline{E}_\rho(u_0, u_1)$ and we deduce that $\overline{E}_\rho(u_0, u_1) \subset Fix(T)$. □

The following corollary explains the relationship between Theorem 2.2 and Theorem 2.6.

Corollary 2.7. *Under the hypothesis of Theorem 2.6, if* $u_1 \in D_{u_0,\rho}$ *then we have* $\overline{E}_\rho(u_0, u_1) \subset D_{u_0,\rho} \cap Fix(T)$ *and* $\overline{E}_\rho(u_0, u_1) \subset D_{u_1,\rho} \cap Fix(T)$. *That is, we have*

$$\overline{E}_\rho(u_0, u_1) \subset D_{u_0,\rho} \cap D_{u_1,\rho} \cap Fix(T).$$

Proof. Considering the hypothesis $u_1 \in D_{u_0,\rho}$, the proof follows by Proposition 2.4 and Corollary 2.3. □

Clearly, under the hypothesis of Theorem 2.6, we have

$$E_\delta(u_0, u_1) \subset Fix(T)$$

for all ellipses $E_\delta(u_0, u_1)$, where $0 \leq \delta \leq \rho$. But, we can achieve this result with less assumption.

Corollary 2.8. *Let* (X, d) *be a metric space,* $T : X \to X$ *be a* $\mathcal{Z}_c$*-contraction with respect to* ζ *with* $u_0 \in X$. *For a point* $u_1 \in X$, *if the condition*

$$0 < d(Tu, u_0) + d(Tu, u_1) \leq \mu \ (0 < \mu \leq \rho)$$

holds for all $u \in \overline{E}_\mu(u_0, u_1) - \{u_0\}$, *then we have*

$$\overline{E}_\mu(u_0, u_1) \subset Fix(T).$$

Remark 2.9. 1) Now we consider the case $\rho = 0$. Then we get $u_0 = u_1$ and $E_\rho(u_0, u_1) = C_{u_0,\rho} = \{u_0\}$. Since T is a $\mathcal{Z}_c$-contraction map with $u_0 \in X$, $u_0 \in Fix(T)$.

2) We note that $u_0, u_1 \in \overline{E}_\rho(u_0, u_1)$ if and only if $d(u_0, u_1) \leq \rho$. Indeed, we have

$$u_1 \in \overline{E}_\rho(u_0, u_1) \Leftrightarrow d(u_1, u_0) + d(u_1, u_1) \leq \rho \Leftrightarrow d(u_1, u_0) \leq \rho,$$

and

$$u_0 \in \overline{E}_\rho(u_0, u_1) \Leftrightarrow d(u_0, u_0) + d(u_0, u_1) \leq \rho \Leftrightarrow d(u_0, u_1) \leq \rho.$$

3) If we replace the condition (2.1) by the one of the following conditions then, under the hypothesis of Theorem 2.6, we have $\overline{E}_\rho(u_0, u_1) \subset Fix(T)$:

$$d(Tu, u) \leq \lambda\,[d(Tu, u_0)] \tag{2.6}$$

for each $u \in X$, where $\lambda \in [0, 1)$.

$$d(Tu, u) \leq d(Tu, u_0) - \varphi\,(d(Tu, u_0)) \tag{2.7}$$

for each $u \in X$, where $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is lower semicontinuous function and $\varphi^{-1}(0) = 0$.

$$d(Tu, u) \leq \varphi\,(d(Tu, u_0))\,(d(Tu, u_0)) \tag{2.8}$$

for each $u \in X$, where $\varphi : \mathbb{R}_0^+ \to [0, 1)$ is a mapping such that $\lim\limits_{t \to r^+} \sup \varphi(t) < 1$, for all $r > 0$.

$$d(Tu, u) \leq \eta\,[d(Tu, u_0)] \tag{2.9}$$

for each $u \in X$, where $\eta \in \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an upper semicontinuous mapping such that $\eta(t) < t$ for all $t > 0$.

$$\int_0^{d(Tu,u)} \phi(t)dt \leq d(Tu, u_0) \tag{2.10}$$

for each $u \in X$, where $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function such that $\int_0^\varepsilon \phi(t)dt$ exists and $\int_0^\varepsilon \phi(t)dt > \varepsilon$, for each $\varepsilon > 0$.

The proof follows by using the simulation functions $\zeta_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}\, (1 \le i \le 5)$ defined by

$$\zeta_1(t,s) = \lambda s - t,$$

$$\zeta_2(t,s) = s - \varphi(s) - t,$$

$$\zeta_3(t,s) = s\varphi(s) - t,$$

$$\zeta_4(t,s) = \eta s - t$$

and

$$\zeta_5(t,s) = s - \int_0^t \phi(u)du$$

for each $s, t \in [0,\infty)$, respectively, in [10].

Now, we give few examples in support of Theorem 2.6.

Example 2.10. Let $X = \mathbb{C}$ be endowed with the usual metric d defined by $d(z,w) = |z-w|$ for every complex numbers z, w. Consider the self-mapping $T : X \to X$ defined by

$$Tz = \begin{cases} z & ; \quad |z| \le 4 \\ \frac{3}{4}z & ; \quad |z| > 4 \end{cases}, \tag{2.11}$$

for all $z = x + iy \in \mathbb{C}$. Then T is a $\mathcal{Z}_c$-contraction with $z_0 = 0$ and the function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{2}{3}s - t$. Indeed, we have $d(z, Tz) = \frac{1}{4}|z| > 0$ for all z with $|z| > 4$ and

$$\begin{aligned} \zeta\left(d(Tz,z), d(Tz,z_0)\right) &= \zeta\left(\frac{1}{4}|z|, \frac{3}{4}|z|\right) = \frac{2}{3}\frac{3}{4}|z| - \frac{1}{4}|z| \\ &= \frac{1}{4}|z| > 0. \end{aligned}$$

We find

$$\begin{aligned} \rho &= \inf\{d(z,Tz) \mid Tz \ne z, z \in \mathbb{C}\} \\ &= \inf\left\{\frac{1}{4}|z| : |z| > 4\right\} = 1. \end{aligned}$$

The condition $0 < d(Tz, 0) \le 1$ holds for all $z \in D_{0,1} - \{0\}$. Consequently, $D_{0,1} = \{z \in \mathbb{C} : |z| \le 1\}$ is a fixed disc of T by Theorem 2.2.

Now, we consider the ellipse $E_1\left(0, \frac{1}{3} + \frac{1}{3}i\right)$. Clearly, T possess the conditions of Theorem 2.6 and we have $\overline{E}_1\left(0, \frac{1}{3} + \frac{1}{3}i\right) \subset Fix(T)$ (see Figure 1 which is drawn using Mathematica, Version 12.0, [21]).

The subsequent example shows that the converse statement of Theorem 2.6 does not hold in general.

Example 2.11. Consider the metric space $(\mathbb{C}, d)$, where d is the usual metric, and the self-mapping $T : \mathbb{C} \to \mathbb{C}$ defined by

$$Tz = \begin{cases} z & ; \quad \left|z - \frac{1}{2}z_0\right| \le \mu \\ \frac{1}{2}z_0 & ; \quad \left|z - \frac{1}{2}z_0\right| > \mu \end{cases}, \tag{2.12}$$

for all $z \in \mathbb{C}$, where $0 < |z_0|$ and $\mu \ge |z_0|$. The self-mapping T is not a $\mathcal{Z}_c$-contraction with respect to any $\zeta \in \mathcal{Z}$ with $\frac{1}{2}z_0 \in \mathbb{C}$. Indeed, by the condition (ζ_2), for all z with

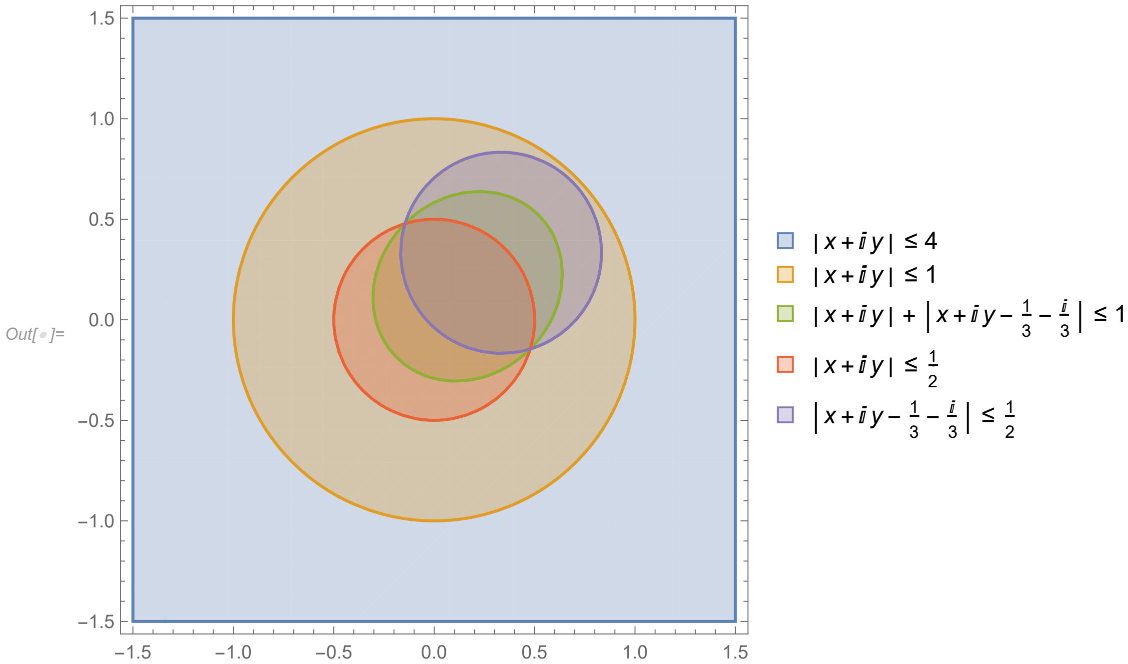

Figure 1. $D_{0,1}$, the fixed disc obtained by Theorem 2.2 for the self-mapping T defined in (2.11), the fixed ellipse $E_1\left(0, \frac{1}{3}+\frac{1}{3}i\right)$ and the discs $D_{0,\frac{1}{2}}$ and $D_{\frac{1}{3}+\frac{1}{3}i,\frac{1}{2}}$.

$\left|z - \frac{1}{2}z_0\right| > \mu$ we have

$$\begin{aligned} \zeta\left(d(Tz,z), d(Tz,z_0)\right) &= \zeta\left(\left|\frac{1}{2}z_0 - z\right|, \left|\frac{1}{2}z_0 - z_0\right|\right) \\ &= \zeta\left(\left|z - \frac{1}{2}z_0\right|, \frac{1}{2}|z_0|\right) < \frac{1}{2}|z_0| - \left|z - \frac{1}{2}z_0\right| \\ &< |z_0| - \left|z - \frac{1}{2}z_0\right| < 0. \end{aligned}$$

However, T fixes the disc $D_{\frac{1}{2}z_0,\mu}$ and any ellipse contained in it.

Remark 2.12. Notice that the radius ρ of the fixed ellipse $E_\rho\left(u_0,u_1\right)$ is independent from the foci u_0 and u_1 in Theorem 2.6. Under the hypothesis of Theorem 2.6, we note that the number ρ defined in (2.3) can produce several fixed ellipses (may be infinitely many) with various foci. For example, if we examine the self-mapping T as in (2.11), then the ellipses $E_1\left(0, \frac{1}{2}i\right)$ and $E_1\left(0, \frac{1}{2}+\frac{1}{2}i\right)$ are also fixed ellipses of T.

2.2. $\mathcal{Z}_c$-contractions and the fixed-Cassini curve problem

In this subsection, we consider the fixed-Cassini curve problem. Let us consider a Cassini curve $C_r(u_0,u_1)$ defined by

$$C_r(u_0,u_1) = \{u \in X : d\left(u,u_0\right) d\left(u,u_1\right) = r\}$$

and the set

$$\overline{C}_r(u_0,u_1) = \{u \in X : d\left(u,u_0\right) d\left(u,u_1\right) \le r\}.$$

Proposition 2.13. *Let (X,d) be a metric space. Consider a set $\overline{C}_r\left(u_0,u_1\right)$ with $u_0,u_1 \in X$. Then we have*

$$\overline{C}_r\left(u_0,u_1\right) \subset D_{u_0,\sqrt{r}} \cup D_{u_1,\sqrt{r}}. \tag{2.13}$$

Proof. Let $u \in \overline{C}_r(u_0, u_1)$ be any point. If $u \notin D_{u_0,\sqrt{r}} \cup D_{u_1,\sqrt{r}}$ then we have

$$d(u, u_0)\, d(u, u_1) > r,$$

which is a contradiction. Therefore, we have $u \in D_{u_0,\sqrt{r}} \cup D_{u_1,\sqrt{r}}$ and hence, the inclusion (2.13) holds. □

Now, by means of the number ρ defined in (2.3) we give a fixed Cassini curve result. Observe that $u_0, u_1 \in \overline{C}_\rho(u_0, u_1)$.

Theorem 2.14. *Let (X, d) be a metric space, and $T : X \to X$ be a $\mathcal{Z}_c$-contraction with respect to ζ with $u_0 \in X$. Suppose $\rho \geq 1$ and for any point $u_1 \in D_{u_0,\rho-\sqrt{\rho}}$, the condition*

$$0 < d(Tu, u_0)d(Tu, u_1) \leq \rho \tag{2.14}$$

holds for each $u \in \overline{C}_\rho(u_0, u_1) - \{u_0\}$. Then $Fix(T)$ contains the Cassini curve $C_\rho(u_0, u_1)$, that is, $C_\rho(u_0, u_1)$ is a fixed Cassini curve of T. Furthermore, we have $\overline{C}_\rho(u_0, u_1) \subset Fix(T)$.

Proof. If $u_0 = u_1$, then the Cassini curve $C_\rho(u_0, u_1)$ becomes the circle $C_{u_0,\sqrt{\rho}}$ and the set $\overline{C}_\rho(u_0, u_1)$ becomes the disc $D_{u_0,\sqrt{\rho}}$. Then by the hypothesis (2.14), we have

$$0 < d(Tu, u_0) \leq \sqrt{\rho}$$

for all $u \in D_{u_0,\sqrt{\rho}} - \{u_0\}$. Clearly, $\sqrt{\rho} \leq \rho$ since $\rho \geq 1$. Then by Corollary 2.3 we have $D_{u_0,\sqrt{\rho}} \subset Fix(T)$.

Now, assume that $u_0 \neq u_1$. The hypothesis (2.14) means that $Tu \in \overline{C}_\rho(u_0, u_1)$ for all $u \in \overline{C}_\rho(u_0, u_1)$ since $u_0 \in Fix(T)$. By Proposition 2.13, we have two cases, that is, $Tu \in D_{u_0,\sqrt{\rho}}$ or $Tu \in D_{u_1,\sqrt{\rho}}$.

Case 1. Let $Tu \in D_{u_0,\sqrt{\rho}}$ and $u \in \overline{C}_\rho(u_0, u_1)$ be arbitrary but fixed. If $Tu \neq u$, then using the condition (ζ_2) together with the definition of the number ρ, we obtain

$$\begin{aligned}
\zeta(d(Tu, u), d(Tu, u_0)) &< d(Tu, u_0) - d(Tu, u)\\
&\leq \sqrt{\rho} - d(Tu, u)\\
&\leq \sqrt{\rho} - \rho\\
&\leq 0,
\end{aligned}$$

and so

$$\zeta(d(Tu, u), d(Tu, u_0)) < 0,$$

which conflicts by the hypothesis (2.1) (since $\rho \geq 1$ we have $\sqrt{\rho} - \rho \leq 0$).

Case 2. Let $Tu \notin D_{u_0,\sqrt{\rho}}$ and $Tu \in D_{u_1,\sqrt{\rho}}$. For any arbitrary but fixed point $u \in \overline{C}_\rho(u_0, u_1)$, if $Tu \neq u$, then using the condition (ζ_2) and triangle inequality together with the definition of the number ρ, we obtain

$$\begin{aligned}
\zeta(d(Tu, u), d(Tu, u_0)) &< d(Tu, u_0) - d(Tu, u)\\
&\leq d(Tu, u_0) - \rho\\
&\leq d(Tu, u_1) + d(u_1, u_0) - \rho\\
&\leq \sqrt{\rho} + (\rho - \sqrt{\rho}) - \rho = 0
\end{aligned}$$

and so

$$\zeta(d(Tu, u), d(Tu, u_0)) < 0,$$

which conflicts by the hypothesis (2.1).

Hence we have $Tu = u$ in both cases. Since $u \in \overline{C}_\rho(u_0, u_1)$ is an arbitrary point then we conclude that $\overline{C}_\rho(u_0, u_1) \subset Fix(T)$. □

Remark 2.15. Now we consider the case $\rho = 0$ separately. If $\rho = 0$, then we get $u_0 = u_1$ and $C_\rho(u_0, u_1) = C_{u_0,\rho} = \{u_0\}$. Since T is a $\mathcal{Z}_c$-contraction with $u_0 \in X$, we know that $u_0 \in Fix(T)$.

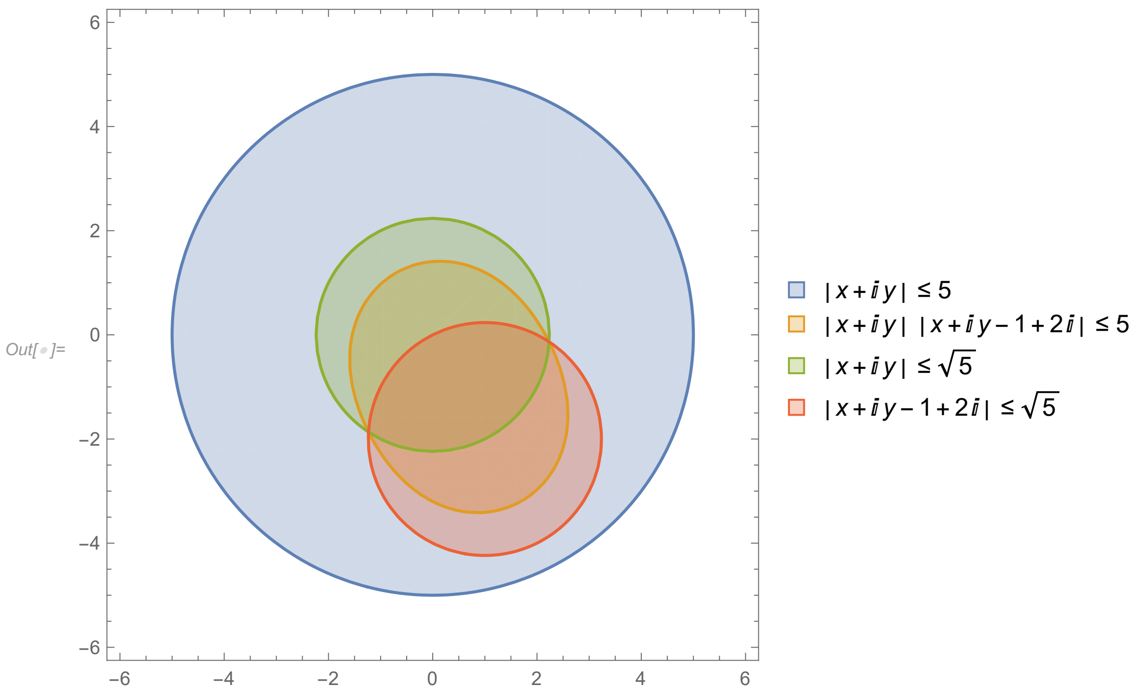

Figure 2. $D_{0,5}$, the fixed disc obtained by Theorem 2.2 for the self-mapping T defined in (2.15), the fixed Cassini curve $C_5(0, 1-2i)$ and the discs $D_{0,\sqrt{5}}$, $D_{1-2i,\sqrt{5}}$.

Example 2.16. Let $X = \mathbb{C}$ be endowed with the usual metric d and consider the self-mapping $T : X \to X$ defined by

$$Tz = \begin{cases} z & ; \quad |z| \leq 5 \\ 2z & ; \quad |z| > 5 \end{cases}, \tag{2.15}$$

for all $z = x + iy \in \mathbb{C}$. Then T is a $\mathcal{Z}_c$-contraction with $z_0 = 0$ and the function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{3}{4}s - t$. Indeed, we have $d(z, Tz) = |z| > 0$ for all z with $|z| > 5$ and

$$\begin{aligned} \zeta(d(Tz,z), d(Tz,z_0)) &= \zeta(|z|, 2|z|) = \frac{3}{4}.2|z| - |z| \\ &= \frac{1}{2}|z| > 0. \end{aligned}$$

We find

$$\begin{aligned} \rho &= \inf\{d(z,Tz) \mid Tz \neq z, z \in \mathbb{C}\} \\ &= \inf\{|z| : |z| > 5\} = 5. \end{aligned}$$

The condition $0 < d(Tz, 0) \leq 5$ holds for all $z \in D_{0,5} - \{0\}$. Consequently, the disc $D_{0,5} = \{z \in \mathbb{C} : |z| \leq 5\}$ is a fixed disc of T by Theorem 2.2.

Now, we consider the Cassini curve $C_5(0, 1-2i)$. It is clear that T satisfies the conditions of Theorem 2.14 and we have $\overline{C}_5(0, 1-2i) \subset Fix(T)$ (see Figure 2).

3. Suzuki type Z_c-contractions and the fixed-figure problem

In this section, we introduce a Suzuki type $\mathcal{Z}_c$-contractive mapping.

Definition 3.1. Let (X,d) be a metric space, $T: X \to X$ be a self-mapping and $\zeta \in \mathcal{Z}$ be any simulation function. Then, T is said to be a Suzuki type $\mathcal{Z}_c$-contraction with respect to the simulation function ζ if there exists an $u_0 \in X$ such that the following condition holds for each $u \in X$ with $u \neq u_0$:

$$\frac{1}{2}d(u,u_0) < d(u,Tu) \Rightarrow \zeta\left(d(Tu,u), d(Tu,u_0)\right) \geq 0. \tag{3.1}$$

If T is a Suzuki type $\mathcal{Z}_c$-contraction with respect to the simulation function ζ, then

$$\frac{1}{2}d(u,u_0) < d(u,Tu) \Rightarrow d(Tu,u) < d(Tu,u_0), \tag{3.2}$$

for each $u \in X$ with $u \neq u_0$. Indeed, if $Tu = u$ then the inequality (3.2) is obvious. If $Tu \neq u$ then $d(Tu,u) > 0$ and by the definition of a Suzuki type $\mathcal{Z}_c$-contraction map and the condition (ζ_2), we get

$$0 \leq \zeta\left(d(Tu,u), d(Tu,u_0)\right) < d(Tu,u_0) - d(Tu,u),$$

and so Equation (3.2) is satisfied.

Remark 3.2. It is clear that every $\mathcal{Z}_c$-contraction map is also a Suzuki type $\mathcal{Z}_c$-contraction map. But, a Suzuki type $\mathcal{Z}_c$-contraction map may not be a $\mathcal{Z}_c$-contraction map (see Example 3.6). Hence, Theorem 3.3 is a generalization of Theorem 2.2.

Theorem 3.3. *If T is a Suzuki type $\mathcal{Z}_c$-contraction with respect to the simulation function ζ with $u_0 \in X$ and the condition $0 < d(Tu,u_0) \leq \rho$ holds for every $u \in D_{u_0,\rho} - \{u_0\}$ then T fixes the disc $D_{u_0,\rho}$.*

Proof. If $\rho = 0$, then $D_{u_0,\rho} = \{u_0\}$. If possible assume that $Tu_0 \neq u_0$ then $d(Tu_0,u_0) > 0$ and as $\frac{1}{2}d(u_0,u_0) < d(Tu_0,u_0)$ then by Definition 3.1 and the condition (ζ_2) of simulation function

$$0 \leq \zeta\left(d(Tu_0,u_0), d(Tu_0,u_0)\right) < d(Tu_0,u_0) - d(Tu_0,u_0) = 0,$$

which contradicts our assumption. Hence it should be $Tu_0 = u_0$.

Let $\rho \neq 0$ and $u \in D_{u_0,\rho}$ be such that $Tu \neq u$. By (2.3) we have $0 < \rho \leq d(Tu,u)$ and so

$$\frac{1}{2}d(u,u_0) \leq \frac{\rho}{2} \leq \frac{d(Tu,u)}{2} < d(Tu,u).$$

Now, by (3.1) and the condition (ζ_2) we find

$$\begin{aligned} 0 \leq \zeta(d(Tu,u), d(Tu,u_0)) &< d(Tu,u_0) - d(Tu,u) \\ &\leq \rho - d(Tu,u) \\ &\leq \rho - \rho = 0. \end{aligned}$$

This is a contradiction. So we get $Tu = u$, for all $u \in D_{u_0,\rho}$. Hence, the Suzuki type $\mathcal{Z}_c$-contraction map T fixes the disc $D_{u_0,\rho}$. □

Corollary 3.4. *Let T be a Suzuki type $\mathcal{Z}_c$-contraction map. If T satisfies the condition $0 < d(Tu,u_0) \leq r$ for each $u \in C_{u_0,r}$ for $r \leq \rho$, then T fixes the circle $C_{u_0,r}$.*

Example 3.5. Let $X = \mathbb{R}$ and d be the usual metric on the set X. Define the self-mapping $T: X \to X$ as

$$Tu = \begin{cases} u & ; \quad u \in [-2,2] \\ 2u & ; \quad u \in (-\infty,-2) \cup (2,\infty) \end{cases},$$

for every $u \in \mathbb{R}$, then the self-mapping T is a Suzuki type $\mathcal{Z}_c$-contraction with $u_0 = 0$ and the simulation function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{2}{3}s - t$.
Indeed, for any u with $Tu \neq u$ and implies $\frac{1}{2}d(u,u_0) = \frac{|u|}{2} < d(u,Tu) = |u|$, we obtain

$$\begin{aligned}\zeta\left(d(Tu,u), d(Tu,u_0)\right) &= \zeta\left(|u|, 2|u|\right) = \frac{2}{3}.2|u| - |u| \\ &= \frac{1}{3}|u| > 0.\end{aligned}$$

Using (2.3) we get

$$\begin{aligned}\rho &= \inf\{d(u,Tu) \mid Tu \neq u, u \in X\} \\ &= \inf\{|u| \mid |u| > 2\} = 2.\end{aligned}$$

By Theorem 3.3, T fixes the disc $D_{0,2}$. It is clear that T is also a $\mathcal{Z}_c$-contraction map with the same simulation function and $u_0 = 0$.

Now, consider the map $T_\mu : X \to X$ as defined below

$$T_\mu u = \begin{cases} u & ; \quad u_0 - \mu \leq u \leq u_0 + \mu \\ 3u_0 & ; \quad \text{otherwise} \end{cases},$$

for every $u \in \mathbb{R}$ with $u_0 > 0$ and $\mu > 4u_0$. By the condition (ζ_2) we can say that T_μ is not a Suzuki type $\mathcal{Z}_c$-contraction with $u_0 \in X$ and for any $\zeta \in \mathcal{Z}$. Because for $T_\mu u \neq u$, $\frac{1}{2}d(u,u_0) < d(u,T_\mu u)$ and by the condition (ζ_2)

$$\begin{aligned}\zeta(d(T_\mu u,u), d(T_\mu u,u_0)) &= \zeta(d(3u_0,u), d(3u_0,u_0)) \\ &= \zeta(|3u_0 - u|, |2u_0|) \\ &< 2u_0 - |3u_0 - u| < 0.\end{aligned}$$

Hence, T_μ is not a Suzuki type $\mathcal{Z}_c$-contraction map, still it fixes the disc $D_{u_0,\rho}$. Which concluding that the converse of Theorem 3.3 may not be true.

Example 3.6. Let $X = \mathbb{R}$ and d be the usual metric on X. Define the self-mapping T on X as

$$Tu = \begin{cases} u & ; \quad u \leq 1 \\ u+1 & ; \quad 1 < u < 2 \\ \frac{u}{2} & ; \quad u \geq 2 \end{cases},$$

for each $u \in \mathbb{R}$. Then T is a Suzuki type $\mathcal{Z}_c$-contraction with $\rho = 1$, $u_0 = 0$ and the simulation function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{4}{5}s - t$. It is obvious that $d(Tu,0) = d(u,0) \leq 1$, for all $u \in D_{0,1}$. Now, for $u \in \mathbb{R}$ with $Tu \neq u$, implies $\frac{1}{2}d(u,0) = \frac{1}{2}|u| < d(Tu,u) = |u+1-u| = 1$, we have

$$\begin{aligned}\zeta(d(Tu,u), d(Tu,u_0)) &= \zeta(1, |u+1|) \\ &= \frac{4}{5}|u+1| - 1 > 0.\end{aligned}$$

Hence by Theorem 3.3, T fixes the disc $D_{0,1}$.

On the other hand, for any $u \in [2,\infty)$ we have $Tu \neq u$. But (2.1) is not satisfied for any simulation function ζ. Indeed, by the condition (ζ_2) we have

$$\begin{aligned}\zeta(d(Tu,u), d(Tu,u_0)) &= \zeta\left(\frac{|u|}{2}, \left|\frac{u}{2} - 0\right|\right) \\ &< \frac{|u|}{2} - \frac{|u|}{2} = 0,\end{aligned}$$

and this implies that (2.1) is not satisfied for all $u \in [2,\infty)$. Hence T is not a $\mathcal{Z}_c$-contraction map.

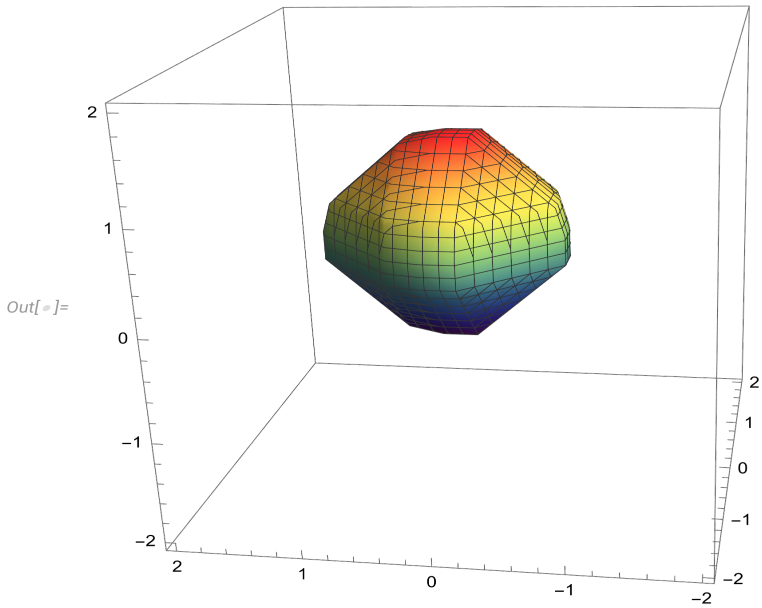

Figure 3. The set $\overline{E}_3((0,0,0),(\frac{1}{2},\frac{1}{2},\frac{1}{2}))$ contained in $Fix(T)$ for the self-mapping T defined in (3.3).

Theorem 3.7. *Let (X,d) be a metric space, $T : X \to X$ be a Suzuki type $\mathcal{Z}_c$-contraction with respect to ζ with $u_0 \in X$. For a point $u_1 \in D_{u_0,\frac{\rho}{2}}$, if the condition*

$$0 < d(Tu,u_0) + d(Tu,u_1) \le \rho$$

holds for all $u \in \overline{E}_\rho(u_0,u_1) - \{u_0\}$, then $Fix(T)$ contains the ellipse $E_\rho(u_0,u_1)$ and the set $\overline{E}_\rho(u_0,u_1)$.

Proof. If $u_0 = u_1$, then ellipse $E_\rho(u_0,u_1)$ becomes the circle $C_{u_0,\frac{\rho}{2}}$ and the set $\overline{E}_\rho(u_0,u_1)$ becomes the disc $D_{u_0,\frac{\rho}{2}}$. By hypothesis we have $0 < d(Tu,u_0) \le \frac{\rho}{2}$, for all $u \in D_{u_0,\frac{\rho}{2}} - \{u_0\}$. Then by Corollary 3.4, we have $D_{u_0,\frac{\rho}{2}} \subset Fix(T)$.
Now, let $u_0 \neq u_1$ and $u \in E_\rho(u_0,u_1)$ be any but fixed. Assume that $Tu \neq u$. As $u \in E_\rho(u_0,u_1)$ i.e. $d(u,u_0)+d(u,u_1) \le \rho$ implies $d(u,u_0) \le \rho \Rightarrow \frac{1}{2}d(u,u_0) < d(u,Tu)$ (by (2.3)). Now using the inequality $0 < \rho \le d(u,Tu)$ and condition (ζ_2) we find

$$\begin{aligned} 0 \le \zeta(d(Tu,u),(Tu,u_0)) &< d(Tu,u_0) - d(Tu,u) \\ &\le d(Tu,u_0) + d(Tu,u_1) - d(Tu,u) \\ &\le \rho - \rho = 0, \end{aligned}$$

which contradicts our assumption. So $Tu = u$, for all $u \in E_\rho(u_0,u_1)$.
By similar arguments, we have also $\overline{E}_\rho(u_0,u_1) \subset Fix(T)$. □

Example 3.8. Let $X = \mathbb{R}^3$ and d be the metric defined by

$$d(u,v) = |u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|,$$

for all $u,v \in \mathbb{R}^3$ where $u = (u_1,u_2,u_3), v = (v_1,v_2,v_3)$. Consider the self-mapping T on the set X defined by

$$Tu = \begin{cases} u & ; \quad \text{if } |u_1|+|u_2|+|u_3| \le 5 \\ \frac{8u}{5} & ; \quad \text{otherwise} \end{cases} \tag{3.3}$$

for all $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. Then T is a Suzuki type $\mathcal{Z}_c$-contraction with $u_0 = \overline{0} = (0,0,0)$ and with the function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{3}{4}s - t$. We have

$$\begin{aligned} d(u,Tu) &= \left|u_1 - \frac{8u_1}{5}\right| + \left|u_2 - \frac{8u_2}{5}\right| + \left|u_3 - \frac{8u_3}{5}\right| \\ &= \left|\frac{3u_1}{5}\right| + \left|\frac{3u_2}{5}\right| + \left|\frac{3u_3}{5}\right| = \frac{3}{5}d(u,\overline{0}) > 0 \end{aligned}$$

and

$$\begin{aligned} \zeta(d(Tu,u), d(Tu,u_0)) &= \zeta\left(\frac{3}{5}d(u,\overline{0}), \frac{8}{5}d(u,\overline{0})\right) \\ &= \frac{3}{5}d(u,\overline{0}) > 0\,, \end{aligned}$$

for all u with $|u_1| + |u_2| + |u_3| > 5$. We find

$$\begin{aligned} \rho &= \inf\{d(u,Tu) \mid Tu \neq u, u \in \mathbb{R}^3\} \\ &= \inf\left\{\frac{3}{5}d(u,\overline{0}) \mid d(u,\overline{0}) > 5\right\} = 3. \end{aligned}$$

Now, consider the ellipse $E_3(\overline{0}, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}))$, clearly T satisfies the condition of Theorem 3.7. Hence, we have $\overline{E}_3(\overline{0}, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) \subset Fix(T)$ (see Figure 3).

Example 3.9. Let $X = \mathbb{C}$ and d be the usual metric. Consider the self-mapping $T : X \to X$ defined by

$$Tz = \begin{cases} z & ; \quad |z| < 3 \\ \frac{15z}{8} & ; \quad \text{otherwise} \end{cases},$$

for all $z \in \mathbb{C}$. Then T is a Suzuki type $\mathcal{Z}_c$-contraction with $z_0 = 0$ and with the simulation function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{4}{5}s - t$. For each z with $|z| \geq 3$, we have

$$d(z,Tz) = \left|z - \frac{15z}{8}\right| = \frac{7}{8}|z| > 0$$

and

$$\begin{aligned} \zeta(d(Tz,z), d(Tz,z_0)) &= \zeta\left(\frac{7}{8}|z|, \frac{15}{8}|z|\right) \\ &= \frac{5}{8}|z| > 0\,. \end{aligned}$$

Then, we find

$$\begin{aligned} \rho &= \inf\{d(z,Tz) \mid Tz \neq z, z \in \mathbb{C}\} \\ &= \inf\left\{\frac{7}{8}|z| \mid |z| \geq 3\right\} = \frac{21}{8}. \end{aligned}$$

Now, consider the ellipse $E_{\frac{21}{8}}(0, \frac{1}{8} + i\frac{1}{8})$, clearly T satisfies the condition of Theorem 3.7. We have $\overline{E}_{\frac{21}{8}}(0, \frac{1}{8} + i\frac{1}{8})) \subset Fix(T)$.

Theorem 3.10. *Let X be a metric space with the metric d and T be a self-mapping on the set X. Suppose that T is a Suzuki type $\mathcal{Z}_c$-contraction with the simulation function ζ and $u_0 \in X$. Suppose $\rho \geq 1$ and for any point $u_1 \in D_{u_0, \rho - \sqrt{\rho}}$, if the condition*

$$0 < d(Tu,u_0)d(Tu,u_1) \leq \rho \tag{3.4}$$

holds for each $u \in \overline{C}_\rho(u_0,u_1) - \{u_0\}$, then $Fix(T)$ contains the Cassini curve $C_\rho(u_0,u_1)$, that is, $C_\rho(u_0,u_1)$ is a fixed Cassini curve of T. Furthermore, we have $\overline{C}_\rho(u_0,u_1) \subset Fix(T)$.

Proof. If $u_0 = u_1$, then the Cassini curve $C_\rho(u_0, u_1)$ becomes the circle $C_{u_0,\sqrt{\rho}}$ and the set $\overline{C}_\rho(u_0, u_1)$ becomes the disc $D_{u_0,\sqrt{\rho}}$. By the hypothesis we have $0 < d(Tu, u_0) \le \sqrt{\rho}$ for all $u \in D_{u_0,\sqrt{\rho}} - \{u_0\}$. Since $\rho \ge 1$, $\sqrt{\rho} \le \rho$ and then by Corollary 3.4 we have $D_{u_0,\sqrt{\rho}} \subset Fix(T)$.
Let $u_0 \ne u_1$ and $u \in \overline{C}_\rho(u_0, u_1)$ be an arbitrary but fixed point. Suppose that $Tu \ne u$. The hypothesis (3.4) means that $Tu \in \overline{C}_\rho\,(u_0, u_1)$ for all $u \in \overline{C}_\rho\,(u_0, u_1)$ since $u_0 \in Fix(T)$. By Corollary 2.13, we have two cases, that is, $Tu \in D_{u_0,\sqrt{\rho}}$ or $Tu \in D_{u_1,\sqrt{\rho}}$.

Case 1. Let $Tu \in D_{u_0,\sqrt{\rho}}$. Since $\rho \ge 1$, $\sqrt{\rho} - \rho \le 0$ and then we have

$$\begin{aligned}\zeta\left(d(Tu,u), d(Tu,u_0)\right) &< d(Tu,u_0) - d(Tu,u)\\ &\le \sqrt{\rho} - d(Tu,u)\\ &\le \sqrt{\rho} - \rho\\ &\le 0\end{aligned}$$

and so

$$\zeta\left(d(Tu,u), d(Tu,u_0)\right) < 0,$$

which conflicts the hypothesis (3.1).

Case 2. Let $Tu \notin D_{u_0,\sqrt{\rho}}$ and $Tu \in D_{u_1,\sqrt{\rho}}$. Then we have

$$\begin{aligned}\zeta\left(d(Tu,u), d(Tu,u_0)\right) &< d(Tu,u_0) - d(Tu,u)\\ &\le (d(Tu,u_1) + d(u_1,u_0)) - d(Tu,u)\\ &\le [\sqrt{\rho} + (\rho - \sqrt{\rho})] - d(Tu,u)\\ &\le \rho - d(Tu,u)\\ &\le \rho - \rho = 0\end{aligned}$$

and so

$$\zeta\left(d(Tu,u), d(Tu,u_0)\right) < 0,$$

which conflicts the hypothesis (3.1).

Hence, we have $Tu = u$. Since $u \in \overline{C}_\rho\,(u_0, u_1)$ is an arbitrary point, then we deduce that $\overline{C}_\rho\,(u_0, u_1) \subset Fix(T)$. □

Example 3.11. Let $X = \mathbb{C}$ and d be the usual metric. Consider the self-mapping $T : X \to X$ defined by

$$Tz = \begin{cases} z & ; \quad |z| < 7 \\ \frac{13}{8}z & ; \quad \text{otherwise} \end{cases},$$

for all $z \in \mathbb{C}$. Then T is a Suzuki type $\mathcal{Z}_c$-contraction with $z_0 = 0$ and with the function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{1}{2}s - t$. We have

$$d(z, Tz) = \left| z - \frac{13}{8}z \right| = \frac{5}{8}\,|z| > 0$$

and

$$\begin{aligned}\zeta(d(Tz,z), d(Tz,z_0)) &= \zeta\left(\frac{5}{8}\,|z|\,, \frac{13}{8}\,|z|\right)\\ &= \frac{3}{16}\,|z| > 0\,,\end{aligned}$$

for all $|z| \ge 7$. We find

$$\begin{aligned}\rho &= \inf\{d(z,Tz) : Tz \ne z, z \in \mathbb{C}\}\\ &= \inf\left\{\frac{5}{8}\,|z| : |z| \ge 7\right\} = \frac{35}{8}.\end{aligned}$$

Now, consider the Cassini curve $C_{\frac{35}{8}}(0,2)$, clearly T satisfies the hypothesis of Theorem 3.10. Hence, $C_{\frac{35}{8}}(0,2)$ is a fixed Cassini curve of T. We have $C_{\frac{35}{8}}(0,2) \subset Fix(T)$.

4. Activation functions having fixed discs

It is well-known that theoretical fixed point results (e.g., Banach fixed point theorem, Brouwer's fixed point theorem, Kakutani's fixed point theorem) are widely used in the study of various types of neural networks (see, for instance, [2, 3] and the references therein). On the other hand, as stated in [12], most common functions used as activation function in neural networks are those mappings having fixed-discs. One of them is the rectified linear unit (or ReLU) activation function defined as

$$\mathrm{ReLU}(x) = \begin{cases} x & ; \quad x \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases},$$

for a real number $x \in \mathbb{R}$. In [6], complex ReLU function zReLU was defined by

$$\mathrm{zReLU}(z) = \begin{cases} z & ; \quad z \in \mathcal{A} \\ 0 & ; \quad \text{otherwise} \end{cases},$$

where $\mathcal{A} \subseteq \mathbb{C}$ is connected, and the case $\mathcal{A} = \{z \in \mathbb{C} : \arg(z) \in [0, \frac{\pi}{2}]\}$ is considered.

Since activation functions are the primary neural networks decision-making units, the selection of the activation function is essential in the design of a neural network. There are a wide range of activation functions. Here, we propose a new type of activation function for complex valued neural networks. We construct a one-parameter generalization of the zReLU function. To do this, first we define the set $\mathcal{A}_\mu$ as

$$\mathcal{A}_\mu = \left\{z \in \mathbb{C} : \arg(z) \in \left[0, \frac{\pi}{2}\right]\right\} \cup \left\{z \in \mathbb{C} : |z| \leq \mu \text{ and } \arg(z) \in \left(\frac{\pi}{2}, 2\pi\right)\right\},$$

where $\mu > 1$. Define the function f_μ by

$$f_\mu(z) = \begin{cases} z & ; \quad z \in \mathcal{A}_\mu \\ \mu z & ; \quad \text{otherwise} \end{cases}, (1 < \mu \leq 2)$$

and consider the simulation function $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ defined as

$$\zeta(t, s) = \lambda s - t, \lambda \in \left[1 - \frac{1}{\mu}, 1\right). \tag{4.1}$$

Then it easy to check that f_μ is a $\mathcal{Z}_c$-contraction (and hence, is a Suzuki type $\mathcal{Z}_c$-contraction) with the simulation function ζ defined in (4.1) and $u_0 = 0$.

The real version of f_μ is the following

$$f_\mu(x) = \begin{cases} x & ; \quad x \geq -\mu \\ \mu x & ; \quad \text{otherwise} \end{cases}, (1 < \mu \leq 2).$$

Besides the theoretical fixed figure results obtained in the previous sections, the fixed disc $D_{0,\rho}$ and the fixed figures contained in it, can have some applications in neural network studies. Such geometric approaches are important for the study of some real-life problems. For example, in [20], a wind prediction system for the wind power generation using ensemble of multiple complex extreme learning machines (C-ELM) was presented via the theory of conformal mapping on the complex plane. Orthogonal families of ellipses and hyperbolas were used to make a better decision making.

5. Concluding remarks

In this manuscript, we have considered the geometric aspects of the fixed point set of a self-mapping with various forms such a fixed disc, fixed ellipse and fixed Cassini curve problems. The general form of these problems is known as the "fixed figure problem". For such kind geometric problems, we have dealt with the $\mathcal{Z}_c$-contraction and Suzuki type $\mathcal{Z}_c$-contraction self maps and using these notions, we have derived the fixed ellipse, fixed Cassini curve and fixed disc results. As future work, using similar approaches, various fixed figures results such as fixed Apollonius circle and fixed hyperbola can be investigated. One can also derive it for multivalued contraction mapping.

Properties of simulation functions have been appeared in the literature in many aspects. For instance, one of the newest generalization of the Banach contraction through the notions of the generalized F-contraction, simulation function, and admissible function was introduced in [4]. The existence and uniqueness of fixed points for a self-mapping on complete metric spaces were investigated by the new constructed contraction. Here, we have proved some existence theorems for some special fixed figures via the help of the number ρ defined in (2.3). Hence, it is natural to investigate some uniqueness theorems for fixed figures.

On the other hand, recently, the notion of a φ-fixed point is introduced as the generalization of a fixed point of a self-mapping in [8]. A φ-fixed point is a fixed point of a mapping T such that it is also a zero of a given function φ (see [8] for more details). Then, φ-fixed point results for self-mappings defined in metric or generalized metric spaces have been intensively studied using different approaches (for example, see [18], [16] and the reference therein). In [16], an open problem concerning to the geometric properties of non-unique φ-fixed points have been considered. The proposed open problem is the investigation of the existence and uniqueness of φ-fixed circles (resp. φ-fixed discs) for various classes of self-mappings (see [14] and [16]). Then, a natural generalization of this open problem is the investigation of the existence and uniqueness of φ-fixed figures (e.g. φ-fixed circle, φ-fixed disc, φ-fixed ellipse, φ-fixed Cassini curve) for various classes of self-mappings. By means of the notions of $\mathcal{Z}_c$-contraction and Suzuki type $\mathcal{Z}_c$-contraction, it is possible to provide a solution to this problem. Applications of these kind theoretical results can also be investigated to various real-life problems.

Acknowledgment. Dhananjay Gopal thanks the Department of Science and Technology, New Delhi, India, for approving the proposal under the FIST Program scheme (Ref. No. SR/FST/MS/2022/122 dated 19/12/2022). He also thanks the administration of GGV Bilaspur for research support.

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