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Research Article

Spectrum, homomorphisms and multipliers of Lau product of Banach algebras

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Abstract

Given Banach algebras A, B and a continuous homomorphism $\theta : B \longrightarrow A$ with $\|\theta\| \leq 1$, we obtain characterization of spectrum, homomorphisms and multipliers of $A \times_\theta B$, which is a strongly splitting Banach algebra extension of B by A. Also we characterize the semisimplicity of these algebras.

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1. Introduction

Let A and B be Banach algebras and $\phi : B \longrightarrow \mathbb{C}$ be a multiplicative linear functional. Then the direct product $A \times B$ equipped with the algebra multiplication

$$(a, b)(u, v) = (au + \phi(b)u + \phi(v)a, bv), \qquad (a, b), (u, v) \in A \times B,$$

and with the l^1-norm is a Banach algebra which is called the Lau product of A and B and is denoted by $A \times_\phi B$.

This type of product was introduced by Lau [9] for certain class of Banach algebras and was extended by Sangani Monfared [11] for the general case.

If we allow $\phi = 0$, then we obtain the usual direct product of Banach algebras, and when $B = \mathbb{C}$ and $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ is the identity map, $A \times_\phi \mathbb{C}$ coincides with the unitization of A.

Some basic properties of $A \times_\phi B$ such as characterization of Gelfand space, topological center, amenability, ideal structure and minimal idempotent are investigated in [11]. Also, characterization of multipliers of these product discussed in [3] and [14]. Additionally, many Banach algebras properties of $A \times_\phi B$ are studied in [1], [6], [8], [10], [13], for example.

Bhatt and Dabhi in [2] introduced a new type of Lau product. Let $\theta : B \longrightarrow A$ be a continuous homomorphism between Banach algebras with $\|\theta\| \leq 1$. Then $A \times B$ with the multiplication

$$(a, b)(u, v) = (au + \theta(b)u + a\theta(v), bv),$$

and with the l^1-norm turns into a Banach algebra, which is denoted by $A \times_\theta B$.

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The product $A \times_\theta B$ provides not only new examples in Banach algebras by themselves, but also it has the potential to serve as counter-examples in different branches of functional and harmonic analysis.

Let $\Delta(A)$ be the set of all multiplicative linear functionals on Banach algebra A. Note that every $\phi \in \Delta(A)$ is continuous and $\|\phi\| \leq 1$, [5]. When A is unital with identity e_A and $\phi \in \Delta(B)$, then $\theta : B \longrightarrow A$ defined by $\theta(b) = \phi(b)e_A$ is a continuous homomorphism with $\|\theta\| \leq 1$. Therefore in this case the Lau product coincides with $A \times_\theta B$.

We remark that in $A \times_\theta B$, we identify $A \times \{0\}$ with A and $\{0\} \times B$ with B. Then A is a closed ideal while B is a closed subalgebra of $A \times_\theta B$, and $(A \times_\theta B)/A$ is isometric isomorphism with B.

Spelitting of Banach algebra extensions has been a major tool in the study of Banach algebras. For example, module extensions as generalizations of Banach algebras extensions where introduced and studied by Gourdeau [7]. On the other hand, $A \times_\theta B$ is a strongly splitting Banach algebra extension of B by A that exhibits many properties that are not shared, in general, by arbitrary strongly splitting extensions. For example, commutativity is not preserved by a general strongly splitting extension. However, $A \times_\theta B$ is commutative if and only if both A and B are commutative, [11, Proposition 2.3].

Let X be an A-bimodule, Y be an B-bimodule and $\theta : B \longrightarrow A$ be a homomorphism. We say that $\sigma : Y \longrightarrow X$ is a right (left) θ-module homomorphism if for all $b \in B$ and $y \in Y$,

$$\sigma(by) = \theta(b)\sigma(y), \qquad (\sigma(yb) = \sigma(y)\theta(b)).$$

In particular, if $X = A$ and $Y = B$, then σ is called right (left) θ-multiplier. It is clear that each multiplier is a special case of a θ-multiplier with $\theta = id$, the identity map on A.

Example 1.1. *Let*

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{C} \right\},$$

and define $\theta, \sigma : A \longrightarrow A$ by

$$\theta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \sigma\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $x, y \in A$,

$$\sigma(xy) = \theta(x)\sigma(y) = \sigma(x)\theta(y).$$

Therefore σ is a θ-multiplier.

The purpose of the present paper is to investigate the spectrum, homomorphisms and multipliers for Banach algebras induced by Lau product of Banach algebras defined by a Banach algebra morphism.

Throughout the paper, we assume that A and B are Banach algebras and $\theta : B \longrightarrow A$ is a continuous homomorphism with $\|\theta\| \leq 1$.

2. Spectrum of Lau product

The spectrum of an element $a \in A$ is defined as

$$Sp_A(a) = \{\lambda \in \mathbb{C} : \ \lambda e_A - a \notin Inv(A)\},$$

where $Inv(A)$ is the set of all invertible elements of A, and the spectral radius $r_A(a)$ of an element a is defined as $r_A(a) = \sup\{|\lambda| : \ \lambda \in Sp_A(a)\}$.

It should be note that if A is not unital, then $Sp_A(a) = Sp_{A^\sharp}(a)$, where $A^\sharp$ stands the unitization of A, [12].

Theorem 2.1. *Let A and B be unital Banach algebras. Then*

$$Inv(A \times_\theta B) \cong Inv(A) \times Inv(B), \qquad (homeomorphism). \tag{2.1}$$

Proof. Let e_A and e_B be unit of A and B, respectively. Then $(e_A - \theta(e_B), e_B)$ is the unit element of $A \times_\theta B$. Define

$$h : Inv(A) \times Inv(B) \longrightarrow Inv(A \times_\theta B),$$

by $h(a, b) = (a - \theta(b), b)$. Let $a \in Inv(A)$ and $b \in Inv(B)$, then there exists $u \in A$ and $v \in B$ such that $au = e_A$ and $bv = e_B$. So

$$(a - \theta(b), b)(u - \theta(v), v) = (au - \theta(bv), bv) = (e_A - \theta(e_B), e_B).$$

Therefore $(a - \theta(b), b)$ has a right inverse.

Similarly, $(a - \theta(b), b)$ has a left inverse and thus h is well-defined. It is easy to check that h is linear and one to one.

Suppose that $(a, b) \in Inv(A \times_\theta B)$. Hence there exist $(u, v) \in Inv(A \times_\theta B)$ such that

$$(au + \theta(b)u + a\theta(v), bv) = (a, b)(u, v) = (e_A - \theta(e_B), e_B). \tag{2.2}$$

It follows from (2.2) that $bv = e_B$ and

$$au + \theta(b)u + a\theta(v) = e_A - \theta(e_B) = e_A - \theta(bv).$$

Therefore

$$au + \theta(b)u + a\theta(v) + \theta(bv) = e_A.$$

Since

$$(a + \theta(b))(u + \theta(v)) = au + \theta(b)u + a\theta(v) + \theta(bv) = e_A,$$

hence $a + \theta(b)$ and b have right inverse in A and B, respectively. Similarly, $a + \theta(b)$ and b have left inverse in A and B, respectively. Thus, $a + \theta(b) \in Inv(A)$, $b \in Inv(B)$ and

$$h(a + \theta(b), b) = (a + \theta(b) - \theta(b), b) = (a, b).$$

This means that h is surjective. For $(a, b) \in Inv(A) \times Inv(B)$, we have

$$\begin{aligned} \|h(a, b)\| &= \|(a - \theta(b), b)\| \\ &= \|a - \theta(b)\| + \|b\| \\ &\le \|a\| + \|\theta\|\|b\| + \|b\| \\ &\le (1 + \|\theta\|)(\|a\| + \|b\|). \end{aligned}$$

Therefore, $\|h\| \le 2$ and h is continuous. Similarly, h^{-1} is continuous. This finishes the proof. □

Our next result concerns the spectrum in $A \times_\theta B$.

Theorem 2.2. *For Banach algebras A and B, we have*

$$Sp_{A \times_\theta B}(a, b) = Sp_A(a + \theta(b)) \cup Sp_B(b). \tag{2.3}$$

Proof. It is enough to show that

$$Sp_{A \times_\theta B}(a, b) = Sp_A(a + \theta(b)) \cup Sp_B(b),$$

when A and B are unital.

Let $\lambda \notin Sp_A(a) \cup Sp_B(b)$. Then $\lambda e_A - a \in Inv(A)$ and $\lambda e_B - b \in Inv(A)$. Hence by Theorem 2.1, we get

$$(\lambda e_A - a - \theta(\lambda e_B - b), \lambda e_B - b) \in Inv(A \times_\theta B), \tag{2.4}$$

which yields that

$$\lambda(e_A - \theta(e_B), e_B) - (a - \theta(b), b) \in Inv(A \times_\theta B).$$

Therefore, $\lambda \notin Sp_{A \times_\theta B}(a - \theta(b), b)$.

Now suppose that $\lambda \notin Sp_A(a+\theta(b)) \cup Sp_B(b)$. Then by the above argument,

$$\lambda \notin Sp_{A\times_\theta B}(a+\theta(b)-\theta(b),b) = Sp_{A\times_\theta B}(a,b).$$

This means that

$$Sp_{A\times_\theta B}(a,b) \subseteq Sp_A(a+\theta(b)) \cup Sp_B(b).$$

For the converse, let $\lambda \notin Sp_{A\times_\theta B}(a,b)$. Then

$$\lambda(e_A-\theta(e_B),e_B)-(a,b) \in Inv(A\times_\theta B). \tag{2.5}$$

But

$$\begin{aligned}\lambda(e_A-\theta(e_B),e_B)-(a,b) &= \big(\lambda e_A-\lambda\theta(e_B)-a,\lambda e_B-b\big)\\ &= \big(\lambda e_A-\lambda\theta(e_B)-a+\lambda\theta(e_B)-\theta(b)-\theta(\lambda e_B-b),\lambda e_B-b\big)\\ &= \big(\lambda e_A-(a+\theta(b))-\theta(\lambda e_B-b),\lambda e_B-b\big).\end{aligned}$$

It follows from (2.5) and the above equality that

$$\lambda e_A-(a+\theta(b)) \in Inv(A), \quad \textit{and} \quad \lambda e_B-b\in Inv(B).$$

Consequently, $\lambda \notin Sp_A(a+\theta(b))$ and $\lambda \notin Sp_B(b)$. Thus,

$$Sp_A(a+\theta(b)) \cup Sp_B(b) \subseteq Sp_{A\times_\theta B}(a,b).$$

This completes the proof. □

The next corollary appeared in [4, Lemma 2.5] for commutative Banach algebras. Here as a consequence of Theorem 2.2, we deduce it for the general case.

Corollary 2.3. *Let A and B be Banach algebras. Then*

$$r_{A\times_\theta B}(a,b) = \max\{r_A(a+\theta(b)),r_B(b)\}.$$

3. Homomorphisms of Lau product

In this section, we assume that E and F are Banach algebras, and $\sigma \in Hom(F,E)$ with $\|\sigma\| \le 1$, where $Hom(F,E)$ denotes the set of all homomorphisms from F into E.

Let $p_A : A\times_\theta B \longrightarrow A$ and $p_B : A\times_\theta B \longrightarrow B$ be the usual projections which are defined by $p_A(a,b)=a$ and $p_B(a,b)=b$, respectively.

Recall that an A-bimodule X is called left (right) *faithful* if the condition $ax=0$ ($xa=0$) for $x\in X$ implies that $x=0$. The Banach algebra A is faithful, if it is faithful as an A-bimodule over itself.

Theorem 3.1. *Suppose that*

$$f_1 : A\longrightarrow E, \quad f_2 : B \longrightarrow F, \quad f : A\times_\theta B \longrightarrow E\times_\sigma F,$$

where $f=(f_1\circ p_A, f_2\circ p_B)$. Then,

(i) *If $f\in Hom(A\times_\theta B, E\times_\sigma F)$, then $f_1\in Hom(A,E)$ and $f_2 \in Hom(B,F)$.*

(ii) *If $f_1\in Hom(A,E)$ and $f_2\in Hom(B,F)$, f_1 is surjective and E is faithful, then $f\in Hom(A\times_\theta B,E\times_\sigma F)$ if and only if $\sigma\circ f_2 = f_1\circ\theta$.*

Proof. (i) Suppose that $f\in Hom(A\times_\theta B,E\times_\sigma F)$, then for all $(a,b),(u,v)\in A\times_\theta B$,

$$f\big((a,b)(u,v)\big) = f(a,b)f(u,v). \tag{3.1}$$

By using $f=(f_1\circ p_A, f_2\circ p_B)$ we get

$$\begin{aligned}f\big((a,b)(u,v)\big) &= f\big(au+\theta(b)u+a\theta(v),bv\big)\\ &= \big(f_1(au)+f_1(\theta(b)u)+f_1(a\theta(v)),f_2(bv)\big).\end{aligned}$$

On the other hand,

$$\begin{aligned} f(a,b)f(u,v) &= (f_1(a), f_2(b))(f_1(u), f_2(v)) \\ &= (f_1(a)f_1(u) + \sigma(f_2(b))f_1(u) + f_1(a)\sigma(f_2(v)), f_2(b)f_2(v)). \end{aligned}$$

Hence by the above two equalities and (3.1), we have $f_2(bv) = f_2(b)f_2(v)$ and

$$f_1(au) + f_1(\theta(b)u) + f_1(a\theta(v)) = f_1(a)f_1(u) + \sigma(f_2(b))f_1(u) + f_1(a)\sigma(f_2(v)). \tag{3.2}$$

By taking $b = v = 0$ in (3.2), we conclude that $f_1(au) = f_1(a)f_1(u)$ for all $a, u \in A$. Thus, f_1 and f_2 are homomorphisms.

(ii) Assume that $f_1 \in Hom(A, E)$, $f_2 \in Hom(B, F)$ and let $f : A \times_\theta B \longrightarrow E \times_\sigma F$ be a homomorphism. Then by the equality (3.2),

$$f_1(\theta(b))f_1(u) + f_1(a)f_1(\theta(v)) = \sigma(f_2(b))f_1(u) + f_1(a)\sigma(f_2(v)). \tag{3.3}$$

If we take $a = 0$ in (3.3), we obtain

$$\big(f_1(\theta(b)) - \sigma(f_2(b))\big)f_1(u) = 0, \quad u \in A, b \in B.$$

Since f_1 is surjective and E is faithful, we get $\sigma \circ f_2 = f_1 \circ \theta$. The converse is similar. □

If $E = F = \mathbb{C}$ and $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ is the identity map, then we get the following result.

Corollary 3.2. *Let*

$$f_1 : A \longrightarrow \mathbb{C}, \quad f_2 : B \longrightarrow \mathbb{C}, \quad f : A \times_\theta B \longrightarrow \mathbb{C}^2,$$

where $f = (f_1 \circ p_A, f_2 \circ p_B)$. *Then,*

(i) *If* $f \in Hom(A \times_\theta B, \mathbb{C}^2)$, *then* $f_1 \in \Delta(A)$ *and* $f_2 \in \Delta(B)$.

(ii) *If* $0 \neq f_1 \in \Delta(A)$ *and* $f_2 \in \Delta(B)$, *then* $f \in Hom(A \times_\theta B, \mathbb{C}^2)$ *if and only if* $f_2 = f_1 \circ \theta$.

It should be pointed out that the topology of $\Delta(A)$ is the induced weak* topology from A^*, the dual space of A. Note that $\Delta(A)$ is a locally compact Hausdorff space and it is compact, whether A is unital, [5].

It is shown in [11, Proposition 2.4] that if A and B are commutative and $\theta \in \Delta(B)$, then $\Delta(A \times_\theta B) = E \cup F$, where

$$E = \{(\phi, \theta) : \ \phi \in \Delta(A)\}, \ \textit{and} \quad F = \{(0, \psi) : \ \psi \in \Delta(B)\}.$$

The following result shows that $\Delta(A)$ and $Hom(A \times_\theta B, \mathbb{C}^2)$ are homeomorphic as a two locally compact Hausdorff spaces.

Corollary 3.3. *For Banach algebras A and B,*

$$\Delta(A) \cong Hom(A \times_\theta B, \mathbb{C}^2).$$

Proof. Define $h : \Delta(A) \longrightarrow Hom(A \times_\theta B, \mathbb{C}^2)$ via

$$h(f_1) = (f_1 \circ p_A, f_1 \circ \theta \circ p_B).$$

Then h is linear and bijective. Moreover, both h and h^{-1} are continuous. Indeed, for each $f_1 \in \Delta(A)$, we have

$$\begin{aligned} \|h(f_1)\| &= \|(f_1 \circ p_A, f_1 \circ \theta \circ p_B)\| \\ &\leq \|f_1 \circ p_A\| + \|f_1 \circ \theta \circ p_B\| \\ &= \|f_1\| + \|f_1 \circ \theta\| \\ &\leq \|f_1\|(1 + \|\theta\|) \leq 2\|f_1\|. \end{aligned}$$

Thus, h is continuous. The continuity of h^{-1} is obvious. □

The *Jacobson radical* of an algebra A, denoted by $radA$, is the intersection of maximal modular left (right) ideals of A. The algebra A is called *semisimple* if $radA = \{0\}$. If A is a commutative Banach algebra, then

$$radA = \bigcap\{ker\phi : \quad \phi \in \Delta(A)\}.$$

Lemma 3.4. *Let A and B be commutative and $(a, b) \in rad(A \times_\theta B)$. Then $f(a, b) = (0, 0)$ for each $f \in Hom(A \times_\theta B, \mathbb{C}^2)$.*

Proof. Let $(a, b) \in rad(A \times_\theta B)$, then $g(a, b) = 0$ for each $g \in \Delta(A \times_\theta B)$. Assume to contrary that there exist $f \in Hom(A \times_\theta B, \mathbb{C}^2)$ such that $f(a, b) \neq (0, 0)$. By Corollary (3.3), there exist $f_1 \in \Delta(A)$ such that

$$(f_1(a), f_1 \circ \theta(b)) = f(a, b) \neq (0, 0).$$

Therefore $f_1(a) \neq 0$ or $f_1 \circ \theta(b) \neq 0$.

Case I: Let $f_1(a) + f_1 \circ \theta(b) \neq 0$.
Let $g : A \times_\theta B \longrightarrow \mathbb{C}$ defined by $g(a, b) = f_1(a) + f_1 \circ \theta(b)$. Then $g \in \Delta(A \times_\theta B)$ and $g(a, b) \neq 0$.

Case II: Let $f_1(a) + f_1 \circ \theta(b) = 0$.
Then $f_1 \circ \theta(b) \neq 0$. Define $g : A \times_\theta B \longrightarrow \mathbb{C}$ via $g(a, b) = f_1 \circ \theta(b)$. Then $g \in \Delta(A \times_\theta B)$ and $g(a, b) \neq 0$.

In both cases we obtain a contradiction. Thus, we reach the desired result. □

The following result is due to Sangani Monfared [11, Theorem 3.1] when $\theta \in \Delta(B)$, see also [2, Corollary 2.2]. Here we outline an alternative proof for it with direct method.

Theorem 3.5. *Let A and B be commutative. Then $A \times_\theta B$ is semisimple if and only if A and B are semisimple.*

Proof. Suppose that $A \times_\theta B$ is semisimple and let $b \in rad(B)$. Then $f_2(b) = 0$, for each $f_2 \in \Delta(B)$. Since $f_1 \circ \theta \in \Delta(B)$, for every multiplicative linear functional f_1 on A, so $(f_1 \circ \theta)(b) = 0$. Therefore, for all $f \in Hom(A \times_\theta B, \mathbb{C}^2)$,

$$f(\theta(b), b) = (f_1(\theta(b)), f_1 \circ \theta(b)) = (0, 0).$$

Noticing that

$$\Delta(A \times_\theta B) \subseteq Hom(A \times_\theta B, \mathbb{C}^2),$$

hence for every $g \in \Delta(A \times_\theta B)$, we have $g(\theta(b), b) = 0$. Thus, $(\theta(b), b) \in rad(A \times_\theta B) = \{0\}$ and hence $b = 0$. Consequently, B is semisimple.

Now we prove that $rad(A) = \{0\}$. To see this, let $a \in rad(A)$, then for each $f_1 \in \Delta(A)$, we have $f_1(a) = 0$. Therefore for all $f \in Hom(A \times_\theta B, \mathbb{C}^2)$,

$$f(a, 0) = (f_1(a), f_1 \circ \theta(0)) = (0, 0),$$

which yields that $g(a, 0) = 0$ for each $g \in \Delta(A \times_\theta B)$. So $(a, 0) \in rad(A \times_\theta B) = \{0\}$ and hence $a = 0$. Therefore, A is semisimple.

For the converse let A and B be semisimple. Suppose that $g \in \Delta(A \times_\theta B)$ is arbitrary and $g(a, b) = 0$. Then by Lemma 3.4, we have $(f_1(a), f_1 \circ \theta(b)) = (0, 0)$ where $f_1 \in \Delta(A)$. So $f_1(a) = f_1 \circ \theta(b) = 0$. It follows from the semisimplicity of A and B that $a = b = 0$. Thus, $A \times_\theta B$ is semisimple. □

Corollary 3.6. *Suppose that A is commutative and semisimple. If θ is one to one, then*

(i) *$A \times_\theta B$ is semisimple,*
(ii) *$\Delta(A \times_\theta B)$ separates the points of $A \times_\theta B$,*
(iii) *$r_{A \times_\theta B}(a, b)$ is a norm on $A \times_\theta B$,*
(iv) *$A \times_\theta B$ has a unique complete norm.*

Proof. (i) Let $b \in rad(B)$. Then $f_2(b) = 0$ for each $f_2 \in \Delta(B)$. Since $f_1 \circ \theta \in \Delta(B)$, for all $f_1 \in \Delta(A)$, so $(f_1 \circ \theta)(b) = 0$. Therefore $\theta(b) \in rad(A) = \{0\}$, and hence $b = 0$. Thus, B is semisimple and by Theorem 3.5, $A \times_\theta B$ is semisimple.

(ii), (iii), and (iv) follows from (i). □

4. Multipliers of Lau product

Let X be an A-bimodule, Y be an B-bimodule and suppose that $\sigma : Y \longrightarrow X$ is a θ-module homomorphism. Consider $X \times_\sigma Y$ as a Banach space, then the action

$$(a, b)(x, y) = (ax + a\sigma(y) + \theta(b)x, by), \qquad (a, b) \in A \times_\theta B, \quad (x, y) \in X \times_\sigma Y.$$

turns $X \times_\sigma Y$ into a left $(A \times_\theta B)$-module. Indeed, for every $(a, b), (u, v) \in A \times_\theta B$ and $(x, y) \in X \times_\sigma Y$ we have

$$\begin{aligned}\big((a, b)(u, v)\big)(x, y) &= (au + \theta(b)u + a\theta(v), bv)(x, y)\\ &= \big(aux + \theta(b)ux + a\theta(v)x + au\sigma(y) + \theta(b)u\sigma(y) + a\theta(v)\sigma(y)\\ &\quad + \theta(bv)x, bvy\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}(a, b)\big((u, v)(x, y)\big) &= (a, b)\big(ux + u\sigma(y) + \theta(v)x, vy\big)\\ &= \big(aux + au\sigma(y) + a\theta(v)x + a\sigma(vy) + \theta(b)ux + \theta(b)u\sigma(y)\\ &\quad + \theta(b)\theta(v)x, bvy\big).\end{aligned}$$

Since σ is a right θ-module homomorphism, by comparing the above two expressions, we obtain

$$\big((a, b)(u, v)\big)(x, y) = (a, b)\big((u, v)(x, y)\big).$$

Similarly, $X \times_\sigma Y$ is a right $(A \times_\theta B)$-module with the module action

$$(x, y)(a, b) = (xa + \sigma(y)a + x\theta(b), yb),$$

and in this case we arrive at

$$(x, y)\big((a, b)(u, v)\big) = \big((x, y)(a, b)\big)(u, v).$$

Theorem 4.1. *Suppose that $\sigma : Y \longrightarrow X$ is a θ-module homomorphism and set*

$$T_1 : A \longrightarrow X, \quad T_2 : B \longrightarrow Y, \quad T : A \times_\theta B \longrightarrow X \times_\sigma Y,$$

where $T = (T_1 \circ p_A, T_2 \circ p_B)$. Then,

(i) *If T is a right multiplier, then T_1 and T_2 are so.*

(ii) *If T_1 and T_2 are right multiplier and X is faithful, then T is right multiplier if and only if $\sigma \circ T_2 = T_1 \circ \theta$.*

Proof. (i) Suppose that T is a right multiplier, then for all $(a, b), (u, v) \in A \times_\theta B$,

$$T\big((a, b)(u, v)\big) = (a, b)T(u, v). \tag{4.1}$$

It follows from (4.1) and our assumption that

$$\begin{aligned}\big(aT_1(u) + a\sigma(T_2(v)) + \theta(b)T_1(u), bT_2(v)\big) &= (a, b)\big(T_1(u), T_2(v)\big)\\ &= (a, b)T(u, v)\\ &= T\big((a, b)(u, v)\big)\\ &= T\big(au + \theta(b)u + a\theta(v), bv\big)\\ &= \big(T_1(au + \theta(b)u + a\theta(v)), T_2(bv)\big).\end{aligned}$$

Therefore, $T_2(bv) = bT_2(v)$ and

$$aT_1(u) + a\sigma(T_2(v)) + \theta(b)T_1(u) = T_1\big(au + \theta(b)u + a\theta(v)\big). \tag{4.2}$$

Setting $b = v = 0$ in (4.2), we get $aT_1(u) = T_1(au)$ for all $a, u \in A$. Thus, T_1 and T_2 are right multipliers.

(ii) Let T_1, T_2 and T are right multipliers. Then (4.2) gives

$$a\sigma(T_2(v)) = T_1(a\theta(v)) = aT_1(\theta(v)), \qquad a \in A, v \in B.$$

Therefore, $a\big(\sigma(T_2(v)) - T_1(\theta(v)\big) = 0$ and since X is faithful, we get $\sigma \circ T_2 = T_1 \circ \theta$.

The converse is immediate. □

Let $\mathfrak{M}_r(A)$ denotes the set of all right multipliers from A into a left A-module X, and let

$$\mathfrak{M}_r(A \times_\theta B) = \{T : A \times_\theta B \longrightarrow X \times_\sigma Y, \ \ T \textit{ is a right multiplier}\}.$$

In the next result we turns our attention to the multipliers of $A \times_\theta B$.

Theorem 4.2. *Suppose that $\sigma : Y \longrightarrow X$ is a invertible θ-module homomorphism. If X is faithful and σ^{-1} is continuous, then*

$$\mathfrak{M}_r(A) \cong \mathfrak{M}_r(A \times_\theta B).$$

Proof. Let $h : \mathfrak{M}_r(A) \longrightarrow \mathfrak{M}_r(A \times_\theta B)$ defined by

$$h(T_1) = (T_1 \circ p_A, \sigma^{-1} \circ T_1 \circ \theta \circ p_B).$$

First note that h is linear and well-defined. To see this, let $T_1 : A \longrightarrow X$ be a right multiplier and take $T_2 = \sigma^{-1} \circ T_1 \circ \theta$. Then for $b_1, b_2 \in B$,

$$\begin{aligned} T_2(b_1b_2) &= \sigma^{-1} \circ T_1 \circ \theta(b_1b_2) \\ &= \sigma^{-1} \circ T_1\big(\theta(b_1)(\theta(b_2)\big) \\ &= \sigma^{-1}\big(\theta(b_1)T_1(\theta(b_2))\big) \\ &= \sigma^{-1}\big(\theta(b_1)\sigma(T_2(b_2))\big) \\ &= b_1T_2(b_2). \end{aligned}$$

The last equality is true, because σ is a θ-module homomorphism. Hence T_2 is a right multiplier from B into Y, so by Theorem 4.1 (ii), $h(T_1) \in \mathfrak{M}_r(A \times_\theta B)$.

Clearly, h is one to one. We show that h is surjective. Let $T : A \times_\theta B \longrightarrow X \times_\sigma Y$ be a right multiplier. Then for all $(a, b) \in A \times_\theta B$,

$$T(a, b) = \big(S_1(a, b), S_2(a, b)\big),$$

where $S_1 : A \times_\theta B \longrightarrow X$ and $S_2 : A \times_\theta B \longrightarrow Y$. Define $T_1 : A \longrightarrow X$ via $T_1 \circ p_A = S_1$ and $T_2 : B \longrightarrow Y$ by $T_2 \circ p_B = S_2$. Then by the preceding theorem T_1 and T_2 are right multipliers. Also the equality $\sigma \circ T_2 = T_1 \circ \theta$ holds true. So

$$h(T_1) = (S_1, S_2) = T.$$

Note that h^{-1} is automatically continuous. In fact, for each $T \in \mathfrak{M}_r(A \times_\theta B)$,

$$\|h^{-1}(T)\| = \|T_1\| \leq \|T_1\| + \|T_2\| = \|T\|,$$

and hence $\|h^{-1}\| \leq 1$.

On the other hand, for each $T_1 \in \mathfrak{M}_r(A)$ we have

$$\begin{aligned} \|h(T_1)\| &= \|(T_1 \circ p_A, \sigma^{-1} \circ T_1 \circ \theta \circ p_B)\| \\ &\leq \|T_1 \circ p_A\| + \|\sigma^{-1} \circ T_1 \circ \theta \circ p_B\| \\ &= \|T_1\| + \|\sigma^{-1} \circ T_1 \circ \theta\| \\ &\leq \|T_1\|(1 + \|\sigma^{-1}\|\|\theta\|). \end{aligned}$$

Consequently, h is continuous. This finishes the proof. □

As a consequence of Theorem 4.2, we deduce the next result.

Corollary 4.3. *Let A be a unital Banach algebra. If θ is invertible and θ^{-1} is continuous, then T_1 is a right multiplier on A if and only if T is a right multiplier on $A \times_\theta B$.*

Let σ be a θ-module homomorphism and define

$$L_X = \{L_x : A \longrightarrow X : \quad L_x(a) = ax, \quad \forall x \in X\},$$

$$L_Y = \{L_y : B \longrightarrow Y : \quad L_y(b) = by, \quad \forall y \in Y\},$$

and

$$(L_X, L_Y) = \{(L_x, L_y) : \quad x \in X, \ y \in Y\}.$$

Moreover, for $x \in X$ and $y \in Y$ we set

$$L_{X \times_\sigma Y} = \{L_{(x,y)} : A \times_\theta B \longrightarrow X \times_\sigma Y : \quad L_{(x,y)}(a, b) = (a, b)(x, y)\}.$$

Then for all $x \in X$ and $y \in Y$, L_x, L_y and $L_{(x,y)}$ are right multipliers.

The next example provided that (L_X, L_Y) is different from $\mathfrak{M}_r(A \times_\theta B)$.

Example 4.4. *Let A be a unital Banach algebra, and let $\theta = \sigma : A \longrightarrow A$ be the identity map. Then by Theorem 4.1, $(L_x, L_y) \in \mathfrak{M}_r(A \times_\theta B)$ if and only if $\sigma \circ L_y = L_x \circ \theta$. However, for $x = e_A$, $y = 2e_A$ and $a = e_A$ we have,*

$$\sigma \circ L_y(a) = \sigma(2e_A) = 2e_A, \qquad L_x \circ \theta(a) = L_x(a) = e_A.$$

Therefore, (L_x, L_y) is not a right multiplier.

Proposition 4.5. *Let $\sigma : Y \longrightarrow X$ be a θ-module homomorphism. If X is faithful and θ is surjective, then $(L_x, L_y) \in \mathfrak{M}_r(A \times_\theta B)$ if and only if $x = \sigma(y)$.*

Proof. Let $(L_x, L_y) \in \mathfrak{M}_r(A \times_\theta B)$. Then by Theorem 4.1, for every $b \in B$,

$$(\sigma \circ L_y)(b) = (L_x \circ \theta)(b),$$

which imply that $\theta(b)\sigma(y) = \sigma(by) = \theta(b)x$. For each $a \in A$ there exist $b \in B$ such that $\theta(b) = a$. Therefore, we have $ax = a\sigma(y)$ and hence $a(x - \sigma(y)) = 0$. Since X is faithful, we conclude that $x = \sigma(y)$. The converse is similar. □

The next corollary follows immediately from preceding result.

Corollary 4.6. *Let A be a unital Banach algebra. Then for all $a \in A$,*

$$(L_a, L_a) \in \mathfrak{M}_r(A \times_\theta A).$$

Lemma 4.7. *Let $\sigma : Y \longrightarrow X$ be a θ-module homomorphism. Then for each $x \in X$ and $y \in Y$,*

$$L_{(x,y)} = \big(L_{(x+\sigma(y))} \circ p_A + L_x \circ \theta \circ p_B, L_y \circ p_B\big).$$

Proof. Let $(a, b) \in (A \times_\theta B)$. Then

$$\begin{aligned} L_{(x,y)}(a, b) &= (a, b)(x, y) \\ &= \big(ax + a\sigma(y) + \theta(b)x, by\big) \\ &= \big(L_{(x+\sigma(y))}(a) + L_x \circ \theta(b), L_y(b)\big). \\ &= \big(L_{(x+\sigma(y))} \circ p_A(a, b) + L_x \circ \theta \circ p_B(a, b), L_y \circ p_B(a, b)\big), \end{aligned}$$

as required. □

Theorem 4.8. *Suppose that $\sigma : Y \longrightarrow X$ is a θ-module homomorphism. If θ is surjective, then*

$$(L_X, L_Y) \cong L_{X \times_\sigma Y}.$$

Proof. Let $h : (L_X, L_Y) \longrightarrow L_{X\times_\sigma Y}$ defined by

$$h((L_x, L_y)) = (L_{(x+\sigma(y))} \circ p_A + L_x \circ \theta \circ p_B, L_y \circ p_B).$$

The mapping h is linear and it is well-defined by Lemma 4.7. Clearly, h is surjective. We show that h is one to one. Let $h((L_x, L_y)) = h((L_s, L_t))$, then

$$L_{(x+\sigma(y))} \circ p_A + L_x \circ \theta \circ p_B = L_{(s+\sigma(t))} \circ p_A + L_s \circ \theta \circ p_B, \tag{4.3}$$

and

$$L_y \circ p_B = L_t \circ p_B. \tag{4.4}$$

It follows from (4.4) that $L_y = L_t$ and hence for each $b \in B$,

$$by = L_y(b) = L_t(b) = bt. \tag{4.5}$$

Since σ is a right θ-module homomorphism, by (4.5) we get

$$\theta(b)\sigma(y) = \sigma(by) = \sigma(bt) = \theta(b)\sigma(t), \tag{4.6}$$

and the surjectivity of θ together (4.6) implies that

$$L_{\sigma(y)}(a) = a\sigma(y) = a\sigma(t) = L_{\sigma(t)}(a), \tag{4.7}$$

for all $a \in A$. From (4.3) we have

$$\begin{aligned} L_{(x+\sigma(y))}(a) &= (L_{(x+\sigma(y))} \circ p_A + L_x \circ \theta \circ p_B)(a, 0) \\ &= (L_{(s+\sigma(t))} \circ p_A + L_s \circ \theta \circ p_B)(a, 0) \\ &= L_{(s+\sigma(t))}(a). \end{aligned}$$

By (4.7) and the above equality, we obtain $L_x = L_s$. Therefore, $(L_x, L_y) = (L_s, L_t)$ and h is one to one. The continuity of h and h^{-1} are obvious. □

From Theorem 4.8, we have the next result.

Corollary 4.9. *If θ is surjective, then $(L_A, L_B) \cong L_{A\times_\theta B}$.*

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