Universal Journal of Mathematics and Applications, 6 (3) (2023) 122-129
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1288015

On the Study of Semilinear Fractional Differential Equations Involving Atangana-Baleanu-Caputo Derivative

Samira Zerbib[1*], Khalid Hilal[1] and Ahmed Kajouni[1]

[1]*Laboratory of Applied Mathematics and Scientific Computing Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco*
[*]*Corresponding author*

Article Info

Keywords: *Atangana-Baleanu-Caputo derivative, Fractional semi-linear differential equation, Krasnoselskii fixed point theorem, Resolvent family*
2010 AMS: *34A12, 35B50, 32G34*
Received: *26 April 2023*
Accepted: *23 September 2023*
Available online: *24 September 2023*

Abstract

This work aims to study the existing results of mild solutions for a semi-linear Atangana-Baleanu-Caputo fractional differential equation with order $0 < \theta < 1$ in an arbitrary Banach space. We rely on some arguments to present the mild solution to our problem in terms of an θ-resolvent family. Then we study the existence of this mild solution by using Krasnoselskii's fixed point theorem. Finally, we give an example to prove our results.

1. Introduction

Fractional calculus has been developed intensively since the first conference on this area in 1974 [1]. Then, it gained popularity and significant consideration mainly due to the numerous applications in various fields of applied sciences and engineering. Its purpose is to extend fractional order derivation or integration using non-integer orders. the fractional calculus has been used in mechanics since 1930 and in electrochemistry since 1960. There are some examples of current applications of fractional calculus: fluid circulation, chemical physics, probability and statistics, viscoelasticity, dynamic processes in structures, optics and processing of signals, etc. See [2]- [5]. several mathematicians and physicists studied differential operators and fractional order systems. In [6], the authors studied the existence and uniqueness results to the linear and nonlinear proposed fractional differential equations involving the Atangana–Baleanu fractional derivative. In [7], optimal control for a fractional-order nonlinear mathematical model of cancer treatment is presented and the fractional derivative is defined in the Atangana–Baleanu Caputo sense. The advantage of the ABC-fractional derivative is that it is non-local and has a non-singular kernel. Therefore, it has many applications to demonstrate different problems including the fractional epidemiological model [8], such as, free motion of a coupled oscillator [9], coronavirus and smoking models [10, 11], etc. For more details on the theory of nonlinear ABC-fractional derivative. See [12]- [16].
In what follows, we discuss the existence of the mild solution of the following Atangana-Baleanu-Caputo fractional semi-linear differential equation

$$\begin{cases} {}^{ABC}D^{\theta}_{0_+}\left(w(t)-Q(t,{}^{AB}I^{\theta}_{0^+}w(t))\right) = A\left(w(t)-Q(t,{}^{AB}I^{\theta}_{0^+}w(t))\right)+Y(t,{}^{AB}I^{\theta}_{0^+}w(t), t\in[0,T] \\ w(0)=w_0, \quad w_0\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $0<\theta<1$, ${}^{ABC}D^{\theta}_{0_+}(.)$ is the Atangana-Baleanu-Caputo fractional derivative of order θ, $A: D(A)\subset X\to X$ is the infinitesimal generator of an θ-resolvent family $\{T_\theta(t)\}_{t\geq 0}$, $\{S_\theta(t)\}_{t\geq 0}$ is solution operator defined on the Banach space $(X, \|\,.\,\|)$, $Q\in C(J\times X, X)$, and $Y\in C(J\times X, X)$.
To the best of our knowledge, this is the first time that the problem (1.1) is being studied.

***Email addresses and ORCID numbers*: zerbib.sam123@gmail.com, 0000-0002-2782-0055 (S. Zerbib), hilalkhalid2005@yahoo.fr, 0000-0002-0806-2623 (K. Hilal), kajjouni@gmail.com, 0000-0001-8484-6107 (A. Kajouni)**
Cite as: S. Zerbib, K. Hilal, A. Kajouni, *On the study of semilinear fractional differential equations involving Atangana-Baleanu-Caputo derivative*, Univers. J. Math. Appl., 6(3) (2023), 122-129.

2. Preliminaries

This section will be devoted to some definitions and lemmas on which we base ourselves to study our problem.
Let $J=[0,T]$ be a finite interval of $\mathbb{R}$. We denote by $C(J,\mathbb{R})$ the Banach space of continuous functions with the norm $\|\Psi\|=max\{|\Psi(t)|: t\in J\}$.

Definition 2.1. *[16]. Let $m\in[1,\infty)$ and B be an open subset of $\mathbb{R}$, the Sobolev space $H^m(B)$ is defined as*

$$H^m(B)=\left\{\Phi\in L^2(B): D^{\delta}\Phi\in L^2(B),\forall|\delta|\leq m\right\}.$$

Lemma 2.2. *(Holder Inequality). Let $\Lambda\subset\mathbb{R}$ and $p,q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\Phi\in L^p(\Lambda,\mathbb{R})$, $\varphi\in L^q(\Lambda,\mathbb{R})$, then $\Phi\varphi\in L^1(\Lambda,\mathbb{R})$, and*

$$\|\Phi\varphi\|_{L^1(\Lambda,\mathbb{R})}\leq\|\Phi\|_{L^p(\Lambda,\mathbb{R})}\|\varphi\|_{L^q(\Lambda,\mathbb{R})}.$$

Definition 2.3. *[17]. The left-sided Riemann-Liouville fractional integral of order $n-1<\theta<n$ of a function Φ, such that $n=[\theta]+1$ is given by*

$$I^{\theta}_{0^+}\Phi(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-s)^{\theta-1}\Phi(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z)=\int_0^{+\infty}e^{-t}t^{z-1}dt, z>0.$$

Definition 2.4. *[6]. We define the left-sided Atangana-Baleanu fractional integral of order $0<\theta<1$ of a function Φ, as follows*

$${}^{AB}I^{\theta}_{0^+}\Phi(t)=\frac{1-\theta}{B(\theta)}\Phi(t)+\frac{\theta}{B(\theta)\Gamma(\theta)}\int_0^t(t-s)^{\theta-1}\Phi(s)ds,$$

where $B(\beta)=1-\beta+\frac{\beta}{\Gamma(\beta)}$ is a normalization function such that $B(0)=B(1)=1$.

Definition 2.5. *[6]. Let $0<\theta<1$ and $\Phi\in H^1(0,T)$. We define the left-sided Atangana-Baleanu fractional derivative of Φ of order θ in Riemann-Liouville sense as follows*

$${}^{ABR}D^{\theta}_{0_+}\Phi(t)=\frac{B(\theta)}{1-\theta}\frac{d}{dt}\int_0^t E_{\theta}\left[-\gamma(t-s)^{\theta}\right]\Phi(s)ds,$$

where $\gamma=\frac{\theta}{1-\theta}$ and E_{θ} is one parameter Mittag-Leffler function defined by [18].

$$E_{\theta}(\lambda)=\sum_{n=0}^{n=\infty}\frac{\lambda^n}{\Gamma(n\theta+1)}.$$

Definition 2.6. *[19]. Let $0<\theta<1$ and $\Phi\in H^1(0,T)$. We define the left-sided Atangana-Baleanu-Caputo fractional derivative of the function Φ of order θ as follows*

$${}^{ABC}D^{\theta}_{0_+}\Phi(t)=\frac{B(\theta)}{1-\theta}\int_0^t E_{\theta}(-\gamma(t-s)^{\theta})\Phi^{'}(s)ds.$$

Lemma 2.7. *[19]. Let $0<\theta<1$, then we have*
${}^{AB}I^{\theta}_{0^+}({}^{ABC}D^{\theta}_{0^+}\Phi(t))=\Phi(t)-\Phi(0)$.

Definition 2.8. *[20]. We denote by $\rho(A)=\{\beta\in\mathbb{C};(\beta-A):D(A)\to X \quad is \quad bijective\}$ the resolvent set. The resolvent $R(\beta,A):=(\beta-A)^{-1}$, $\beta\in\rho(A)$, is a bounded operator on X.*

Definition 2.9. *[20]. We say that A is a sectorial operator if the following conditions satisfies*

i) A is linear and closed operator.
ii) there exist constants $M>0$, $v\in\mathbb{R}$, and $\beta\in[\frac{\pi}{2},\pi]$, such that
$\Sigma_{(\beta,v)}=\{\lambda\in\mathbb{C};\lambda\neq v,|\lambda-v|<\beta\}\subset\rho(A)$
iii) $\|R(\lambda,A)\|\leq\frac{M}{|\lambda-v|}$, $\lambda\in\Sigma_{(\beta,v)}$.

Theorem 2.10. *[21]. Let S be a convex, closed, and nonempty subset of the Banach algebra X. Suppose that $P,F:S\to X$ are two operators such that:*

a) $Pw+Fv\in S$ for all $w,v\in S$.
b) P is a contraction on S.
c) F completely continuous on S.
Then, the operator $w=Pw+Fw$ has a solution in S.

3. Main Results

In this section, we give and prove an existence theorem of the mild solution to the ABC fractional semi-linear differential equation (1.1). The first, we give the following remark on which we will rely to prove our major results.

Remark 3.1. *To give the mild solution of the problem* (1.1)*, we rely on the same arguments that the authors used in [22], [23] to determine the solution to the following Cauchy problem*

$$\begin{cases} {}^{ABC}D_t^{\theta}u(t) = Au(t) + g(t), \quad t \in [0,T], 0 < \theta < 1 \\ u(0) = u_0 \in X. \end{cases} \tag{3.1}$$

The problem (3.1) *has a mild solution given by*

$$u(t) = \chi T_\theta(t)u_0 + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}g(s)ds + \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)g(s)ds,$$

where χ and φ are linear operators such that:
$\chi = \zeta(\zeta I - A)^{-1}$ *and* $\varphi = -\gamma A(\zeta I - A)^{-1}$ *with* $\zeta = \frac{B(\theta)}{1-\theta}$, $\gamma = \frac{\theta}{1-\theta}$, *and*

$$T_\theta(t) = E_\theta(-\varphi t^\theta) = \frac{1}{2\pi i}\int_\Gamma e^{t\tau}\tau^{(\theta-1)}(\tau^\theta I - \varphi)^{-1}d\tau$$

$$S_\theta(t) = t^{\theta-1}E_{\theta,\theta}(-\varphi t^\theta) = \frac{1}{2\pi i}\int_\Gamma e^{t\tau}(\tau^\theta I - \varphi)^{-1}d\tau,$$

where Γ is a certain path lying on $\Sigma_{(\beta,\nu)}$ and $g \in C(J,X)$. See [24].

Based on the above arguments, we give the following definition

Definition 3.2. *Let $Q \in C(J\times X, X)$, $J = [0,T]$ and $Y \in C(J\times X, X)$. Then the problem* (1.1) *admits a mild solution given by*

$$\begin{aligned} w(t) &= \chi T_\theta(t)(w_0 - Q(0, {}^{AB}I^\theta w_0)) + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s, {}^{AB}I^\theta w(s))ds \\ &+ \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)Y(s, {}^{AB}I^\theta w(s))ds, \end{aligned}$$

where $\chi = \zeta(\zeta I - A)^{-1}$ and $\varphi = -\gamma A(\zeta I - A)^{-1}$ with $\zeta = \frac{B(\theta)}{1-\theta}$ and $\gamma = \frac{\theta}{1-\theta}$.

Lemma 3.3. *[22]. If $A \in A^\theta(\beta_0, \nu_0)$ then $\| T_\theta(t) \| \leq Me^{t\nu}$ and $\| S_\theta(t) \| \leq Ce^{t\nu}(1+t^{\theta-1})$, for all $t > 0$, $\nu > \nu_0$.*

According to the Lemma above if we set $L_1 = \sup_{t\geq 0} \| T_\theta(t) \|$ and $L_2 = \sup_{t\geq 0} Ce^{t\nu}(1+t^{\theta-1})$. We get $\| T_\theta(t) \| \leq L_1$ and $\| S_\theta(t) \| \leq t^{\theta-1}L_2$. For more details see [22].
Next, we introduce the following assumptions:
(A_1) Both operators $T_\theta(t)$ and $S_\theta(t)$ are compact operators, $\forall t \in J$.
(A_2) There is a constant δ such that for each $p, q \in X$, and $t \in J$ we have: $|Y(t,p) - Y(t,q)| \leq \delta|p-q|$.
(A_3) The function $Y(t,.) : X \to X$ is continuous, for all $t \in J$ and the function $Y(.,p) : X \to X$ is strongly measurable, $\forall p \in X$.
(A_4) There exists a constant $\alpha \in (0,\theta]$ and $h \in L^{\frac{1}{\alpha}}(J,\mathbb{R}^+)$, for all $p \in X$, and $t \in J$ we have

$$|Y(t,p)| \leq h(t).$$

Define $S = \{v \in X, \|v\| \leq R\}$, such that:

$$R = \| \chi \| \left(L_1(|w_0| + |Q(0, {}^{AB}I^\theta w_0)|) + (\frac{\| \varphi \| (1-\theta)}{|\Gamma(\theta)|} + L_2 \| \chi \|)\frac{T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \right),$$

where $C = \frac{\theta-1}{1-\alpha} \in (-1,0)$.
It is easy to see that S is a convex, closed, and nonempty subset of the Banach algebra X. Define the operators $P : S \to X$ and $F : S \to X$, for each $t \in J$ by:

$$Pw(t) = \chi T_\theta(t)(w_0 - Q(0, {}^{AB}I^\theta w_0)) + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s, {}^{AB}I^\theta w(s))ds.$$

$$Fw(t) = \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)Y(s, {}^{AB}I^\theta w(s))ds.$$

We consider the mapping $G : S \to X$ defined by

$$Gw(t) = Pw(t) + Fw(t), \quad t \in J.$$

Theorem 3.4. *If assumptions* (A_1)*-*(A_4) *hold. Then, the fractional semi-linear differential equation* (1.1) *admits a mild solution* $w \in X$ *provided:*

$$\frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|}\delta\Theta < 1, \tag{3.2}$$

where

$$\Theta = \frac{(1-\theta)}{B(\theta)} + \frac{T^{\theta}}{\Gamma(\theta+1)B(\theta)}. \tag{3.3}$$

Proof. To prove the Theorem 3.4 is equivalent to proving that the mapping G has a fixed point, we show that the operators P and F satisfy the conditions of the Theorem 2.10.
Before proceeding to the proof of the Theorem 3.4, we will need the following lemma

Lemma 3.5. *If there exists a constant* $\alpha \in (0,\theta]$ *and* $h \in L^{\frac{1}{\alpha}}([0,T],\mathbb{R}^+)$ *such that* $|Y(t,p)| \le h(t)$ *for all* $p \in X$*, and almost all* $t \in [0,T]$*, we have the following inequality*

$$\int_0^t |(t-s)^{\theta-1}Y(s,{}^{AB}I^{\theta}w(s))|ds \le \frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]}.$$

Proof. Assuming that all the conditions of the above lemma are satisfied. By a direct calculation, we get $(t-s)^{\theta-1} \in L^{\frac{1}{1-\alpha}}[0,t]$ for $t \in J$ and $\alpha \in (0,\theta]$. Then by using Lemma 2.2, we have

$$\begin{aligned}\int_0^t |(t-s)^{\theta-1}Y(s,{}^{AB}I^{\theta}w(s))|ds &\le \int_0^t |(t-s)^{\theta-1}||h(s)|ds \\ &\le \left(\int_0^t (t-s)^{\frac{\theta-1}{1-\alpha}}ds\right)^{1-\alpha}\left(\int_0^t |h(s)|^{\frac{1}{\alpha}}ds\right)^{\alpha} \\ &\le \frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]},\end{aligned}$$

where $C = \frac{\theta-1}{1-\alpha} \in (-1,0)$. □

We now move on to continue the proof of the theorem, then the proof is as follows:
Step 1:
Let $w,v \in S$, then for all $t \in [0,T]$ according to the assumptions A_1, A_4, Lemma 3.3, and the Lemma 3.5, we have:

$$\begin{aligned}&|(Pw(t)+Fv(t))| \\ &= \left|\chi T_{\theta}(t)(w_0 - Q(0,{}^{AB}I^{\theta}w_0)) + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s,{}^{AB}I^{\theta}w(s))ds + \frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}v(s))ds\right| \\ &\le \| \chi \| \| T_{\theta}(t) \| (|w_0| + |Q(0,{}^{AB}I^{\theta}w_0)|) + \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\int_0^t |(t-s)^{\theta-1}||Y(s,{}^{AB}I^{\theta}w(s))|ds \\ &+ \frac{\theta \| \chi^2 \|}{|B(\theta)|}\int_0^t \| S_{\theta}(t-s) \| |Y(s,{}^{AB}I^{\theta}v(s))|ds \\ &\le \| \chi \| L_1(|w_0| + |Q(0,{}^{AB}I^{\theta}w_0)|) + \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\int_0^t |(t-s)^{\theta-1}||Y(s,{}^{AB}I^{\theta}w(s))|ds + \frac{\| \chi^2 \|}{|B(\theta)|}L_2\int_0^t |(t-s)^{\theta-1}||Y(s,{}^{AB}I^{\theta}v(s))|ds \\ &\le \| \chi \| L_1(|w_0| + |Q(0,{}^{AB}I^{\theta}w_0)|) + \left(\frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} + \frac{L_2 \| \chi \|^2}{|B(\theta)|}\right)\frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \\ &\le \| \chi \| \left\{L_1(|w_0| + |Q(0,{}^{AB}I^{\theta}w_0)|) + \left(\frac{\| \varphi \| (1-\theta)}{|\Gamma(\theta)|} + L_2 \| \chi \|\right)\frac{T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]}\right\},\end{aligned}$$

this gives:

$$\| (Pw(t)+Fv(t)) \| \le R,$$

this implies, $(Pw(t)+Fv(t)) \in S$ for all $w,v \in S$.
Step 2: P is a contraction on S:
Let $w,v \in S$, then from assumption (A_2), we have

$$\begin{aligned}|Pw(t)-Pv(t))| &= \left|\frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s,{}^{AB}I^{\theta}w(s))ds - \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s,{}^{AB}I^{\theta}v(s))ds\right| \\ &\le \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\int_0^t (t-s)^{\theta-1}|Y(s,{}^{AB}I^{\theta}w(s)) - Y(s,{}^{AB}I^{\theta}v(s))|ds \\ &\le \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\delta\int_0^t (t-s)^{\theta-1}|{}^{AB}I^{\theta}w(s)) - {}^{AB}I^{\theta}v(s))|ds.\end{aligned} \tag{3.4}$$

We have

$$\begin{aligned}\left|{}^{AB}I^{\theta}w(s)) - {}^{AB}I^{\theta}v(s))\right| &= \frac{(1-\theta)}{B(\theta)}|w(s)-v(s)| + \frac{\theta}{B(\theta)}{}^{RL}I^{\theta}|w(s)-v(s)| \\ &\leq \frac{(1-\theta)}{B(\theta)}|w(s)-v(s)| + \frac{1}{\Gamma(\theta)B(\theta)}\int_0^s (s-\tau)^{\theta-1}|w(\tau)-v(\tau)|d\tau \\ &\leq \| w-v \| \left(\frac{(1-\theta)}{B(\theta)} + \frac{1}{\Gamma(\theta)B(\theta)}\int_0^s (s-\tau)^{\theta-1}d\tau\right) \\ &\leq \| w-v \| \left(\frac{(1-\theta)}{B(\theta)} + \frac{T^{\theta}}{\Gamma(\theta+1)B(\theta)}\right) \\ &\leq \| w-v \| \Theta,\end{aligned}$$

where Θ is given by (3.3).
Therefore

$$\begin{aligned}(3.4) &\leq \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\delta\Theta \| w-v \| \int_0^t (t-s)^{\theta-1} \\ &\leq \frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|}\delta\Theta \| w-v \|,\end{aligned}$$

this implies that

$$\| Pw(t) - Pv(t)) \| \leq \frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|}\delta\Theta \| w-v \| .$$

Then, according to the condition (3.2) the operator P is a contraction on S.
Step 3: F is completely continuous:
i) F is continuous.
Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of S such that $w_n \to w$ as $n \to \infty$ in S. We prove that $Fw_n \to Fw$ as $n \to \infty$ in S. By using Lemma 3.3, we get

$$\begin{aligned}|Fw_n(t) - Fw(t)| &= \left|\frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w_n(s))ds - \frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w(s))ds\right| \\ &\leq \frac{\theta \| \chi^2 \|}{|B(\theta)|}\int_0^t \| S_{\theta}(t-s) \| |Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds \\ &\leq \frac{\theta \| \chi^2 \|}{|B(\theta)|}L_2\int_0^t (t-s)^{\theta-1}|Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds \\ &\leq \frac{\| \chi^2 \|}{|B(\theta)|}T^{\theta}L_2 \sup_{s\in[0,T]}|Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds.\end{aligned}$$

By using the assumption A_3 and Lebesgue dominated convergence theorem, we get:
$\|Fw_n - Fw\| \to 0$ as $n \to \infty$.
This implies that the operator $F : S \to X$ is continuous.
ii) $F(S) = \{Fw : w \in S\}$ is uniformly bounded.
Using assumptions (A_1) and (A_4), Lemma 3.3, and the Lemma 3.5, for any $w \in S$ and $t \in [0,T]$, we have:

$$\begin{aligned}|Fw(t)| &= \left|\frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w(s))ds\right| \\ &\leq \frac{\| \chi^2 \|}{|B(\theta)|}L_2\int_0^t |(t-s)^{\theta-1}||Y(s,{}^{AB}I^{\theta}v(s))|ds \\ &\leq \frac{L_2 \| \chi \|^2 T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} .\end{aligned}$$

Therefore,

$$\| Fw(t) \| \leq \frac{L_2 \| \chi \|^2 T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]},$$

this proves that $F(S) = \{Fw : w \in S\}$ is uniformly bounded.
iii) $F(S)$ is equicontinuous

Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and $w \in S$, then using assumptions (A_1) and (A_2), we have

$$
\begin{aligned}
&|Fw(t_2) - Fw(t_1)| \\
&= \left| \frac{\theta\chi^2}{B(\theta)} \int_0^{t_2} S_\theta(t_2 - s) Y(s, {}^{AB}I^\theta w(s)) ds - \frac{\theta\chi^2}{B(\theta)} \int_0^{t_1} S_\theta(t_1 - s) Y(s, {}^{AB}I^\theta w(s)) ds \right| \\
&\leq \frac{\| \chi^2 \|}{|B(\theta)|} \left| \int_0^{t_2} |S_\theta(t_2 - s)||Y(s, {}^{AB}I^\theta w(s))| ds - \int_0^{t_1} |S_\theta(t_1 - s)||Y(s, {}^{AB}I^\theta w(s))| ds \right| \\
&\leq \frac{\| \chi^2 \|}{|B(\theta)|} L_2 \left| \int_0^{t_1} (t_2 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds + \int_{t_1}^{t_2} (t_2 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds - \int_0^{t_1} (t_1 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds \right| \\
&\leq \frac{\| \chi^2 \|}{|B(\theta)|} L_2 \left| \int_{t_1}^{t_2} (t_2 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds \right| + \frac{\| \chi^2 \|}{|B(\theta)|} L_2 \left| \int_0^{t_1} ((t_2 - s)^{\theta-1} - (t_1 - s)^{\theta-1}) |Y(s, {}^{AB}I^\theta w(s))| ds \right|
\end{aligned}
$$

If we set

$$
I = \left| \int_{t_1}^{t_2} (t_2 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds \right|,
$$

$$
J = \left| \int_0^{t_1} ((t_2 - s)^{\theta-1} - (t_1 - s)^{\theta-1}) |Y(s, {}^{AB}I^\theta w(s))| ds \right|,
$$

and by using the arguments of lemma 3.5 we get

$$
\begin{aligned}
I &= \left| \int_{t_1}^{t_2} (t_2 - s)^{\theta-1} |Y(s, {}^{AB}I^\theta w(s))| ds \right| \\
&\leq \frac{(t_2 - t_1)^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]},
\end{aligned}
$$

and

$$
\begin{aligned}
J &= \left| \int_0^{t_1} ((t_2 - s)^{\theta-1} - (t_1 - s)^{\theta-1}) |Y(s, {}^{AB}I^\theta w(s))| ds \right| \\
&\leq \int_0^{t_1} ((t_1 - s)^{\theta-1} - (t_2 - s)^{\theta-1}) |h(s)| ds \\
&\leq \left(\int_0^{t_1} ((t_1 - s)^{\theta-1} - (t_2 - s)^{\theta-1})^{\frac{1}{1-\alpha}} ds \right)^{1-\alpha} \left(\int_0^{t_1} |h(s)|^{\frac{1}{\alpha}} ds \right)^\alpha \\
&\leq \left(\int_0^{t_1} ((t_1 - s)^C - (t_2 - s)^C) ds \right)^{1-\alpha} \| h \|_{L^{\frac{1}{\alpha_1}}[0,t]} \\
&\leq \frac{\left(t_1^{1+C} - t_2^{1+C} + (t_2 - t_1)^{1+C} \right)^{1-\alpha_1}}{(1+C)^{1-\alpha_1}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \\
&\leq \frac{(t_2 - t_1)^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
|Fw(t_2) - Fw(t_1)| &\leq \frac{\| \chi^2 \|}{|B(\theta)|} L_2 (I + J) \\
&\leq \frac{2L_2 \| \chi^2 \| (t_2 - t_1)^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]}.
\end{aligned}
$$

Therefore, if $|t_1 - t_2| \to 0$ then $|Fw(t_2) - Fw(t_1)| \to 0$, this implies that $F(S)$ is equicontinuous.
According to parts (ii), (iii), and Arzela-Ascoli theorem, it is deduced that $F(S)$ is relatively compact. And according to the part (i) deduce that it is completely continuous.
According to steps 1, 2, and 3, we notice that all the conditions of Theorem 2.10 hold. Then operator G admits a fixed point in S. This proves that the problem (1.1) admits a mild solution in $C(J, X)$. □

4. Example

This section is devoted to an illustrative example that shows the results of this work.
Let $X = L^2([0,1])$, $w(t) = w(.,t)$, $Y(t, {}^{AB}I_{0_+}^{\frac{1}{2}} w(.,t)) = \frac{1}{19}(1 + \cos({}^{AB}I_{0_+}^{\frac{1}{2}} w(.,t))$, $Q(t, {}^{AB}I_{0_+}^{\frac{1}{2}} w(.,t)) = 1 + t^{AB}I_{0_+}^{\frac{1}{2}} w(.,t))$, $J = [0,1]$.

We Consider the following problem

$$\begin{cases} {}^{ABC}\partial_t^{\frac{1}{2}}\left(w(x,t)-(1+t^{AB}I_{0_+}^{\frac{1}{2}}w(x,t))\right)=\Delta\left(w(x,t)-(1+t^{AB}I_{0_+}^{\frac{1}{2}}w(x,t))\right) \\ +\frac{1}{19}(1+\cos({}^{AB}I_{0_+}^{\frac{1}{2}}w(x,t)), \quad t\in[0,1], x\in[0,1] \\ w(0,t)=w(1,t)=0, \quad t\in[0,1] \\ w(x,0)=w_0(x), \quad x\in[0,1], \end{cases} \tag{4.1}$$

where ${}^{ABC}\partial_t^{\frac{1}{2}}$ is the ABC-fractional partial derivative of order $\frac{1}{2}$, and $A : D(A) \subset X \longrightarrow X$ be an operator defined by

$$D(A) := H^2(0,1)\cap H^1(0,1) \quad \textit{and} \quad Au = \Delta u.$$

The operator A generates a uniformly bounded semi-group $T(t)_{t\geq 0}$ in X. See [20].
Let $v(t) =^{AB} I_{0^+}^{\frac{1}{2}} w(.,t)$, then $Y(t,v(t)) = \frac{1}{19}(1+\cos(v(t))$. It is clear that
$|Y(t,u(t))-Y(t,v(t))| \leq \frac{1}{19}|u-v|$ and $|Y(t,u(t))| \leq 1$. We take $h(t) = 1$, $(\| \chi \|, \| \varphi \| \leq 1)$, and $\delta = \frac{1}{19}$. then the assumptions A_2, A_3 and A_4 are satisfied.
Now we check for condition (3.2). We have $T = 1$, $\theta = \frac{1}{2}$, then after some calculations, we find

$$\begin{aligned} \frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta_1+1)|}\delta\Theta &\leq \frac{1-\frac{1}{2}}{19B(\frac{1}{2})\Gamma(\frac{1}{2}+1)}\left(\frac{1-\frac{1}{2}}{B(\frac{1}{2})}+\frac{1}{B(\frac{1}{2})\Gamma(\frac{1}{2}+1)}\right) \\ &\leq \frac{1}{19B(\frac{1}{2})\Gamma(\frac{3}{2})}\left(\frac{1}{B(\frac{1}{2})}+\frac{1}{B(\frac{1}{2})\Gamma(\frac{3}{2})}\right) \\ &\simeq 0,114 < 1. \end{aligned}$$

Then from the results above, deduce that the ABC-fractional semi-linear differential problem (4.1) has a mild solution w in $C([0,1]\times[0,1],X)$.

5. Conclusion

In this paper, we have studied the existence of the mild solutions of a fractional semi-linear differential equation involving Atangana-Baleanu-Caputo fractional derivative with order $0 < \theta < 1$ by using the Krasnoselskii fixed point theorem. In the end, an illustrative example is presented to demonstrate our results.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives, Translated from the 1987 Russian original*, (1993).
[2] R. L. Bagley, P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., **27**.3 (1983), 201-210.
[3] N. Chefnaj, T. Abdellah, K. Hilal, S. Melliani, A. Kajouni, *Boundary Problems for Fractional Differential Equations Involving the Generalized Caputo-Fabrizio Fractional Derivative in the λ-Metric Space*, Turkish Journal of Science, **8**(1) (2023), 24-36.
[4] M. D. Bayrak, A. Demir, *On the challenge of identifying space dependent coefficient in space-time fractional diffusion equations by fractional scaling transformations method*, Turkish Journal of Science, **7**(2) (2022) 132-145.
[5] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue, V. Feliu-Batlle., *Fractional-Order Systems and Controls: Fundamentals and Applications*, Springer Science & Business Media, 2010.
[6] M. I. Syam, M. Al-Refai, *Fractional differential equations with Atangana–Baleanu fractional derivative: Analysis and applications*, Chaos Solitons Fractals: X, **2** (2019), 100013.
[7] N. Sweilam, S. Al-Mekhlafi, T. Assiri, A. Atangana, *Optimal control for cancer treatment mathematical model using Atangana–Baleanu–Caputo fractional derivative*, Adv. Difference Equ., **2020**(1) (2020), 1-21.
[8] J. Singh, D. Kumar, Z. Hammouch, A. Atangana, *A fractional epidemiological model for computer viruses pertaining to a new fractional derivative*, Appl. Math. Comput., **316** (2018), 504-515.
[9] A. Jajarmi, D. Baleanu, S. S Sajjadi, J. H. Asad, *A new feature of the fractional Euler–Lagrange equations for a coupled oscillator using a nonsingular operator approach*, Frontiers in Physics, **7** (2019), 196.

[10] S. Uçar, E. Uçar, N. Özdemir, Z. Hammouch, *Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative*, Chaos Solitons Fractals, **118** (2019), 300-306.
[11] M. Abdo, K. Shah, H. Wahash, S. Panchal, *On a comprehensive model of the novel coronavirus (COVID-19) under Mittag-Leffler derivative*, Chaos Solitons Fractals, **135** (2020), 109867.
[12] M. I. Abbas, M. A. Ragusa, *Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel*, AIMS Math, **7**(11) (2022), 20328-20340.
[13] F. Jarad, T. Abdeljawad, Z. Hammouch, *On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative*, Chaos Solitons Fractals, **117** (2018), 16-20.
[14] A. O. Akdemir, A. Karaoğlan, M. A. Ragusa, *Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions*, J. Funct. Spaces, **2021** (2021), 1-10.
[15] M. A. Dokuyucu, *Analysis of a novel finance chaotic model via ABC fractional derivative* Numer. Methods Partial Differential Equations, **37**(2) (2021), 1583-1590.
[16] M. I. Syam, M. Al-Refai, *Fractional differential equations with Atangana-Baleanu fractional derivative: Analysis and applications*, Chaos Solitons Fractals: X, **2** (2019), 100013.
[17] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.
[18] K. Diethelm, N. J. Ford, *The Analysis of Fractional Differential equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, **2004**. Berlin: Springer, 2010.
[19] A. Atangana, I. Koca, *Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order*, Chaos Solitons Fractals, **89** (2016), 447-454.
[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, **44**, Springer Science & Business Media, 2012.
[21] T. A. Burton, *A fixed-point theorem of Krasnoselskii*, Appl. Math. Lett., **11**(1) (1998), 85-88.
[22] X. B. Shu, Y. Lai, Y. Chen, *The existence of mild solutions for impulsive fractional partial differential equations.*Nonlinear Analysis: Theory, Methods & Applications, **74**(5) (2011), 2003-2011.
[23] E. G. Bajlekova, *Fractional Evolution Equations in Banach Spaces*, Eindhoven: Technische Universiteit Eindhoven, 2001.
[24] G. M. Bahaa, A. Hamiaz, *Optimality conditions for fractional differential inclusions with nonsingular Mittag-Leffler kernel*, Adv. Difference Equ., **2018**(2018), 1-26.