# NUMERICAL APPROXIMATION WITH THE SPLITTING ALGORITHM TO A SOLUTION OF THE MODIFIED REGULARIZED LONG WAVE EQUATION

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ABSTRACT. In this article, a Lie-Totter splitting algorithm, which is highly reliable, flexible and convenient, is proposed along with the collocation finite element method to approximate solutions of the modified regular long wave equation. For this article, quintic B-spline approximation functions are used in the implementation of collocation methods. Four numerical examples including a single solitary wave, the interaction of two- three solitary waves, and a Maxwellian initial condition are presented to test the closeness of the solutions obtained by the proposed algorithm to the exact solutions. The solutions produced are compared with those in some studies with the same parameters that exist in the literature. The fact that the present algorithm produces results as intended is a proof of how useful, accurate and reliable it is. It can be stated that this fact will be very useful the application of the presented technique for other partial differential equations, with the thought that it may lead the reader to obtain superior results from this study.

## 1. INTRODUCTION

Nonlinear partial differential equations play an important role in the modeling of many disciplines. The generalized regularized long wave (GRLW), presented in the form below, is among these equations

$$U_t + U_x - \mu U_{xxt} + \epsilon U^p U_x = 0 \tag{1}$$

in which $p$ is positive integer, $\mu$ and $\epsilon$ non-negative constants. The solutions of this equation, which have an important place in the propagation of nonlinear dispersion waves, are among the solitary wave types, which are packets or pulses propagating in a nonlinear dispersion medium. They have shapes that are not affected by

collisions. These waves preserve their stable wave form since the nonlinear and dispersive effects have dynamical balance. Sometimes it is not easy to obtain analytical solutions of all partial differential equations. In this case, various numerical methods have been developed to obtain approximate solutions to such problems. Some authors have proposed various approaches to solve Eq.(1) numerically. A few of them can be given as Petrov-Galerkin finite element method, Petrov-Galerkin scheme and lumped Galerkin method based on cubic B-splines, respectively, by Refs. [6, 37, 46], and a collocation method with cubic, septic and quintic B-splines, respectively, by Refs. [10, 20, 47], and also Chebyshev-spectral collocation scheme by Ref. [14], parabolic Monge-Ampere moving mesh and uniform by Ref. [2], an approximate quasilinearization approach by Ref. [33], basis of reproducing kernel space by Ref. [27], exponential B-spline collocation scheme by Ref. [28] and element-free kp-Ritz method by Ref. [12]. When $p = 1$ in Eq.(1), it becomes the regularized long wave (RLW) equation used to model a significant number of physical phenomena with weak nonlinearity and dispersion waves containing longitudinal dispersive waves in elastic rods, phonon packets in non-linear crystals, the transverse waves in shallow water, a pressure wave in liquid's gas bubbles, ion-acoustic waves and magnetohydrodynamic wave in plasma. This equation was first introduced by [32] then [5] worked. Many studies can be found in the literature for the approximate solution of the RLW equation. Some of those can be given, Refs. [3, 8, 15, 23] with the finite difference method, Refs. [9, 24, 41] with the Galerkin and Petrov Galarkin methods and Refs. [7, 36, 39, 40, 44, 45] with the collocation algorithm as the finite element method. Additionally, it were worked on methods explicit multistep by Ref. [25] and Haar wavelet by Ref. [30] for RLW equation. In the present studied, the modified regularized long wave equation will be discussed

$$U_t + U_x - \mu U_{xxt} + 6U^2 U_x = 0 \qquad (2)$$

given with initial condition

$$U(x,0) = g(x), \quad x_L \leq x \leq x_R \qquad (3)$$

and boundary conditions

$$\begin{aligned}
U(x_L,t) &= U(x_R,t) = 0, \\
U_x(x_L,t) &= U_x(x_R,t) = 0, \\
U_{xx}(x_L,t) &= U_{xx}(x_R,t) = 0.
\end{aligned} \qquad (4)$$

It can be seen that the approximate solutions of the MRLW equation have been calculated by many methods in the literature. For example, as the finite element method, while the Galerkin and Petrov Galerkin approaches were studied by Refs. [16, 18, 37], the collocation algorithm with B-splines was studied by Refs. [11, 13, 16, 19, 21, 22, 34, 36], At the same time, [17] used finite difference scheme for the MRLW, [1] solved the equation with mesh free collocation method using radial basis function and [38] acquired the solutions of the equation with the help of Butcher's fifth-order Runge-Kutta (BFRK) scheme.

In this paper, the numerical algorithm of the MRLW equation has been obtained by obtaining two numerical schemes with the help of the Lie-Trotter splitting algorithm and the quintic B-spline collocation method has been applied to each scheme. Thanks to this algorithm, the motion of a single solitary wave, the interaction of two and three solitary waves and the Maxwell initial state have been examined and thus numerical solutions produced with a hybrid approach have been obtained as targeted. Furthermore, Linear stability analysis has been investigated with the help of Von Neumann method.

## 2. The Splitting Algorithm

One of the developed methods to produce numerical solutions of partial differential equations is operator splitting methods. A time-dependent partial differential equation, which usually represents complex physical phenomena such as convection, diffusion, reaction in chemical phenomena, or diffusion, may consist of a combination of one or more operators. Although the computational power of computers has increased rapidly in recent years, good results may not be obtained even if a lot of time is spent in obtaining numerical solutions of a complex problem. Operator splitting methods can be a good approach to numerical solution of such problems. One of these methods is the first-order Lie-Trotter splitting method according to time. This method is the simplest splitting method that reduces the solution of the Cauchy problem given as below to the successive solution of two subproblems

$$\frac{dU(t)}{dt} = \Lambda U(t), \quad U(0) = U_0, t \geq 0, \tag{5}$$

where operator $\Lambda$ can be written as the sum of operators $\hat{A}$ and $\hat{B}$. In this case, equation (5) can be written in the following form

$$\frac{dU(t)}{dt} = \hat{A}U(t) + \hat{B}U(t), \quad U(0) = U_0, t \geq 0, \tag{6}$$

in which $U_0 \in X$ is the vector obtained from the initial condition, $u(x, t)$ is solution vector, the operators $\Lambda, \hat{A}, \hat{B}$ are bounded or unbounded operators in a finite or infinite Banach space $X$. To solve the equation (6) numerically, firstly splitting technique split the equation into as follows

$$\frac{dU(t)}{dt} = \hat{A}U(t), \quad \frac{dU(t)}{dt} = \hat{B}U(t). \tag{7}$$

Here, let $\rho_{\Delta t}^{[\hat{A}]}$ and $\rho_{\Delta t}^{\hat{B}}$ be the numerical solutions of the equations containing the expressions $\hat{A}$ and $\hat{B}$ in expression (7), and let the exact solution of (6) be given as $\psi_{\Delta t}$. The simplest splitting methods are introduced as follows

$$\rho_{\Delta t}^{[\hat{B}]} o \rho_{\Delta t}^{[\hat{A}]} = e^{\Delta t \hat{B}} e^{\Delta t \hat{A}} \quad \text{or} \quad \rho_{\Delta t}^{[\hat{A}]} o \rho_{\Delta t}^{[\hat{B}]} = e^{\Delta t \hat{A}} e^{\Delta t \hat{B}}.$$

and it is known as the Lie-Trotter splitting technique [42] in the literature. Using the Taylor series, It can be stated that the following approximation for an initial

value $U_0$ is a first-order approximation to the solution of equation (6)

$$\psi_{\Delta t}(U_0) = (\rho_{\Delta t}^{[\hat{A}]} o \rho_{\Delta t}^{[\hat{B}]})(U_0) + 0(\Delta t^2).$$

Let the formal solution of (6) be given in the form

$$U(t_{n+1}) = e^{\Lambda \Delta t} U(t_n) = e^{(\hat{A}+\hat{B})\Delta t} U(t_n). \tag{8}$$

Unfolding Taylor series for this solution can be given in the following form

$$U(t_{n+1}) = e^{\Delta t(\hat{A}+\hat{B})} U(t_n) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \big( \hat{A}(u(t))\frac{\partial}{\partial U} + \hat{B}(u(t))\frac{\partial}{\partial U} \big)^k U(t_n).$$

By calculating the sum of the operators $\hat{A}$ and $\hat{B}$ instead of $\Lambda$, a new approach to Equation (6) can be obtained, presented in the form below

$$U(t_{n+1}) = e^{\hat{A}\Delta t} e^{\hat{B}\Delta t} U(t_n). \tag{9}$$

An error occurs if (9) is used instead of the equation (8). This is a local splitting error given as follows

$$\begin{aligned} Te &= \frac{1}{\Delta t}(e^{\Delta t(\hat{A}+\hat{B})} - e^{\Delta t \hat{B}} e^{\Delta t \hat{A}})U(t_n) \\ &= \frac{1}{\Delta t}[\frac{\Delta t^2}{2}(\hat{A}\hat{B} - \hat{B}\hat{A})U(t_n) + O(\Delta t^3)] \\ &= \frac{1}{\Delta t}[\hat{A}, \hat{B})U(t_n) + O(\Delta t^2)] \end{aligned}$$

To explain in more detail: Splitting technique splits the given original problem into two parts according to the time. As a result, subproblems with a simpler structure are obtained. Thus, the solution of the original problem is obtained from the solution of the subproblems. In the Lie-Trotter schemes, the first sub-problem with operator $\hat{A}$ is solved using the original initial condition given with the problem. Then, the solutions generated with the operator $\hat{A}$ are utilized as the initial condition for the solution of the second sub-problem given with the operator $\hat{B}$ and presented as the solution of the main problem in the first time step. In this way, approximate solutions at the next time levels are obtained similarly to those in the first time step. Algorithm of the mentioned technique with $t_0 = 0$ and $t_N = T$,

$$\frac{dU^*(t)}{dt} = \hat{A}U^*(t), \quad U^*(t_n) = U_n^0 \quad , \quad t \in [t_n, t_{n+1}],$$
$$\frac{dU^{**}(t)}{dt} = \hat{B}U^{**}(t), \quad U^{**}(t_n) = U^*(t_{n+1}) \quad , \quad t \in [t_n, t_{n+1}].$$

where $U_n^0$ is the original initial condition given in (5), $\Delta t$ is the time step, $\Delta t = t_{n+1} - t_n, n = 0, 1, ..., N - 1$. Thus, the targeted solutions are obtained with $U(t_{n+1}) = u^{**}(t_{n+1})$. This scheme is called as $(\hat{A} - \hat{B})$-shaped splitting scheme. It can be stated here that solving the sub-problems separately is more advantageous in terms of computational cost rather than solving the whole problem [26, 43].

### 3. The Quintic B-splines

In order to make approximate calculations of the MRLW equation, the solution region is limited to the interval $x_R \leq x \leq x_N$. This range is partitioned by nodes $x_j$ into uniformly finite elements of length $h$ such that $x_L = x_0 \leq x_1 \leq ... \leq x_N = x_R$ and $h = x_{j+1} - x_j$. The set of quintic B-splines $\varphi_j(x)$ for $j = -2(1)N + 2$ forming a base on the interval $[x_L, x_R]$ at nodes $x_j$ is presented as follows by [31]

$$
\varphi_j(x) = \frac{1}{h^5} \begin{cases}
p_0 = (x - x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}] \\
p_1 = p_0 - 6(x - x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}] \\
p_2 = p_1 - 6(x - x_{j-2})^5 + 15(x - x_{j-1})^5, & x \in [x_{j-1}, x_j] \\
p_3 = p_2 - 6(x - x_{j-2})^5 - 20(x - x_j)^5, & x \in [x_j, x_{j+1}] \\
p_4 = p_3 - 6(x - x_{j-2})^5 + 15(x - x_{j+1})^5, & x \in [x_{j+1}, x_{j+2}] \\
p_5 = p_4 - 6(x - x_{j-2})^5 - 6(x - x_{j+2})^5, & x \in [x_{j+2}, x_{mj3}] \\
0, & \text{otherwise.}
\end{cases}
\tag{10}
$$

The numerical solution $U_N(x, t)$ corresponding to the exact solution $U(x, t)$ is searched in terms of quintic B-splines in the following form

$$
U_N(x, t) = \sum_{j=-2}^{N+2} \varphi_j(x) \delta_j(t) \tag{11}
$$

Here, $\delta_j(t)$ are the unknown time parameters determined with both boundary and collocation conditions. When the trial function (10) is substituted in the equation (11), The knot values $U_j, U_j', U_j''$ at nodes $x_j$ are acquired in terms of the parameter $\delta_j(t)$ with form

$$
\begin{aligned}
U_j &= \delta_{m-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2}, \\
U_j' &= \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}), \\
U_j'' &= \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta)_{j+1} + \delta_{j+2}),
\end{aligned}
\tag{12}
$$

Here, the first and second derivatives with respect to $x$ are denoted by the symbols $'$ and $''$. The all of quintic B-spline base functions are zero outside of $\phi_{j-2}, \varphi_{j-1}, \varphi_j$, $\varphi_{j+1}, \varphi_{j+2}$ and $\varphi_{j+3}$.

### 4. The Implementation of Collocation Method

In this section, firstly, the MRLW equation with the initial-boundary value problem is split. In an other saying, the main problem is divided into sub-equations as follows to obtain two partial differential equations, one linear and the other nonlinear, with respect to time

$$
U_t - \mu U_{xxt} + U_x = 0 \tag{13}
$$

$$U_t - \mu U_{xxt} + 6U^2 U_x = 0. \tag{14}$$

When $U_j$ and its first derivatives $U_j'$ and $U_j''$ given in the (12) equation are substituted in equations (13) and (14), system of ordinary differential equations given in the following form are obtained for $j = 0(1)N$ in the entire solution region

$$\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2}$$
$$-\mu \frac{20}{h^2}(\dot{\delta}_{m-2} + 2\dot{\delta}_{m-1} - 6\dot{\delta}_m + 2\dot{\delta}_{m+1} + \dot{\delta}_{m+2})$$
$$+\frac{5}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) = 0, \tag{15}$$

$$\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2}$$
$$-\mu \frac{20}{h^2}(\dot{\delta}_{m-2} + 2\dot{\delta}_{m-1} - 6\dot{\delta}_m + 2\dot{\delta}_{m+1} + \dot{\delta}_{m+2})$$
$$+\frac{5z_j}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) = 0 \tag{16}$$

in which the first derivative according to time $t$ is shown with symbol $''.''$ and $z_j$ is gotten as

$$z_j = 6(\delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2})^2$$

to linearize the (16) system. Then, by applying $\dfrac{\delta_j^{n+1} + \delta_j^n}{2}$ for spatial discretization and $\dfrac{\delta_j^{n+1} - \delta_j^n}{\Delta t}$ for time discretization to these two systems, two numerical schemes are obtained with form

$$k_1\delta_{m-2}^{n+1} + k_2\delta_{m-1}^{n+1} + k_3\delta_m^{n+1} + k_4\delta_{m+1}^{n+1} + k_5\delta_{m+2}^{n+1}$$
$$= k_6\delta_{m-2}^n + k_7\delta_{m-1}^n + k_8\delta_m^n + k_9\delta_{m+1}^n + k_{10}\delta_{m+2}^n, \tag{17}$$

$$l_1\delta_{m-2}^{n+1} + l_2\delta_{m-1}^{n+1} + l_3\delta_m^{n+1} + l_4\delta_{m+1}^{n+1} + l_5\delta_{m+2}^{n+1}$$
$$= l_6\delta_{m-2}^n + l_7\delta_{m-1}^n + l_8\delta_m^n + l_9\delta_{m+1}^n + l_{10}\delta_{m+2}^n \tag{18}$$

in which $k_i, l_i(i = 1(1)10)$ and $z_j$ are $z_j = 6U^2$

$$k_1 = 1 - \frac{20\mu}{h^2} - \frac{5\Delta t}{2h}, k_2 = 26 - \frac{40\mu}{h^2} - \frac{25\Delta t}{h}, k_3 = 66 + \frac{120\mu}{h^2},$$
$$k_4 = 26 - \frac{40\mu}{h^2} + \frac{25\Delta t}{h}, k_5 = 1 - \frac{20\mu}{h^2} + \frac{5\Delta t}{h}$$

$$k_6 = 1 - \frac{20\mu}{h^2} + \frac{5\Delta t}{2h}, k_7 = 26 - \frac{40\mu}{h^2} - \frac{25\Delta t}{h}, k_8 = 66 + \frac{120\mu}{h^2},$$
$$k_9 = 26 - \frac{40\mu}{h^2} + \frac{25\Delta t}{h}, k_{10} = 1 - \frac{20\mu}{h^2} - \frac{5\Delta t}{h}$$

$$l_1 = 1 - \frac{20\mu}{h^2} - \frac{5z_j\Delta t}{2h}, l_2 = 26 - \frac{40\mu}{h^2} - \frac{25z_j\Delta t}{h}, l_3 = 66 + \frac{120\mu}{h^2},$$

$$l_4 = 26 - \frac{40\mu}{h^2} + \frac{25z_j\Delta t}{h}, l_5 = 1 - \frac{20\mu}{h^2} + \frac{5z_j\Delta t}{2h}.$$

$$l_6 = 1 - \frac{20\mu}{h^2} + \frac{5z_j\Delta t}{2h}, l_7 = 26 - \frac{40\mu}{h^2} + \frac{25z_j\Delta t}{h}, l_8 = 66 + \frac{120\mu}{h^2},$$

$$l_9 = 26 - \frac{40\mu}{h^2} - \frac{25z_j\Delta t}{h}, l_{10} = 1 - \frac{20\mu}{h^2} - \frac{5z_j\Delta t}{2h}.$$

(17) and (18) are systems consisting of $(N+1)$ equations with $(N+5)$ unknowns. These systems contain four additional element parameters $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ outside the solution region of the problem. To obtain the only solution of systems (17) and (18), the parameters that are not in the solution region must be eliminated from these systems. For this purpose, the nodal values of $U_j$ and $(U_j)'$ in the equation (12) and the boundary conditions $U(x_L, t) = U(x_R, t) = 0$ and $U_x(x_L, t) = U_x(x_R, t) = 0$ are used. Thus, systems (17) and (18) are reduced to the $(N+1)$ x $(N+1)$ matrix system.

For approximate solutions of the (17) and (18) systems, it is necessary to find the initial vector $\delta_j^0$. This required initial vector is found by solving the system of algebraic equations given in the following form, using the initial condition $U(x_j, 0) = U_N(x_j, 0) = g_0(x_j)$, and the approach $U_N(x, 0) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j^0(0)$

$$U_m = \delta_{j-2}^0 + 26\delta_{j-1}^0 + 66\delta_j^0 + 26\delta_{j+1}^0 + \delta_{j+2}^0, j = 0(1)N,$$
$$U_0 = \delta_{-2}^0 + 26\delta_{-1}^0 + 66\delta_0^0 + 26\delta_1^0 + \delta_2^0,$$
$$U_1 = \delta_{-1}^0 + 26\delta_0^0 + 66\delta_1^0 + 26\delta_2^0 + \delta_3^0,$$
$$\cdot$$
$$\cdot \qquad\qquad (19)$$
$$\cdot$$
$$U_{N-1} = \delta_{N-3}^0 + 26\delta_{N-2}^0 + 66\delta_{N-1}^0 + 26\delta_N^0 + \delta_{N+1}^0,$$
$$U_N = \delta_{N-2}^0 + 26\delta_{N-1}^0 + 66\delta_N^0 + 26\delta_{N+1}^0 + \delta_{N+2}^0.$$

with unknown element parameters $\delta_j^0$. By using the boundary conditions $U_x(x_L, t) = U_x(x_R, t) = 0$ and $U_{xx}(x_L, t) = U_{xx}(x_R, t) = 0$ for these systems, $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ are eliminated so that the following matrix equation is obtained

$$\lambda b^0 = d$$

for the initial vector $\delta_j^0$ in which

$$
\lambda = \begin{bmatrix}
54 & 60 & 6 \\
25.25 & 67.5 & 26.25 & 1 \\
1 & 26 & 66 & 26 & 1 \\
& & & \ddots \\
& & 1 & 26 & 66 & 26 & 1 \\
& & & 1 & 26.25 & 67.5 & 25.25 \\
& & & & 6 & 60 & 54
\end{bmatrix},
$$

$$
b^0 = \left( \delta^0{}_0, \delta^0{}_1, \delta^0{}_2...\delta^0{}_{N-2}, \delta^0{}_{N-1}, \delta^0{}_N \right)^T
$$

and

$$
d = \left( U_0, U_1, U_2, ..., U_{N-2}, U_{N-1}, U_N \right)^T.
$$

Also, due to the non-linear $z_j$ term in system (18) to make numerical solutions better, an inner iteration three or five times given in the following form throughout computer work has been applied $(\delta^*)^n = \delta^n + \dfrac{1}{2}(\delta^n - \delta^{n-1})$.

## 5. STABILITY ANALYSIS OF NUMERICAL ALGORITHM

The stability analysis of the Lie-Trotter splitting algorithm applied to the MRLW equation with the help of von Neumann theory is examined. In the previous section, (17) and (18) systems were obtained by dividing the MRLW equation into linear and non-linear sub-equations given as (13) and (14), and applying the collocation method. For stability analysis based on Von Neumann theory of systems (17) and (18), let the growth factors of a typical Fourier mode be defined as follows, with $\gamma$ a mode number and $h$ the element size, and

$$
\delta_j^n = \varrho_1^n e^{ij\gamma h}, \tag{20}
$$

$$
\Psi_j^n = \varrho_2^n e^{ij\gamma h}, \tag{21}
$$

substitute 23 for scheme 17 and 20 for scheme 18. Since the system (18) contains the nonlinear term $6U^2 U_x$, the Fourier mode method cannot be applied to this system, first of all, if the amount of $6U^2$ in the nonlinear term is taken as a local constant like $z_j$, the term $6U^2 U_x$ is linearized and after that Von Neumann method is applied to the mentioned system. It can be stated Von Neumann analysis is one of the most used techniques to analyze the stability analysis of approximate

schemes for linear or linearized partial differential equations. Via the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$, growth factors $\varrho_1$ and $\varrho_2$ submitted as follows are acquired

$$\varrho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \varrho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{22}$$

$$A_1 = (2 - \frac{40\mu}{h^2})cos(2\gamma h) + (52 - \frac{80\mu}{h^2})cos(\gamma h) + (66 + \frac{120\mu}{h^2}),$$

$$B = \frac{5\Delta t}{h}sin(2\gamma h) + \frac{50\Delta t}{h}sin(\gamma h),$$

and

$$C = \frac{5z_m\Delta t}{h}sin(2\gamma h) + \frac{50z_m\Delta t}{h}sin(\gamma h).$$

For $k_1, k_2, ..., k_9, k_{10}$ and $l_1, l_2, ..., l_9, l_{10}$ founded in section 3. $|\varrho_1| = |\varrho_2| = 1$ from Equation (22) and hence, for the whole system with Lie Trotter-Splitting algorithm can be written as $|\varrho_1|.|\varrho_2| = 1$. Because the conditions $|\varrho_1| \leq 1$, and $|\varrho_2| \leq 1$ according to the von Neumann theory are satisfied, it can be clearly said that the systems (17) and (18) are unconditionally stable.

## 6. NUMERICAL EXPERIMENTS AND DISCUSSION

For numerical calculations of main problem are considered to the movement of single solitary wave, two and three solitary wave interactions and the Maxwellian initial condition. The difference between the exact and approximate solutions is calculated by choosing some specific times to match the studies in the literature. For this, the following error norms are used

$$L_2 = ||U - U_N||_2 = \sqrt{h\sum_{j=0}^{N}(U - U_N)^2},$$

and

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|.$$

To check the conservation of numerical schemes during the simulation of solitary wave motion, the invariants $I_1, I_2$ and $I_3$ are calculated, which correspond to the conservation of mass, momentum and energy proved by Olver [29] and presented as follows

$$I_1 = \int_{x_L}^{x_R} U(x,t)dx,$$

$$I_2 = \int_{x_L}^{x_R} [U^2(x,t) + \mu U_x^2(x,t)]dx,$$

$$I_3 = \int_{x_L}^{x_R} [U^4(x,t) - \mu U_x^2(x,t)]dx.$$

6.1. **Example I: The movement of a single solitary wave.** This example considers the MRLW equation by taking into accounting boundary condition $U \to 0$ when $x \to \pm\infty$ and initial condition

$$U(x,0) = \sqrt{c}\,sech[s(x - x_0)].$$

The exact solution for this problem is presented in the following form

$$U(x,t) = \sqrt{c}\,sech[s(x - (c+1)t - x_0)].$$

Here, $c$ and $x_0$ are arbitrary constants and $s = \sqrt{\frac{c}{\mu(c+1)}}$. The exact values of the conservation quantities of a single solitary wave with width $s$ and amplitude $\sqrt{c}$ as in [11] are given as follows

$$I_1 = \int_{x_L}^{x_R} U(x,t)dx = \frac{\pi\sqrt{c}}{s},$$

$$I_2 = \int_{x_L}^{x_R} [U^2(x,t) + \mu U_x^2(x,t)]dx = \frac{2c}{s} + \frac{2\mu sc}{3}, \tag{23}$$

$$I_3 = \int_{x_L}^{x_R} [U^4(x,t) - \mu U_x^2(x,t)]dx = \frac{4c^2}{3s} - \frac{2\mu sc}{3}.$$

For solitary wave motion with amplitude 1, All of calculations and comparisons in Table 1-2 are done with $\mu = 1, x_0 = 40, c = 1, \Delta t = 0.025$ and $h = 0.2$ over $[0, 100]$ to match those in $[11, 16, 21, 22, 36, 37]$. Table 1 reports the invariant and error norm amounts of the current approach from $t = 0$ to 10 with one increment value. This table shows that the calculated invariants are compatible with each other and gratifying because the error norms $L_2$ and $L_\infty$ are quite small. Furthermore, it can be seen from Table 1 that the the changing of invariants $I_1, I_2, I_3$ are less than $0.8\text{x}10^{-7}$, $1.1\text{x}10^{-7}$, $1.26\text{x}10^{-5}$, respectively. The comparison of the ones of the previously recorded methods with the results of the proposed technique is given in Table 2 at time $t = 10$. Looking at the table, it can be easily seen that the current approach produces the best results for error norms and the computed invariant values are in agreement with the analytical ones $I_1 = 4.4428829, I_2 = 3.2998316$ and $I_3 = 1.4142135$. The motion of a single solitary wave at various time levels with parameters $\Delta t = 0.025, h = 0.2, c = 1$ is plotted in Fig.1 and this figure shows that the soliton shifts to the right at a constant velocity with an almost unchanged amplitude even as time increases, as hoped. At $t = 0$, the amplitude is 1 which is situated at $x = 40$ and $x = 60$.

For Table 3-4, the parameters $\mu = 1, x_0 = 40, c = 0.3, \Delta t = 0.01$ and $h = 0.1$ over $[0, 100]$ are selected as in n Refs. [21]. Thus, The amplitude of the solitary wave is 0.547723. Table 3 displays the invariant and error norm amounts of the present approach from $t = 0$ to 20 with two increment value. From this table, it can be observed that very small and pleasing solutions are obtained with the Lie-Trotter splitting technique. Invariants $I_1, I_3$ are compatible with each other and $I_2$ remains constant. Furthermore, it can be seen from Table 3 that the the

TABLE 1. The error norms and invariants of the single solitary wave with $\Delta t = 0.025, h = 0.2$, for $c = 1$ on the region $[0, 100]$

| t | $I_1$ | $I_2$ | $I_3$ | $L_2$ x $10^3$ | $L_\infty$ x $10^3$ |
|---|---|---|---|---|---|
| 0 | 4.44288294 | 3.29983161 | 1.41421360 | 0.00000000 | 0.00000000 |
| 1 | 4.44288293 | 3.29983161 | 1.41420559 | 1.56533150 | 1.02551887 |
| 2 | 4.44288292 | 3.29983159 | 1.41419884 | 2.41711399 | 1.40020924 |
| 3 | 4.44288292 | 3.29983158 | 1.41419842 | 2.67081261 | 1.22652109 |
| 4 | 4.44288291 | 3.29983157 | 1.41419926 | 2.68157587 | 1.04668636 |
| 5 | 4.44288290 | 3.29983156 | 1.41419995 | 2.62782252 | 1.10627938 |
| 6 | 4.44288289 | 3.29983155 | 1.41420040 | 2.56931532 | 1.11288942 |
| 7 | 4.44288289 | 3.29983153 | 1.41420067 | 2.52306335 | 1.07889684 |
| 8 | 4.44288288 | 3.29983152 | 1.41420084 | 2.49240521 | 1.02360259 |
| 9 | 4.44288287 | 3.29983151 | 1.41420094 | 2.47654106 | 0.95806880 |
| 10 | 4.44288286 | 3.29983150 | 1.41420100 | 2.47366508 | 0.89643465 |

TABLE 2. Comparisons of the error norms and invariants of the single solitary wave with $\Delta t = 0.025, h = 0.2$, for $c = 1$ on the region $[0, 100]$ at $t = 10$

| method | $I_1$ | $I_2$ | $I_3$ | $L_2$ x $10^3$ | $L_\infty$ x $10^3$ |
|---|---|---|---|---|---|
| Exact | 4.4428829 | 3.2998316 | 1.4142135 | 0 | 0 |
| Present | 4.4428829 | 3.2998315 | 1.4142010 | 2.47366508 | 0.89643465 |
| [21] | 4.4428661 | 3.2997108 | 1.4143165 | 2.58891 | 1.35164 |
| [22] | 4.44288 | 3.29983 | 1.41420 | 9.30196 | 5.43718 |
| [16] | 4.4431919 | 3.3003022 | 1.4146930 | 2.41750 | 1.08099 |
| [36] | 4.445176 | 3.302476 | 1.417411 | 0.8644 | 1.2475 |
| [11]1 | 4.442 | 9.299 | 1.413 | 19.39 | 9.24 |
| [11]2 | 4.440 | 3.296 | 1.411 | 20.3 | 11.2 |
| [37] | 4.44288 | 3.29981 | 1.41416 | 3.00533 | 1.68749 |

changing of invariants $I_1, I_3$ are less than $1.4 \times 10^{-7}$, $0.9 \times 10^{-7}$, respectively and 0 of $I_2$ during the execution of the program. Table 4 presents a comparison of the results of the proposed study and other registered ones by calculating error norms $L_2$ and $L_\infty$ and invariant values $I_1, I_2$ and $I_3$. All comparisons are made for time t=20. As a result of this comparison, it can easily be seen that the present technique produces more satisfactory results. Invariant values are consistent with those compared. The motion of a single solitary wave at various time levels with parameters $\Delta t = 0.01, h = 0.1, c = 0.3$ is plotted in Fig.2. Fig.3 shows the graphs of the error distributions of the solitary wave with amplitude of 1 and 0.3 respectively, at $t = 10$ and 20.
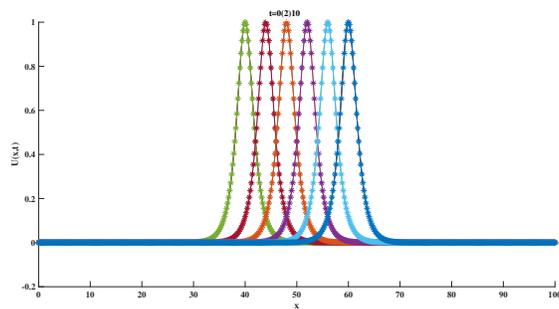
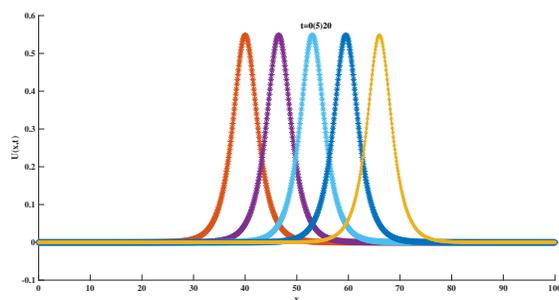FIGURE 1. Movement of a single solitary wave at t = 0(2)10 for MRLW equation



FIGURE 2. Movement of a single solitary wave at t = 0(5)20 for MRLW equation



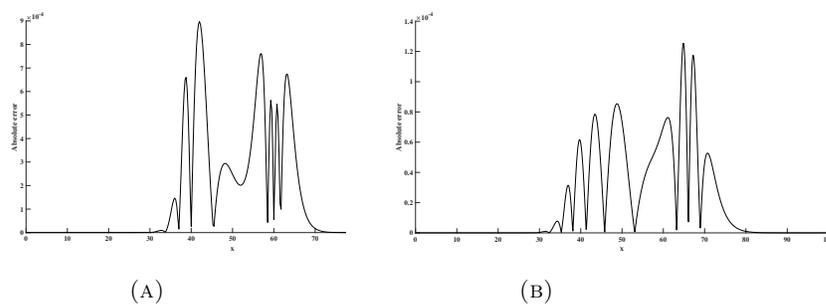(A)                                   (B)

FIGURE 3. Error distribution graphs for a) $\Delta t = 0.025, h = 0.2, c = 1$ and b) $\Delta t = 0.01, h = 0.1, c = 0.3$ over $[0, 100]$.

TABLE 3. The error norms and invariants of the single solitary wave with $\Delta t = 0.01, h = 0.1$, for $c = 0.3$ on the region $[0, 100]$

| t | $I_1$ | $I_2$ | $I_3$ | $L_2$ x $10^3$ | $L_\infty$ x $10^3$ |
|---|---|---|---|---|---|
| 0 | 3.58196673 | 1.34507649 | 0.15372303 | 0.00000000 | 0.00000000 |
| 2 | 3.58196674 | 1.34507649 | 0.15372299 | 0.17472238 | 0.09673881 |
| 4 | 3.58196675 | 1.34507649 | 0.15372294 | 0.28047087 | 0.14367750 |
| 6 | 3.58196676 | 1.34507649 | 0.15372293 | 0.32725879 | 0.14066701 |
| 8 | 3.58196677 | 1.34507649 | 0.15372293 | 0.34590142 | 0.13215052 |
| 10 | 3.58196678 | 1.34507649 | 0.15372293 | 0.35279733 | 0.13158877 |
| 12 | 3.58196678 | 1.34507649 | 0.15372294 | 0.35465095 | 0.13057054 |
| 14 | 3.58196679 | 1.34507649 | 0.15372294 | 0.35426140 | 0.12926338 |
| 16 | 3.58196678 | 1.34507649 | 0.15372294 | 0.35288262 | 0.12791317 |
| 18 | 3.58196674 | 1.34507649 | 0.15372294 | 0.35110940 | 0.12661543 |
| 20 | 3.58196659 | 1.34507649 | 0.15372294 | 0.34923321 | 0.12540476 |

TABLE 4. Comparisons of the error norms and invariants of the single solitary wave with $\Delta t = 0.01, h = 0.1$, for $c = 0.3$ on the region $[0, 100]$ at $t = 20$

| method | $I_1$ | $I_2$ | $I_3$ | $L_2$ x $10^4$ | $L_\infty$ x $10^4$ |
|---|---|---|---|---|---|
| Present | 3.58196659 | 1.34507649 | 0.15372294 | 0.34923321 | 0.12540476 |
| [21] | 3.5820204 | 1.3450974 | 0.1537250 | 0.8112594 | 0.3569076 |
| [22] | 3.58197 | 1.34508 | 0.153723 | 6.06885 | 2.96650 |
| [16] | 3.5820206 | 1.3450944 | 0.1537284 | 1.2273638 | 0.4472294 |
| [36] | 3.582265 | 1.345182 | 0.1538901 | 3.379583 | 7.672911 |
| [1]MQ | 3.5819665 | 1.3450764 | 0.153723 | 0.51498 | 0.22551 |
| [1]TPS | 3.5819663 | 1.3450759 | 0.153723 | 0.51498 | 0.26605 |
| [13] | 3.581967 | 1.345076 | 0.153723 | 0.5089274 | 0.2222848 |

6.2. **Example II: Interaction of two solitary waves.** This example considers the problem of interaction of two solitary waves with various amplitudes and for Eq.(2), the following initial conditions are written as the linear sum of two well-separated solitary waves with different amplitudes

$$U(x, 0) = \sum_{i=1}^{2} a_i sech[s_i(x - x_i)],$$

in which $c_i$ and $x_i$ are arbitrary constants, $a_i = \sqrt{c_i}, s_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}, i = 1(1)2$. The exact values of the conservation quantities are given as follows [11]

$$I_1 = \sum_{i=1}^{2} \frac{\pi\sqrt{c_i}}{s_i},$$

$$I_2 = \sum_{i=1}^{2} \frac{2c_i}{s_i} + \frac{2\mu s_i c_i}{3},$$

$$I_3 = \sum_{i=1}^{2} \frac{4c_i^2}{3s_i} - \frac{2\mu s_i c_i}{3}.$$

Numerical simulation is done on the $[0, 250]$ region by selecting the parameters $\mu = 1, x_1 = 25, x_2 = 55, c_1 = 4, c_2 = 1, \Delta t = 0.025, h = 0.2$ as in Ref. [21]. The experimental results obtained by running the numerical experiments at t=0(2)20 times are shown in Table 5. The exact values of the invariants are $I_1 = 11.467698, I_2 = 14.629243$ and $I_3 = 22.880466$. Table 5 submits a comparison of the solutions in the proposed method with those in the references [1, 16, 19, 21, 22, 34, 37, 38] and this table displays that the invariant quantities $I_1, I_2$ and $I_3$ are quite conservative and the values found are consistent with their exact values throughout the operation of the computer program. Fig.4 reports the interactions of two solitary waves at various time levels. It can be clearly seen from this figure that at $t = 0$, the wave with the smaller amplitude is to the right of the wave with the larger amplitude. Since the shorter wave moves slower than the longer one, the longer wave catches the short wave at $t = 12$ and collides. Later, it fars away from the shorter one with the advancing time. At $t = 20$, while the amplitude of the smaller wave becomes 1.014 at $x = 84.2$ the amplitude of the larger wave becomes 1.998 at $x = 97.4$.

6.3. **Example III:Interaction of three solitary waves.** This example deals with the problem of interaction of three solitary waves with different amplitudes and advancing in the same direction and for MRLW equation, the following initial conditions are written as the linear sum of three well-separated solitary waves with different amplitudes

$$U(x, 0) = \sum_{i=1}^{3} a_i sech[s_i(x - x_i)],$$

in which $c_i$ and $x_i$ are arbitrary constants, $a_i = \sqrt{c_i}, s_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}, i = 1(1)3$. The exact values of the conservation quantities obtained from Eq.(18) are given as follows

$$I_1 = \sum_{i=1}^{3} \frac{\pi\sqrt{c_i}}{s_i},$$

TABLE 5. Comparison of invariants of two solitary waves with values $\Delta t = 0.025, h = 0.2$, for $x_1 = 25, x_2 = 55, c_1 = 4, c_2 = 1$ on the region $[0, 250]$ at $t = 0(2)20$ with those in [13]

| $t$ | method | | | [13] | | |
|---|---|---|---|---|---|---|
| | $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ |
| 0 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467698 | 14.629277 | 22.880432 |
| 2 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467698 | 14.624259 | 22.860365 |
| 4 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467698 | 14.619226 | 22.840279 |
| 6 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467699 | 14.614169 | 22.820069 |
| 8 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467700 | 14.606821 | 22.787857 |
| 10 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467700 | 14.603687 | 22.771773 |
| 12 | 11.46788404 | 14.62924273 | 22.88046615 | 11.467699 | 14.603056 | 22.775766 |
| 14 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467699 | 14.598059 | 22.756029 |
| 16 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467700 | 14.593048 | 22.736127 |
| 18 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467700 | 14.588061 | 22.716289 |
| 20 | 11.46769804 | 14.62924273 | 22.88046615 | 11.467701 | 14.583089 | 22.696510 |
| 20 [19]1 | 11.4676541 | 14.6292088 | 22.8803901 | | | |
| 20 [19]2 | 11.4676452 | 14.6309639 | 22.8786025 | | | |
| 20 [34] | 11.4677 | 14.6299 | 22.8806 | | | |
| 20 [38] | 11.4676977 | 14.62927316 | 22.8804154 | | | |
| 20 [21] | 11.4691886 | 14.6331334 | 22.8764330 | | | |
| 20 [16] | 11.4662207 | 14.6253125 | 22.8650456 | | | |
| 20 [37] | 11.4677 | 14.6299 | 22.8806 | | | |
| 20 [22] | 11.4677 | 14.6292 | 22.8809 | | | |
| 20 [1]MQ | 11.467698 | 14.583052 | 22.696539 | | | |
| 20 [1]TPS | 11.467742 | 14.582424 | 22.694269 | | | |

(A)

(B)

(C)

(D)
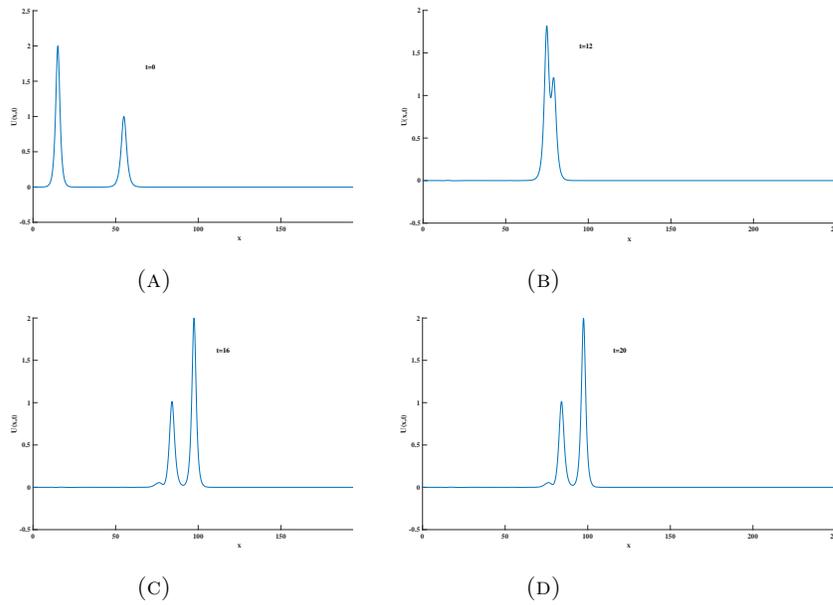
FIGURE 4. The interactions of two solitary waves at various time levels of MRLW equation

$$I_2 = \sum_{i=1}^{3} \frac{2c_i}{s_i} + \frac{2\mu s_i c_i}{3},$$

$$I_3 = \sum_{i=1}^{3} \frac{4c_i^2}{3s_i} - \frac{2\mu s_i c_i}{3}.$$

During the numerical simulation, the calculation process is performed by taking the parameters $\mu = 1, x_1 = 15, x_2 = 45, x_3 = 60 c_1 = 4, c_2 = 1, c_3 = 0.25 \Delta t = 0.025, h = 0.2$ on the region $[0, 250]$. This process is carried out at times $0(5)45$ and the exact values of the invariants here are $I_1 = 14.9801, I_2 = 15.8218$ and $I_3 = 22.9923$. Table 6 presents a comparison of the solutions in the suggested method with those in the references $[1, 21, 22, 34]$ and this table displays that the invariant quantities $I_1$, $I_2$ and $I_3$ are quite conservative and the values found are consistent with their exact values throughout the operation of the computer program. Here, the interaction of solitary waves using various times is shown in Fig.5. This figure indicates that the interaction started at approximately $t = 10$. there were overlaps at time $t = 40$, and then the waves returned to their original state at $t = 40$.

TABLE 6. Comparison of invariants of three solitary waves with values $\Delta t = 0.025, h = 0.2$, for $x_1 = 15, x_2 = 45, x_3 = 60, c_1 = 4, c_2 = 1, c_3 = 0.25$ on the region $[0, 250]$ at $t = 0(5)45$ with those in $[13]$

| $t$ | method $I_1$ | $I_2$ | $I_3$ | [13] $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|---|---|---|
| 0 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980099 | 15.837528 | 23.008136 |
| 5 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980105 | 15.837528 | 22.957891 |
| 10 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980109 | 15.807025 | 22.877972 |
| 15 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980106 | 15.807032 | 22.885947 |
| 20 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980106 | 15.795022 | 22.837454 |
| 25 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980107 | 15.782840 | 22.788852 |
| 30 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980107 | 15.770634 | 22.740419 |
| 35 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980108 | 15.758480 | 22.692279 |
| 40 | 14.97251076 | 15.82181232 | 22.99226955 | 14.980108 | 15.746389 | 22.644448 |
| 45 | 14.97251076 | 15.82181232 | 22.99226955 | 14.968030 | 15.734374 | 22.596591 |
| 20 [34] | 14.930390 | 15.822500 | 22.964190 | | | |
| 20 [21] | 14.7145273 | 15.4927592 | 23.3529062 | | | |
| 20 [22] | 13.7043 | 15.6563 | 22.9303 | | | |
| 20 [1]MQ | 14.96814 | 15.73434 | 22.596625 | | | |
| 20 [1]TPS | 14.96824 | 15.73376 | 22.594494 | | | |

6.4. **Example IV: The Maxwellian initial condition.** This last example examines the improvement of the following the Maxwell initial condition in the sequence of solitary waves

$$U(x, 0) = exp(-(x - 40)^2).$$

Here, the solution behavior for the Maxwellian condition presented above is evaluated with the values of $\mu$. Approximate values of invariants are shown in Table 7. All figures are depicted in Fig. 6 at time 14.5. At the end of the study for values
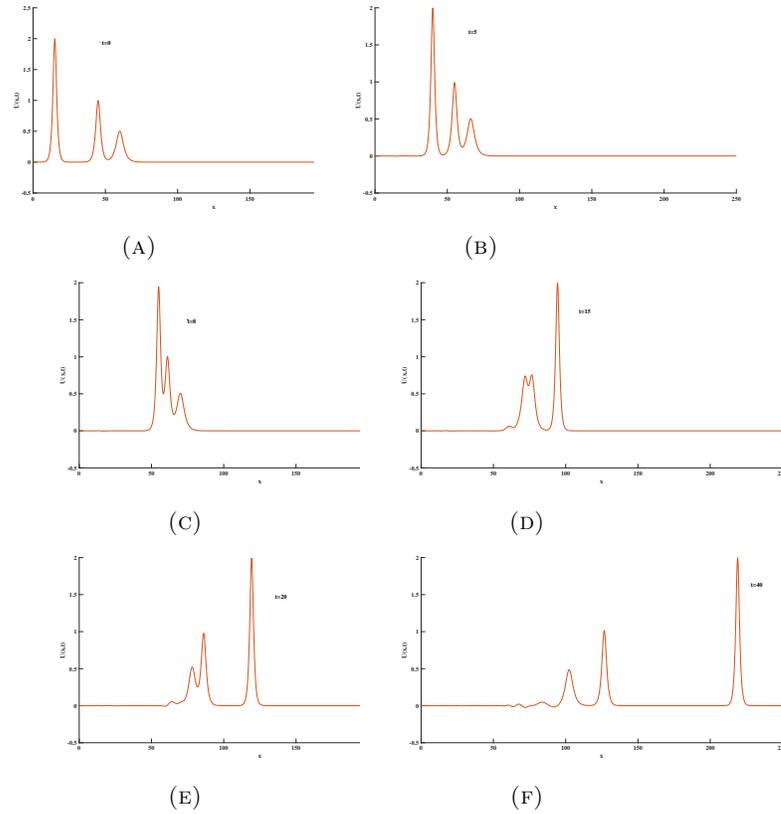
FIGURE 5. The interactions of three solitary waves at various time
levels of MRLW equation

$\mu = 0.1, 0.04, 0.015$ and $\mu = 0.01.$, it is observed that only a single soliton move-
ment is followed for $\mu = 0.1$ and this is pictured with a). When $\mu = 0.04, 0.015$ are
taken, it is shown with b) and c) that two and three stable solitons are occured and
when the value $\mu = 0.01$ is selected, the Maxwellian initial condition decomposes
into four solitary waves and it is plotted with d). In all the figures in this example,
the presence of a small oscillating tail formed behind the last wave is observed.
The peaks of the well-developed wave whose speeds are linearly dependent on their
amplitudes lie on a straight line.

## 7. Conclusion

In this article, Lie-trotter splitting algorithm with collocation finite element
method has been presented. Four experimental examples are given to measure

Table 7. Values of the invariants of the MRLW equation for the Maxwellian initial condition

| $t$ | $\mu$ | $I_1$ | $I_2$ | $I_3$ | $\mu$ | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.1 | 1.77245385 | 1.37864519 | 0.76089588 | 0.015 | 1.77245385 | 1.27211379 | 0.86742727 |
| 3 | | 1.77242409 | 1.37853754 | 0.66726850 | | 1.74470267 | 1.23369811 | 0.72548127 |
| 6 | | 1.77237678 | 1.37846448 | 0.66768464 | | 1.71890317 | 1.17440856 | 0.73866510 |
| 9 | | 1.77232940 | 1.37839090 | 0.66787415 | | 1.70124953 | 1.13869004 | 0.74633211 |
| 12 | | 1.77228192 | 1.37831554 | 0.66798301 | | 1.68830339 | 1.11560302 | 0.75073526 |
| 15 | | 1.77223441 | 1.37823955 | 0.66805774 | | 1.68128767 | 1.11897777 | 0.74641698 |
| 0 | 0.01 | 1.77245385 | 1.26584724 | 0.87369382 | 0.04 | 1.77245385 | 1.30344656 | 0.83609451 |
| 3 | | 1.71307229 | 1.14513104 | 0.76148916 | | 1.77091565 | 1.30148045 | 0.69106562 |
| 6 | | 1.68854075 | 1.12482375 | 0.75995674 | | 1.76851865 | 1.29612046 | 0.69320291 |
| 9 | | 1.66818971 | 1.08010404 | 0.76885908 | | 1.76632858 | 1.29227683 | 0.69439710 |
| 12 | | 1.65804755 | 1.07686486 | 0.76703741 | | 1.76426994 | 1.28925313 | 0.69497936 |
| 15 | | 1.64143143 | 1.02666184 | 0.77926968 | | 1.76221840 | 1.28586712 | 0.69570702 |

(A)

(B)

(C)

(D)
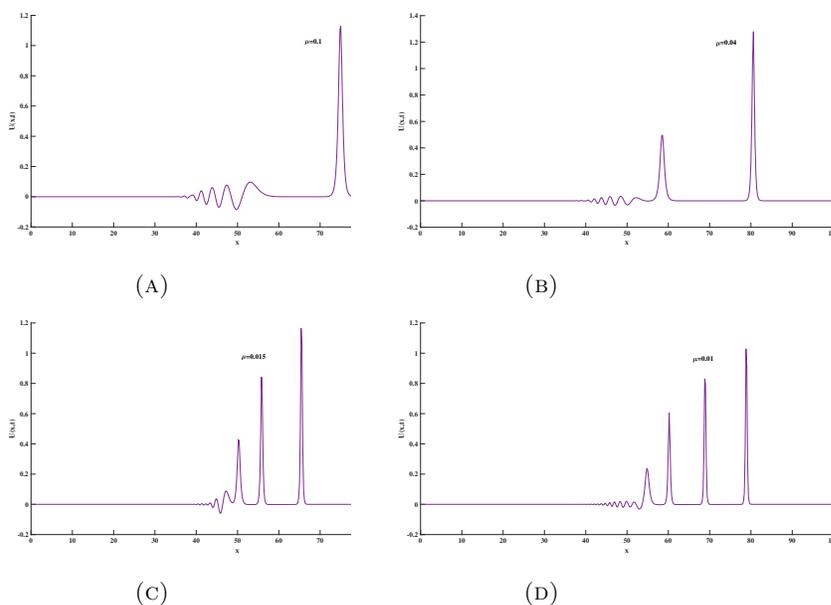
Figure 6. The interactions of two solitary waves at various time levels of MRLW equation

the reliability and performance of the method in this study. The error norms $L_2$ and $L_\infty$ are calculated to show that results are superior to those of the methods in the literature and the results produced have been compared in tables and figures. The error norms are, as hoped, smaller than the results in the literature and thus closer to the analytic solution. The invariants $I_1, I_2$ and $I_3$ are satisfactorily well preserved throughout the entire computer run. The computed solutions displays it

can be easily said that the current algorithm will be beneficial in applying to other nonlinear equation types such as MRLW.

**Declaration of Competing Interests** The author declares that there is no competing interest regarding the publication of this paper.

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