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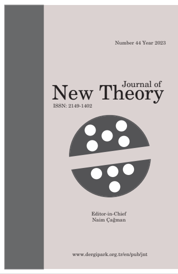

BL-Algebras with Permuting Tri-Derivations

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Abstract — Basic logic algebras (BL-algebras) were introduced by Hajek. Multi-value algebras (MV-algebras), Gödel algebras, and product algebras are particular cases of BL-algebras. Moreover, BL-algebras are algebraic structures, and their principal examples are the real interval $[0,1]$ with the structure given by a continuous t-norm and abelian l-groups. In this article, we consider a type of derivation structure on BL-algebras. We study $(\odot,\vee)$-permuting tri-derivations of BL-algebras and their examples and basic properties. We obtain results regarding the trace of $(\odot,\vee)$-permuting derivations on Gödel BL-algebras. Finally, the article presents that the results herein can be generalized in future research.

Keywords *BL-algebra, permuting tri-derivation, Boolean algebra, isotone, trace*

Mathematics Subject Classification (2020) 08A05, 03G25

1. Introduction

The interest in multi-value algebras (MV-algebras) introduced by Chang [1] continues to increase. One of the most important examples of MV-algebra is the interval $[0,1]$ of the commutative l-group: $(M, \max, \min, +, ', 0)$ equipped with a continuous t-conorm and continuous t-norm defined by $m \oplus n = \min\{m+n, 1\}$ and $m \odot n = \max\{m+n-1, 0\}$ and with the negation defined by $m^{'} = 1-m$. After MV-algebras, Gödel algebras and product algebras have been investigated [2–5]. These three structures form the most important algebraic model structure for fuzzy logic, which are Lukasiewicz logic, Gödel logic and product logic, respectively. These logics studying these algebras are of logical interest, as well as their connection to some mathematical structures, as they correspond to the most important continuous t-norms on $[0,1]$ and their associated residues. Basic logic algebras (BL-algebras) [6] has been introduced by Hajek. MV-algebras, Gödel algebras, and product algebras are particular cases of BL-algebras. Various derivation studies [7–9] have been done on BL-algebras. In this article, we investigate a derivation type defined by some authors [10–13] in rings, lattices, and MV-algebras. More precisely, we define a type of permuting tri-derivations on BL-algebras and study some of its properties.

This article is organized as follows: The next section reminds some results and basic properties of BL-algebras. Section 3 defines $(\odot,\vee)$-permuting tri-derivation structure in BL-algebras and obtains their some results. Moreover, it explores many properties by the trace of the $(\odot,\vee)$-permuting tri-derivation on BL-algebras. Finally, the conclusion briefly overviews this type of algebra and discusses future studies.

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2. Preliminaries

This section provides some basic notions to be needed for the next section. A t-norm $\otimes$ on the real interval $[0,1]$ is a commutative and associative operation on $[0,1]$ such that

i. If $\beta \leq \delta$, then $\beta \otimes \eta \leq \delta \otimes \eta$, for all $\beta, \delta, \eta \in [0,1]$

ii. $1 \otimes \beta = \beta$, for all $\beta \in [0,1]$

From i and ii, $0 \otimes \beta = 0$, for all $\beta \in [0,1]$. A t-norm $\otimes$ on $[0,1]$ is continuous if it is continuous in the usual sense as a function $\otimes : [0,1] \times [0,1] \to [0,1]$. On $[0,1]$, Lukasiewicz t-norm, Gödel t-norm, and Product t-norm, the most important continuous t-norms, are defined as follows, respectively:

$$\beta \odot \delta = \max\{\beta + \delta - 1, 0\}$$

$$\beta \odot \delta = \min\{\beta, \delta\}$$

and

$$\beta \odot \delta = \beta\delta$$

Note that every continuous t-norm $\otimes$ on $[0,1]$ induces a binary operation $\to$ defined by

$$\beta \to \delta = \max\{\eta \in [0,1] |\, \beta \otimes \eta \leq \delta\}$$

called the associated residuum. The binary operation satisfies the statement $\beta \otimes \delta \leq \eta$ if and only if (iff) $\beta \leq \delta \to \eta$, for all $\beta, \delta, \eta \in [0,1]$. These three norms above are related to fuzzy logic along with the residues Lukasiewicz implication, Gödel implication, and Product implication, defined as follows, respectively:

$$\beta \to \delta = \min\{1, 1 - \beta - \delta\}$$

$$\beta \to \delta = \begin{cases} 1, & \beta \leq \delta \\ \delta, & \text{otherwise} \end{cases}$$

and

$$\beta \to \delta = \begin{cases} 1, & \beta \leq \delta \\ \frac{\delta}{\beta}, & \text{otherwise} \end{cases}$$

If $\otimes$ is a continuous t-norm on $[0,1]$ and $\to$ is the associated residuum, then the structure

$$([0,1], \max, \min, \otimes, \to, 0, 1)$$

is the starting point in describing and investigating Basic Logic and corresponds to the basic logic system: BL-algebras.

Definition 2.1. [6] A basic logic algebra (BL-algebra) is a structure $(\Lambda, \wedge, \vee, \odot, \to, 0, 1)$ such that

i. $(\Lambda, \wedge, \vee, 0, 1)$ is a bounded lattice

ii. $(\Lambda, \odot, 1)$ is a commutative monoid

iii. $\beta \leq \delta \to \eta$ iff $\delta \odot \beta \leq \eta$ (residuation)

iv. $\beta \wedge \eta = \beta \odot (\beta \to \eta)$ (divisibility)

v. $(\beta \to \eta) \vee (\eta \to \beta) = 1$ (prelinearity)

For any $\beta, \eta \in \Lambda$, we define

$$\beta^* = \beta \to 0, \quad \beta \oplus \eta = \left(\beta^* \odot \eta^*\right)^*, \quad \text{and} \quad \beta \ominus \eta = \beta \odot \eta^*$$

We recall that a BL-algebra Λ is an MV-algebra iff $(\beta^*)^* = \beta$, for all $\beta \in \Lambda$ [4]. A Gödel algebra is a

BL-algebra Λ satisfying the condition $\beta \odot \beta = \beta$, for all $\beta \in \Lambda$ [6]. Moreover, a product algebra is a BL-algebra Λ satisfying the conditions *i.* $\beta \wedge \beta^* = 0$ and *ii.* $(\delta^*)^* \odot (\beta \odot \delta \rightarrow \eta \odot \delta) \leq \beta \rightarrow \eta$, for all $\beta, \eta, \delta \in \Lambda$ [6].

Proposition 2.2. [2] Suppose that Λ is a BL-algebra and $\beta, \eta, \delta \in \Lambda$. Thus, the followings hold:

i. $\beta \leq \eta$ iff $\beta \rightarrow \eta = 1$

ii. $\beta \rightarrow (\eta \rightarrow \delta) = (\beta \odot \eta) \rightarrow \delta = \eta \rightarrow (\beta \rightarrow \delta)$

iii. If $\beta \leq \eta$, then $\eta \rightarrow \delta \leq \beta \rightarrow \delta$, $\delta \rightarrow \beta \leq \delta \rightarrow \eta$, $\beta \odot \delta \leq \eta \odot \delta$, and $\eta^* \leq \beta^*$

iv. $\beta \leq (\eta \rightarrow \beta) \rightarrow \beta$, $\eta \leq (\eta \rightarrow \beta) \rightarrow \beta$, and $\beta \vee \eta = ((\beta \rightarrow \eta) \rightarrow \eta) \vee ((\eta \rightarrow \beta) \rightarrow \beta)$

v. $\beta \odot \eta \leq \beta$, $\beta \odot \eta \leq \eta$, $\beta \odot \eta \leq \beta \wedge \eta$, $\beta \odot 0 = 0$, and $\beta \odot \beta^* = 0$

vi. $1 \rightarrow \beta = \beta$, $\beta \rightarrow \beta = 1$, $\beta \leq \eta \rightarrow \beta$, $\beta \rightarrow 1 = 1$, and $0 \rightarrow \beta = 1$

vii. $\beta \odot \eta = 0$ iff $\beta \leq \eta^*$

viii. $\beta \odot (\eta \wedge \sigma) = (\beta \odot \eta) \wedge (\beta \odot \sigma)$

ix. $\beta \odot (\eta \vee \sigma) = (\beta \odot \eta) \vee (\beta \odot \sigma)$

A BL-algebra satisfying $\beta \vee \beta^* = 1$ is called a Boolean algebra. For a BL-algebra Λ, if

$$B(\Lambda) = \{\beta \in \Lambda : \beta \oplus \beta = \beta\} = \{\beta \in \Lambda : \beta \odot \beta = \beta\}$$

then $(B(\Lambda), \oplus, *, 0)$ is the largest subalgebra of Λ and a Boolean algebra. Hence, $B(\Lambda)$ is called Boolean center of Λ.

Theorem 2.3. [2] Let Λ be a BL-algebra and $\beta, \eta \in \Lambda$. Then, the following conditions are equivalent:

i. $\beta \in B(\Lambda)$

ii. $\beta \odot \beta = \beta$ and $\beta^{**} = \beta$

iii. $\beta \odot \beta = \beta$ and $\beta^* \rightarrow \beta = \beta$

iv. $\beta^* \vee \beta = 1$

v. $(\beta \rightarrow \eta) \rightarrow \beta = \beta$

vi. $\beta \wedge \eta = \beta \odot \eta$

3. $(\odot, \vee)$-Permuting Tri-derivations on BL-algebras

This section defines $(\odot, \vee)$-permuting tri-derivation structures in BL-algebras and explores their some results. It investigates the properties provided by the trace of the $(\odot, \vee)$-permuting tri-derivation on BL-algebras. Throughout this paper, Λ denotes a BL-algebra unless otherwise specified.

Definition 3.1. A map $\Phi : \Lambda \times \Lambda \times \Lambda \rightarrow \Lambda$ is called a permuting mapping if $\Phi(\beta, \delta, \theta) = \Phi(\beta, \theta, \delta) = \Phi(\delta, \beta, \theta) = \Phi(\delta, \theta, \beta) = \Phi(\theta, \beta, \delta) = \Phi(\theta, \delta, \beta)$, for all $\beta, \delta, \theta \in \Lambda$.

Furthermore, a map $\varphi : \Lambda \rightarrow \Lambda$ defined by $\varphi(\beta) = \Phi(\beta, \beta, \beta)$ is referred to as the trace of Φ such that Φ is a permuting mapping. Hereinafter, for brevity, we use the notation $\varphi\beta$ instead of $\varphi(\beta)$.

Definition 3.2. Let $\Phi : \Lambda \times \Lambda \times \Lambda \rightarrow \Lambda$ be a permuting mapping. If Φ satisfies the following condition

$$\Phi(\beta \odot \eta, \delta, \theta) = (\Phi(\beta, \delta, \theta) \odot \eta) \oplus (\beta \odot \Phi(\eta, \delta, \theta))$$

for all $\beta, \eta, \delta, \theta \in \Lambda$, then Φ is called a permuting tri-derivation. Clearly, if Φ is a permuting tri-derivation on Λ, then the following relations hold:

$$\Phi(\beta, \delta \odot \eta, \theta) = (\Phi(\beta, \delta, \theta) \odot \eta) \oplus (\delta \odot \Phi(\beta, \eta, \theta))$$

and

$$\Phi(\beta, \delta, \theta \odot \eta) = (\Phi(\beta, \delta, \theta) \odot \eta) \oplus (\theta \odot \Phi(\beta, \delta, \eta))$$

for all $\beta, \eta, \delta, \theta \in \Lambda$.

Example 3.3. Let $\Lambda = \{0, \beta, \theta, 1\}$ such that $0 < \beta < \theta < 1$ and the binary operations $\odot$ and $\rightarrow$ be defined as follows:

$\odot$	0	β	θ	1
0	0	0	0	0
β	0	β	β	β
θ	0	β	θ	θ
1	0	β	θ	1

and

$\rightarrow$	0	β	θ	1
0	1	1	1	1
β	0	1	1	1
θ	0	β	1	1
1	0	β	θ	1

Then, $(\Lambda, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Define a map $\Phi : \Lambda \times \Lambda \times \Lambda \rightarrow \Lambda$ by

$$\Phi\left(x_1, x_2, x_3\right) = \begin{cases} \beta, & (x_1, x_2, x_3) \in \left\{ \begin{array}{c} (\beta,\beta,\beta), (\beta,\beta,\theta), (\beta,\theta,\beta), (\theta,\beta,\beta), (\beta,\beta,1), \\ (\beta,1,\beta), (1,\beta,\beta), (\theta,\theta,1), (\theta,1,\theta), (1,\theta,\theta) \end{array} \right\} \\ \theta, & (x_1, x_2, x_3) = (\theta,\theta,\theta) \\ 0, & \text{otherwise} \end{cases}$$

It is easy to observe that Φ is a $(\odot, \vee)$-permuting tri-derivation on Λ.

Proposition 3.4. Assume that Φ is a $(\odot, \vee)$-permuting tri-derivation on Λ, φ is the trace of Φ, and $\beta, \delta, \theta \in \Lambda$. Then, the following conditions are valid:

i. $\varphi 0 = 0$

ii. $\varphi\beta \odot \beta^* = \beta \odot \varphi\beta^* = 0$

iii. $\varphi\beta = \varphi\beta \vee (\beta \odot \Phi(\beta, \beta, 1))$

iv. If $\beta \in B(\Lambda)$, then $\beta \leq (\Phi(\beta, \beta, \beta^*))^*$

v. If $\beta \in B(\Lambda)$, then $\Phi(\beta, \delta, \theta) \leq \beta$ and $\Phi(\beta^*, \delta, \theta) \leq \beta^*$

Proof.

i. For all $\beta \in \Lambda$,

$$\begin{aligned} \Phi(\beta, \beta, 0) &= \Phi(\beta, \beta, 0 \odot 0) \\ &= (\Phi(\beta, \beta, 0) \odot 0) \vee (0 \odot \Phi(\beta, \beta, 0)) \\ &= 0 \vee 0 = 0 \end{aligned}$$

Since φ is the trace of Φ,

$$\begin{aligned} \varphi 0 &= \Phi(0, 0, 0) \\ &= \Phi(0 \odot 0, 0, 0) \\ &= (\Phi(0, 0, 0) \odot 0) \vee (0 \odot \Phi(0, 0, 0)) \\ &= 0 \vee 0 = 0 \end{aligned}$$

ii. For all $\beta \in \Lambda$,

$$\begin{aligned} 0 &= \Phi(\beta, \beta, 0) \\ &= \Phi(\beta, \beta, \beta \odot \beta^*) \\ &= (\Phi(\beta, \beta, \beta) \odot \beta^*) \vee (\beta \odot \Phi(\beta, \beta, \beta^*)) \end{aligned}$$

Hence, $\varphi\beta \odot \beta^* = 0$ and $\beta \odot \Phi(\beta,\beta,\beta^*) = 0$. Similarly, $\beta \odot \varphi\beta^* = 0$, for all $\beta \in \Lambda$.

iii. For all $\beta \in \Lambda$,

$$\begin{aligned}
\varphi\beta &= \Phi(\beta,\beta,\beta)\\
&= \Phi(\beta,\beta,\beta \odot 1)\\
&= (\Phi(\beta,\beta,\beta) \odot 1) \vee (\beta \odot \Phi(\beta,\beta,1))\\
&= \varphi\beta \vee (\beta \odot \Phi(\beta,\beta,1))
\end{aligned}$$

iv. Let $\beta \in B(\Lambda)$. Since $\Phi(\beta,\beta,\beta^*) \odot \beta = 0$, then $\Phi(\beta,\beta,\beta^*) \leq \beta^*$. From Theorem 2.3, $\beta^{**} = \beta$ because $\beta \in B(\Lambda)$. From Proposition 2.2 *iii*, $\beta \leq (\Phi(\beta,\beta,\beta^*))^*$.

v. Let $\beta \in B(\Lambda)$. For all $\delta, \theta \in \Lambda$,

$$\Phi(\beta \odot \beta^*, \delta, \theta) = (\Phi(\beta,\delta,\theta) \odot \beta^*) \vee (\beta \odot \Phi(\beta^*,\delta,\theta)$$

which implies that $\Phi(\beta,\delta,\theta) \odot \beta^* = 0$ and $\beta \odot \Phi(\beta^*,\delta,\theta) = 0$. Thus, $\Phi(\beta,\delta,\theta) \leq \beta$ and $\Phi(\beta^*,\delta,\theta) \leq \beta^*$.

□

Proposition 3.5. Let Φ be a $(\odot,\vee)$-permuting tri-derivation on Λ, φ be the trace of Φ, and $\beta, \theta \in \Lambda$ such that $\beta \leq \theta$. Then, the followings hold:

i. $\varphi(\beta \odot \theta^*) = 0$

ii. $\varphi\theta^* \leq \beta^*$

iii. If $\beta \in B(\Lambda)$, then $\varphi\beta \odot \varphi\theta^* = 0$

PROOF.

i. Since $\beta \leq \theta$, then $\beta \odot \theta^* \leq \theta \odot \theta^* = 0$. Hence, $\beta \odot \theta^* = 0$. Thus, $\varphi(\beta \odot \theta^*) = 0$.

ii. Since $\beta \leq \theta$, then $\beta \odot \varphi\theta^* \leq \theta \odot \varphi\theta^* = 0$. Hence, $\varphi\theta^* \odot \beta = 0$. From Proposition 2.2 *vii*, $\beta \in B(\Lambda)$. Then, $\varphi\theta^* \leq \beta^*$.

iii. Let $\beta \in B(\Lambda)$. For all $\theta \in \Lambda$,

$$0 = \Phi(\beta \odot \beta^*, \theta, \theta) = (\Phi(\beta,\theta,\theta) \odot \beta^*) \vee (\beta \odot \Phi(\beta^*,\theta,\theta))$$

Hence, $\Phi(\beta,\theta,\theta) \odot \beta^* = 0$. Therefore, $\Phi(\beta,\theta,\theta) \leq \beta$. Thus, we replace θ by β in the last relation, $\varphi\beta \leq \beta$. Therefore, $\varphi\beta \odot \varphi\theta^* \leq \beta \odot \varphi\theta^* = 0$.

□

Proposition 3.6. Let Φ be a $(\odot,\vee)$-permuting tri-derivation on Λ, φ be the trace of Φ, and $\beta \in B(\Lambda)$. Then,

i. $\varphi\beta \odot \varphi\beta^* = 0$

ii. $\varphi\beta^* = (\varphi\beta)^*$ iff φ is the identity on Λ

PROOF.

i. By Proposition 3.5 *iii*, as $\beta \leq \beta$ and $\beta \in B(\Lambda)$, then $\varphi\beta \odot \varphi\beta^* = 0$.

ii. $(\Rightarrow)$: Since $\beta \odot \varphi\beta^* = 0$, then $\beta \odot (\varphi\beta)^* = 0$. Thus, $\varphi\beta \leq \beta$ and $\beta \leq \varphi\beta$ which implies that $\varphi\beta = \beta$. Hence, φ is identity on Λ.

$(\Leftarrow)$: If φ is an identity on Λ, then $\varphi\beta^* = (\varphi\beta)^*$, for all $\beta \in \Lambda$.

□

Definition 3.7. Let Φ be a $(\odot,\vee)$-permuting tri-derivation on Λ. If $\beta \le \delta$ implies $\Phi(\beta,\theta,\eta) \le \Phi(\delta,\theta,\eta)$, for all $\beta,\theta,\eta \in \Lambda$, then Φ is called an isotone. If φ is the trace of Φ, and Φ is an isotone, then $\beta \le \delta$ implies $\varphi\beta \le \varphi\delta$, for all $\beta,\delta \in \Lambda$.

Example 3.8. Let Λ be a BL-algebra and Φ be a $(\odot,\vee)$-permuting tri-derivation in Example 3.3. Then, Φ is not isotone, because $\Phi(\beta,\beta,1) \nleq \Phi(1,\beta,1)$.

Example 3.9. Consider $\Lambda = \{0,\beta,\theta,1\}$ in Example 3.3. Define a map $\Phi : \Lambda\times\Lambda\times\Lambda \to \Lambda$ by

$$\Phi\left(x_1,x_2,x_3\right)=\begin{cases}\beta, (x_1,x_2,x_3)\in\left\{\begin{array}{c}(\beta,\beta,\beta),(\beta,\beta,\theta),(\beta,\theta,\beta),(\theta,\beta,\beta),(\beta,\beta,1),(\beta,1,\beta),(1,\beta,\beta),\\(\beta,\theta,\theta),(\theta,\beta,\theta),(\theta,\theta,\beta),(\beta,\theta,1),(\beta,1,\theta),(\theta,\beta,1),(\theta,1,\beta),\\(1,\beta,\theta),(1,\theta,\beta),(\beta,1,1),(1,\beta,1),(1,1,\beta)\end{array}\right\}\\ \theta, \quad (x_1,x_2,x_3)\in\left\{(\theta,\theta,\theta),(\theta,\theta,1),(\theta,1,\theta),(1,\theta,\theta),(\theta,1,1),(1,\theta,1),(1,1,\theta)\right\}\\ 1, \quad (x_1,x_2,x_3)=(1,1,1)\\ 0, \quad \text{otherwise}\end{cases}$$

Then, Φ is an isotone $(\odot,\vee)$-permuting tri-derivation on Λ.

Proposition 3.10. Suppose that Φ is a $(\odot,\vee)$-permuting tri-derivation on Λ and φ is the trace of Φ. If $\varphi\beta^{*} = \varphi\beta$, for all $\beta \in \Lambda$, then the followings hold:

i. $\varphi 1 = 0$

ii. $\varphi\beta \odot \varphi\beta = 0$

iii. If Φ is an isotone on Λ, then $\varphi = 0$

PROOF.

i. Replacing β by 0 in the relation $\varphi\beta^{*} = \varphi\beta$, since $0^{*} = 1$ and $\varphi 0 = 0$, then $\varphi 1 = 0$.

ii. Using hypothesis, $\varphi\beta \odot \varphi\beta = \varphi\beta \odot \varphi\beta^{*} = 0$, for all $\beta \in \Lambda$.

iii. Let Φ be an isotone on Λ. For $\beta \in \Lambda$, $\varphi\beta = 0$, since $\varphi\beta \le \varphi 1 = 0$. Thus, $\varphi = 0$.

□

Definition 3.11. Let Φ be a $(\odot,\vee)$-permuting tri-derivation on Λ. If $\Phi(\beta\odot\theta,\delta,\eta) = \Phi(\beta,\delta,\eta)\odot\Phi(\theta,\delta,\eta)$, for all $\beta,\theta,\delta,\eta \in \Lambda$, then Φ is called a tri-multiplicative $(\odot,\vee)$-permuting tri-derivation on Λ.

Theorem 3.12. Suppose that Φ is a tri-multiplicative $(\odot,\vee)$-permuting tri-derivation on Λ and φ is the trace of Φ. Hence, $\varphi(B(\Lambda)) \subseteq B(\Lambda)$.

PROOF.
Let $\beta \in \varphi(B(\Lambda))$. Then, $\beta = \varphi(\theta)$, for some $\theta \in B(\Lambda)$. Thus, $\beta\odot\beta = \varphi\theta\odot\varphi\theta = \Phi(\theta\odot\theta,\theta,\theta) = \varphi\theta = \beta$. Therefore, $\beta \in B(\Lambda)$. Hence, $\varphi(B(\Lambda)) \subseteq B(\Lambda)$. □

Theorem 3.13. Let Φ be a $(\odot,\vee)$-permuting tri-derivation on Λ. If there exists a $\beta \in \Lambda$ such that $\beta\odot\Phi(\theta,\delta,\eta) = 1$, for all $\theta,\delta,\eta \in \Lambda$, then $\beta = 1$.

PROOF.
Assume that there exists a $\beta \in \Lambda$ such that $\Phi(\theta,\delta,\eta)\odot\beta = 1$, for all $\theta,\delta,\eta \in \Lambda$. Since Φ is a $(\odot,\vee)$-permuting tri-derivation on Λ,

$$\begin{aligned} 1 &= \Phi(\theta \odot \beta, \delta, \eta) \odot \beta \\ &= ((\Phi(\theta, \delta, \eta) \odot \beta) \vee (\theta \odot \Phi(\beta, \delta, \eta)) \odot \beta \\ &= (1 \vee (\theta \odot \Phi(\beta, \delta, \eta)) \odot \beta \\ &= 1 \odot \beta \\ &= \beta \end{aligned}$$

□

Theorem 3.14. Let Φ be a $(\odot, \vee)$-permuting tri-derivation on $B(\Lambda)$. Thus,

i. Φ is a permuting tri-derivation on a lattice, i.e.,

$$\Phi(\beta \wedge \eta, \theta, \delta) = (\Phi(\beta, \theta, \delta) \wedge \eta) \vee (\beta \wedge \Phi(\eta, \theta, \delta)), \quad \text{for all } \beta, \theta, \delta, \eta \in B(\Lambda)$$

ii. $\Phi(\beta, \theta, \delta) = \Phi(\beta, \theta, \delta) \wedge \beta$, for all $\beta, \theta, \delta \in B(\Lambda)$

PROOF.

i. For all $\beta, \theta, \delta, \eta \in B(\Lambda)$,

$$\begin{aligned} \Phi(\beta \wedge \eta, \theta, \delta) &= \Phi(\beta \odot \eta, \theta, \delta) \\ &= (\Phi(\beta, \theta, \delta) \odot \eta) \vee (\beta \odot \Phi(\eta, \theta, \delta)) \\ &= (\Phi(\beta, \theta, \delta) \wedge \eta) \vee (\beta \wedge \Phi(\eta, \theta, \delta)) \end{aligned}$$

ii. For all $\beta, \theta, \delta \in B(\Lambda)$,

$$\begin{aligned} \Phi(\beta, \theta, \delta) &= \Phi(\beta \odot \beta, \theta, \delta) \\ &= (\Phi(\beta, \theta, \delta) \odot \beta) \vee (\beta \odot \Phi(\beta, \theta, \delta)) \\ &= \Phi(\beta, \theta, \delta) \odot \beta \\ &= \Phi(\beta, \theta, \delta) \wedge \beta \end{aligned}$$

□

Theorem 3.15. Suppose that Φ is a $(\odot, \vee)$-permuting tri-derivation on Gödel BL-algebra Λ, φ is the trace of Φ, and $\beta, \theta \in \Lambda$. Then, the followin conditions are valid:

i. $\varphi\beta \leq \beta$

ii. If $\beta \leq \Phi(\beta, \beta, 1)$, then $\varphi\beta = \beta$

iii. If $\beta \geq \Phi(\beta, \beta, 1)$, then $\Phi(\beta, \beta, 1) \leq \varphi\beta$

iv. If $\beta \leq \theta$, then $\varphi\beta = \beta$ or $\varphi\beta \geq \Phi(\beta, \beta, \theta)$

PROOF.

i.

$$\begin{aligned} \varphi\beta &= \Phi(\beta, \beta, \beta) = \Phi(\beta \odot \beta, \beta, \beta) \\ &= (\varphi\beta \odot \beta) \vee (\beta \odot \varphi\beta) = \varphi\beta \odot \beta \\ &= \min\{\varphi\beta, \beta\} \end{aligned}$$

Thus, $\varphi\beta \leq \beta$.

ii. Let $\beta \leq \Phi(\beta, \beta, 1)$, for $\beta \in \Lambda$. Thus,

$$\begin{aligned}
\varphi\beta &= \Phi(\beta, \beta, \beta) \\
&= \Phi(\beta \odot 1, \beta, \beta) \\
&= (\varphi\beta \odot 1) \vee (\beta \odot \Phi(1, \beta, \beta)) \\
&= \varphi\beta \vee \min\{\beta, \Phi(1, \beta, \beta)\} \\
&= \varphi\beta \vee \beta \\
&= \beta, \text{ by } i.
\end{aligned}$$

iii. Let $\beta \geq \Phi(\beta, \beta, 1)$, for $\beta \in \Lambda$. Hence,

$$\begin{aligned}
\varphi\beta &= \Phi(\beta, \beta, \beta) \\
&= \Phi(\beta \odot 1, \beta, \beta) \\
&= \varphi\beta \vee \min\{\beta, \Phi(1, \beta, \beta)\} \\
&= \varphi\beta \vee \Phi(1, \beta, \beta)
\end{aligned}$$

Thus, $\Phi(\beta, \beta, 1) \leq \varphi\beta$.

iv. Let $\beta \leq \theta$. From i, $\varphi\beta \leq \beta \leq \theta$. Thus, $\varphi\beta \leq \theta$. Thereby,

$$\begin{aligned}
\varphi\beta &= \Phi(\beta, \beta, \beta) \\
&= \Phi(\beta \odot \theta, \beta, \beta) \\
&= (\varphi\beta \odot \theta) \vee (\beta \odot \Phi(\theta, \beta, \beta)) \\
&= \varphi\beta \vee (\beta \odot \Phi(\theta, \beta, \beta))
\end{aligned}$$

If $\beta \leq \Phi(\theta, \beta, \beta)$, then $\varphi\beta = \beta$ by i. If $\beta \geq \Phi(\theta, \beta, \beta)$, then $\varphi\beta = \varphi\beta \vee \Phi(\theta, \beta, \beta)$. Hence, $\varphi\beta \geq \Phi(\beta, \beta, \theta)$.

□

4. Conclusion

BL-algebras were introduced by Hajek [6] to investigate many-valued logic. One of the reasons for his motivations for introducing BL-algebras was providing an algebraic counterpart of propositional logic, called Basic Logic (BL-logic), which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic, and Product Logic. This BL-logic is proposed as "the most general" many-valued logic with truth values in $[0, 1]$. Another reason to work was to provide an algebraic mean for the study of continuous t-norms on $[0, 1]$. In this work, we introduce one type of permuting tri-derivation on BL-algebras and investigate its some properties. Moreover, we propose many of the basic properties that the trace of $(\odot, \vee)$-permuting tri-derivation provides. In the future, different permuting tri-derivations can be defined in these algebras, and generalized permuting tri-derivations can be studied in BL-algebras.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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