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A New Insight on Rectifying-Type Curves in Euclidean 4-Space

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Abstract

In this study, our purpose is to determine the generalized rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. Also, some characterizations of them are given. We construct some correlations between curvatures and invariants of generalized rectifying-type curves. Additionally, we obtain an illustrative example with respect to the rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4.

Keywords: *Myller configuration, generalized rectifying-type curves, Euclidean-4 space.*
AMS Subject Classification (2020): *Primary: 58A04.*

1. Introduction

In differential geometry, curve theory is a quite fundamental work-frame and attracts several researchers. Several types special curves have been studied in the existing literature such as; osculating, rectifying curves, and normal curves (spherical curves), which satisfy Cesaro's fixed point condition (cf. [43]), [1–5,7–9,12–22,45, 46] in Euclidean and Minkowski 3-space and 4-space. The curve $C : I \to E_3$ for which the position vector of the curve C always lies in their rectifying plane, is named rectifying curves [3,5,15]. Similarly, if the position vector of the curve C always lies in its normal plane, the curve is named normal curve [12,13], and if the position vector of the curve C always lies in its osculating plane, the curve is named osculating curve [15,16].

One determines some versor fields such as; tangent, principal normal, and binormal, as well as some plane fields such as; rectifying, osculating, and normal planes along the curve C in Euclidean 3-space E_3. As a generalization, a versor field is denoted by (C, ζ), and a plane field is denoted by (C, π). The couple $\{(C, \zeta), (C, \pi)\}$ where $\zeta \in \pi$ is named a Myller configuration denoted by $\mathcal{M}(C, \overline{\zeta}, \pi)$ in Euclidean 3-space E_3. Miron [34] examined the Myller configuration in 1960. If the plane π is tangent to the curve C, we have a tangent Myller configuration denoted by $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ [31,33]. When literature is examined, lots of studies have been done on the Myller configuration for different areas and in different spaces. In 1922, Myller examined the parallelism of the versor field $(C, \overline{\zeta})$ in the plane field (C, π) developing a generalization of Levi-Civita parallelism on the curved surfaces. Mayer examined also these investigations which have new invariants for $\mathfrak{M}(C, \overline{\zeta}, \pi)$. Additionally, the importance of these studies was underlined by Levi-Civita [29,31,33].

Provided that C is a curve on the surface S, the geometry of the field (C, ζ) on surface S is the geometry of the associated Myller configurations $\mathcal{M}_t(C, \overline{\zeta}, \pi)$ [33]. It is underlined that the geometry of tangent Myller configuration $\mathcal{M}_t(C, \overline{\zeta}, \pi)$ is a special case of general Myller configuration $\mathcal{M}(C, \overline{\zeta}, \pi)$ [33]. Myller configuration is examined by several researchers in different concepts such as; in Riemannian geometry [36–39], in symplectic geometry [44], in Finsler space [6], in Minkowski space [35,42], in Hamilton, Lagrange, and Finsler spaces [28] (see also the studies [40,41]), respectively [33]. Thereafter, versor fields along a curve in Lorentz 4-space were examined in [11].

In recent studies, Macsim et al. [30] studied the special curves in a Myller configuration and examined their properties, as well. Macsim et al. [31, 32] determined the rectifying-type curves and Bertrand curves with

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Frenet-type frame for E_3 with Myller configuration, respectively. Since the theory and geometry of versor fields along a curve with Myller configuration in E_3 is a generalization of the ordinary theory and geometry of curves in classical Euclidean space, the rectifying-type curves with Frenet-type frame in Myller configuration in E_3 are a generalization of rectifying curves in Frenet frame in E_3 (cf. [11, 31, 33]). In a similar way, the Bertrand curves with Frenet-type frame in Myller configuration in E_3 are a generalization of Bertrand curves with Frenet frame in E_3. Keskin and Yaylı studied the rectifying-type curves and rotation minimizing frame with Myller configuration [25]. Also, İşbilir and Tosun introduced the osculating-type curves with Frenet-type frame in Myller configuration for Euclidean 3-space E_3 [23] and osculating-type curves in Myller configuration for Euclidean 4-space E_4 [24]. A remarkable study that includes the theory of Myller configurations $\mathcal{M}(C,\overline{\zeta},\pi)$ and tangent Myller configuration $\mathcal{M}_t(C,\overline{\zeta},\pi)$ has been done in [33].

We aim to get the results and interpretations concerning investigations of the rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. For this purpose, we started this interesting notion since the geometry of versor fields along a curve with Frenet-type curve in Myller configuration in E_4 is a generalization of the classical theory of curves with Frenet frame in E_4 (cf. [11, 31, 33]), the rectifying curves with Frenet frame in E_4 are one of the special cases of the rectifying-type curves with Frenet-type frame in Myller configuration in E_4. According to the natural construction of the Frenet-type frame in Myller configuration for the Euclidean 4-space E_4, we present relations between the rectifying curves with Frenet frame in E_4 and rectifying-type curves with Frenet-type frame in Myller configuration for E_4.

In this manuscript, we determine the generalized rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. We examine some characterizations and give some correlations between curvatures and invariants of the generalized rectifying-type curves with Frenet-type frame in Myller configuration for E_4. Also, we give a numerical example in order to support constructed materials.

2. Basic Concepts

In this part of this study, we give some necessary information and reminders related to the Frenet-type frame in Myller configuration for Euclidean 4-space E_4. The Frenet-type frame with Myller configuration in four-dimensional Lorentz space examined by Heroiu in [11]. Thanks to the studies [11, 33], the followings are satisfied for Frenet-type frame in Myller configuration for Euclidean 4-space E_4:

Let $(C,\overline{\zeta})$ be a versor field in E_4 and $\overline{r}(s)$ is a position vector of the curve C where s is the arc-length on the curve C. Frenet-type frame $\mathcal{R}_F=\{P,\overline{\zeta}_1(s),\overline{\zeta}_2(s),\overline{\zeta}_3(s),\overline{\zeta}_4(s)\}$ of the versor field $(C,\overline{\zeta})$ can be constructed. If $\overline{\zeta}_1^{'}(s)\neq 0$, then $\langle\overline{\zeta}_1^{'}(s),\overline{\zeta}_1(s)\rangle=0$. Then, $\overline{\zeta}_2(s)$ can be taken in the direction of $\overline{\zeta}_1^{'}(s)$. Because of the fact that $\dfrac{\overline{\zeta}_1^{'}(s)}{\|\overline{\zeta}_1^{'}(s)\|}=\overline{\zeta}_2(s)$, $\overline{\zeta}_2(s)$ is then the normalized vector field corresponding to $\overline{\zeta}_1^{'}(s)$. In that case, we get $\overline{\zeta}_1^{'}(s)=\mathcal{K}_1(s)\overline{\zeta}_2(s)$ where $\mathcal{K}_1(s)=\|\overline{\zeta}_1^{'}(s)\|$. The versor field $\overline{\zeta}_3(s)$ is in the direction of the normal component of $\overline{\zeta}_2^{'}(s)$ with according to plane $\{\overline{\zeta}_1(s),\overline{\zeta}_2(s)\}$. Moreover, $\overline{\zeta}_3(s)$ is the normalized vector field of the normal component and $\overline{\zeta}_4$ is the unique unit vector field perpendicular to the 3-dimensional subspace $\{\overline{\zeta}_1(s),\overline{\zeta}_2(s),\overline{\zeta}_3(s)\}$. Then, Frenet-type derivative formulas in Myller configuration for E_4 are written as follows:

$$\begin{cases}\overline{\zeta}_1^{'}(s)=\mathcal{K}_1(s)\overline{\zeta}_2(s),\\ \overline{\zeta}_2^{'}(s)=-\mathcal{K}_1(s)\overline{\zeta}_1(s)+\mathcal{K}_2(s)\overline{\zeta}_3(s),\\ \overline{\zeta}_3^{'}(s)=-\mathcal{K}_2(s)\overline{\zeta}_2(s)+\mathcal{K}_3(s)\overline{\zeta}_4(s),\\ \overline{\zeta}_4^{'}(s)=-\mathcal{K}_3(s)\overline{\zeta}_3(s),\end{cases}\tag{2.1}$$

where $\mathcal{K}_1(s)>0$, $\mathcal{K}_2(s)$ and $\mathcal{K}_3(s)$ are curvatures. Also, the following equation can be expressed:

$$\overline{r}^{'}(s)=\varrho_1(s)\overline{\zeta}_1(s)+\varrho_2(s)\overline{\zeta}_2(s)+\varrho_3(s)\overline{\zeta}_3(s)+\varrho_4(s)\overline{\zeta}_4(s)\tag{2.2}$$

where $\varrho_1^2(s)+\varrho_2^2(s)+\varrho_3^2(s)+\varrho_4^2(s)=1$. It is seen that, if $\varrho_1(s)=1$, $\varrho_2(s)=0$, $\varrho_3(s)=0$ and $\varrho_4(s)=0$, we have Frenet equations of a curve in Euclidean 4-space E_4 (cf. [10, 14, 27]). Thanks to the studies [33] and [11], the fundamental theorem of invariants for versor field $(C,\overline{\zeta})$ for Euclidean 4-space E_4 is given as follows:

Theorem 2.1. *If the functions $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s),\varrho_1(s),\varrho_2(s),\varrho_3(s),\varrho_4(s)$ with $\varrho_1^2(s)+\varrho_2^2(s)+\varrho_3^2(s)+\varrho_4^2(s)=1$ are smooth functions for $s\in[a_1,a_2]$, then there exist a curve $C:[a_1,a_2]\to E_4$ parameterized by arc-length s and a versor* *field $(C,\overline{\zeta})$. In that case, there exists a versor field $\overline{\zeta}$ tangent to the curve C such that $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ are curvatures and $\varrho_i, i=1,2,3,4$ are the components of the versor field $(C,\overline{\zeta})$ in the Frenet-type frame associated to the $(C,\overline{\zeta})$.*

3. Rectifying-Type Curves with Frenet-Type Frame in Euclidean 4-Space with Myller Configuration

In this section, we determine the rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean space E_4. Also, we construct the conditions for being rectifying-type curves with Frenet-type frame in Myller configuration for E_4. Then, some characterizations with respect to them are given. One can see that the rectifying-type curves with Frenet-type frame in Myller configuration for E_4 are a generalization of rectifying curves with Frenet frame in E_4.

Definition 3.1. Let $\overline{r}(s): I \to E_4$ be defined rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4 if

$$\overline{r}(s) = \eta(s)\overline{\zeta}_1(s) + \omega(s)\overline{\zeta}_3(s) + \vartheta(s)\overline{\zeta}_4(s), \tag{3.1}$$

where $\eta(s), \omega(s)$ and $\vartheta(s)$ are smooth functions.

First of all, let us write some preparations before starting the relevant theorems:
By differentiating the Eq. (3.1), we get:

$$\begin{aligned}\varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s) + \varrho_3(s)\overline{\zeta}_3(s) + \varrho_4(s)\overline{\zeta}_4(s) =& \eta'(s)\overline{\zeta}_1(s) + \eta(s)\mathcal{K}_1(s)\overline{\zeta}_2(s) \\ &+ \omega'(s)\overline{\zeta}_3(s) + \omega(s)\left(-\mathcal{K}_2(s)\overline{\zeta}_2(s) + \mathcal{K}_3(s)\overline{\zeta}_4(s)\right) \\ &+ \vartheta'(s)\overline{\zeta}_4(s) + \vartheta(s)\left(-\mathcal{K}_3(s)\overline{\zeta}_3(s)\right) \\ =& \eta'(s)\overline{\zeta}_1(s) + \left(\eta(s)\mathcal{K}_1(s) - \omega(s)\mathcal{K}_2(s)\right)\overline{\zeta}_2(s) \\ &+ \left(\omega'(s) - \vartheta(s)\mathcal{K}_3(s)\right)\overline{\zeta}_3(s) \\ &+ \left(\vartheta'(s) + \omega(s)\mathcal{K}_3(s)\right)\overline{\zeta}_4(s).\end{aligned} \tag{3.2}$$

Then, we have:

$$\begin{cases} \eta'(s) = \varrho_1(s), \\ \eta(s)\mathcal{K}_1(s) - \omega(s)\mathcal{K}_2(s) = \varrho_2(s), \\ \omega'(s) - \vartheta(s)\mathcal{K}_3(s) = \varrho_3(s), \\ \vartheta'(s) + \omega(s)\mathcal{K}_3(s) = \varrho_4(s). \end{cases} \tag{3.3}$$

Now, let us give the necessary and sufficient condition for being a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space.

Special Case 3.1. Let $\overline{r}(s): I \to E_4$ is a curve with the non-zero curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s)$ and $\mathcal{K}_3(s)$ with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If we take $\varrho_1(s) = 1, \varrho_2(s) = \varrho_3(s) = \varrho_4(s) = 0$ in the equations (3.2) and (3.3), we get the following equations (cf. [14]):

$$\overline{\zeta}_1(s) = \eta'(s)\overline{\zeta}_1(s) + \left(\eta(s)\mathcal{K}_1(s) - \omega(s)\mathcal{K}_2(s)\right)\overline{\zeta}_2(s) + \left(\omega'(s) - \vartheta(s)\mathcal{K}_3(s)\right)\overline{\zeta}_3(s) + \left(\vartheta'(s) + \omega(s)\mathcal{K}_3(s)\right)\overline{\zeta}_4(s)$$

and

$$\eta'(s) = 1, \quad \eta(s)\mathcal{K}_1(s) - \omega(s)\mathcal{K}_2(s) = 0, \quad \omega'(s) - \vartheta(s)\mathcal{K}_3(s) = 0, \quad \vartheta'(s) + \omega(s)\mathcal{K}_3(s) = 0.$$

This is a characterization of Euclidean 4-space for Frenet frame studied by İlarslan and Nešović in [14].

Theorem 3.1. *Let $\overline{r}(s): I \to E_4$ is a curve with the non-zero curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s)$ and $\mathcal{K}_3(s)$ with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. Then, $\overline{r}(s)$ is a rectifying-type curve if and only if*

$$\begin{aligned}\overline{r}(s) =& \left(\int \varrho_1(s)ds\right)\overline{\zeta}_1 + \left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\int \varrho_1(s)ds - \frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)\overline{\zeta}_3(s) \\ &+ \left(\frac{1}{\mathcal{K}_3(s)}\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int \varrho_1(s)ds + \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s) - \frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)' - \frac{\varrho_3(s)}{\mathcal{K}_3(s)}\right)\overline{\zeta}_4(s).\end{aligned} \tag{3.4}$$

Proof. Assume that $\overline{r}(s)$ is a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space. Differentiating the Eq. (3.1), we obtain the Eqs. (3.2) and (3.3). Then, we have:

$$\begin{cases} \eta(s) = \int \varrho_1(s)ds, \\ \omega(s) = \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)} \int \varrho_1(s)ds - \frac{\varrho_2(s)}{\mathcal{K}_2(s)}, \\ \vartheta(s) = \frac{1}{\mathcal{K}_3(s)} \left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)' \int \varrho_1(s)ds + \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s) - \frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)' - \frac{\varrho_3(s)}{\mathcal{K}_3(s)}. \end{cases} \tag{3.5}$$

If we substitute the functions $\eta(s), \omega(s)$ and $\vartheta(s)$ written in the Eq. (3.5) in the equation (3.1), we get what is desired. Contrarily, suppose that the Eq. (3.4) holds. Then, we can write:

$$\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle = \int \varrho_1(s)ds. \tag{3.6}$$

Differentiating the (3.6), we get:

$$\varrho_1(s) + \mathcal{K}_1(s) \left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle = \varrho_1(s).$$

Since $\mathcal{K}_1(s) \neq 0$, we get $\left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle = 0$. Therefore, $\overline{r}(s)$ is a rectifying-type curve with Frenet-type frame in Myller configuration for E_4. □

Special Case 3.2. Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\varrho_1(s) = 1$, $\varrho_2(s) = \varrho_3(s) = \varrho_4(s) = 0$ are written in the Eq. (3.5), we get the functions $\eta(s), \mu(s)$ and $\vartheta(s)$ as found for rectifying curves with Frenet frame in E_4:

$$\begin{cases} \eta(s) = s + c, \\ \omega(s) = \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}(s+c), \\ \vartheta(s) = \frac{\mathcal{K}_1(s)\mathcal{K}_2(s) + (s+c)\left(\mathcal{K}_1'(s)\mathcal{K}_2(s) - \mathcal{K}_1(s)\mathcal{K}_2'(s)\right)}{\mathcal{K}_2^2(s)\mathcal{K}_3(s)}, \end{cases}$$

where $c \in \mathbb{R}$ (cf. [14]). This is a characterization of Euclidean 4-space for Frenet frame studied by İlarslan and Nešović in [14].

Theorem 3.2. *Let $\overline{r}(s)$ be a curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. Then, $\overline{r}(s)$ is a rectifying-type curve if and only if non-zero curvatures $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, $\mathcal{K}_3(s)$ and the functions $\varrho_1(s), \varrho_2(s), \varrho_3(s), \varrho_4(s)$ satisfy the following relation:*

$$\begin{aligned} &\left(\frac{1}{\mathcal{K}_3(s)} \left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)' \int \varrho_1(s)ds + \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s) - \frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)' - \frac{\varrho_3(s)}{\mathcal{K}_3(s)}\right)' \\ &+ \frac{\mathcal{K}_1(s)\mathcal{K}_3(s)}{\mathcal{K}_2(s)} \int \varrho_1(s)ds - \frac{\varrho_2(s)\mathcal{K}_3(s)}{\mathcal{K}_2(s)} = \varrho_4(s). \end{aligned} \tag{3.7}$$

Proof. Suppose that $\overline{r}(s)$ is a rectifying-type curve with Frenet-type frame in Myller configuration for E_4. If the functions $\eta(s), \omega(s)$, and $\vartheta(s)$, which are written in the Eq. (3.5), are substituted in the last equation of the Eqn. (3.3) we get the Eq. (3.7). Contrarily, assume that $\overline{r}(s)$ is a curve satisfying the equation (3.7). Differentiating the

Eq. (3.4) and by using the Eqs. (2.1), (2.2), and (3.4), we get:

$$\begin{aligned}
&\frac{d}{ds}\left[\begin{array}{l}\overline{r}(s)-\left(\int \varrho_1(s)ds\right)\overline{\zeta}_1-\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\int \varrho_1(s)ds-\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)\overline{\zeta}_3(s)\\
-\left(\frac{1}{\mathcal{K}_3(s)}\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int \varrho_1(s)ds+\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s)-\frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)'-\frac{\varrho_3(s)}{\mathcal{K}_3(s)}\right)\overline{\zeta}_4(s)\end{array}\right]\\
&=\overline{r}'(s)-\varrho_1(s)\overline{\zeta}_1-\mathcal{K}_1(s)\left(\int \varrho_1(s)ds\right)\overline{\zeta}_2-\left[\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int \varrho_1(s)ds+\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\varrho_1(s)-\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)'\right]\overline{\zeta}_3(s)\\
&\quad+\mathcal{K}_1(s)\left(\int \varrho_1(s)ds\right)\overline{\zeta}_2(s)-\varrho_2(s)\overline{\zeta}_2(s)-\mathcal{K}_3(s)\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\left(\int \varrho_1(s)ds\right)-\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)\overline{\zeta}_4(s)\\
&\quad-\left(\frac{1}{\mathcal{K}_3(s)}\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int \varrho_1(s)ds+\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s)-\frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)'-\frac{\varrho_3(s)}{\mathcal{K}_3(s)}\right)'\overline{\zeta}_4(s)\\
&\quad+\left(\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int \varrho_1(s)ds+\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\varrho_1(s)-\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)'-\varrho_3(s)\right)\overline{\zeta}_3(s)\\
&=0.
\end{aligned}$$

Therefore, $\overline{r}(s)$ is a rectifying-type curve with Frenet-type frame in Myller configuration for E_4. □

Special Case 3.3. Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\varrho_1(s)=1$, $\varrho_2(s)=\varrho_3(s)=\varrho_4(s)=0$ are written in the equation (3.7), then we get the following equation for rectifying curves with Frenet frame in E_4:

$$\left(\frac{\mathcal{K}_1(s)\mathcal{K}_2(s)+(s+c)\left(\mathcal{K}_1'(s)\mathcal{K}_2(s)-\mathcal{K}_1(s)\mathcal{K}_2'(s)\right)}{\mathcal{K}_2^2(s)\mathcal{K}_3(s)}\right)'+\frac{\mathcal{K}_1(s)\mathcal{K}_3(s)\,(s+c)}{\mathcal{K}_2(s)}=0, \tag{3.8}$$

where $c\in\mathbb{R}$ (cf. [14]). This is a characterization of Euclidean 4-space for Frenet frame studied by İlarslan and Nešović in the study [14].

Theorem 3.3. *Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. There is no exists rectifying-type curve with non-zero constant curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ and functions $\varrho_1(s),\varrho_2(s),\varrho_3(s),\varrho_4(s)$.*

Proof. Suppose that $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4 such that $\varrho_1(s),\varrho_2(s),\varrho_3(s),\varrho_4(s)$ and curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ are non-zero constant. In that case, we get a contradiction in the equation (3.7). Therefore, we have the desired result. □

Special Case 3.4. Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\varrho_1(s)=1$, $\varrho_2(s)=\varrho_3(s)=\varrho_4(s)=0$, we can say that there is no rectifying curve lying fully in E_4, with non-zero constant curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ (cf. [14]). This is a characterization of E_4 for Frenet frame which is examined by İlarslan and Nešović in the study [14].

Theorem 3.4. *Let $\overline{r}(s)$ be a rectifying-type curve with non-zero curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\overline{r}(s)$ is a rectifying-type curve, then the distance function is given as*

$$\begin{cases} p(s)=||\overline{r}(s)||,\\ p^2(s)=\left(\int \varrho_1(s)ds\right)^2+2\int \dfrac{\omega'(s)\varrho_4(s)-\vartheta'(s)\varrho_3(s)}{\mathcal{K}_3(s)}ds.\end{cases} \tag{3.9}$$

Proof. Assume that $\overline{r}(s)$ is a rectifying-type curve. Then, by using the Eq. (3.1), we obtain:

$$p^2(s)=\langle\overline{r}(s),\overline{r}(s)\rangle=\eta^2(s)+\omega^2(s)+\vartheta^2(s).$$

If we multiply the third equation of the Eq. (3.3) by $-\vartheta'(s)$ and the last equation of the Eq. (3.3) by $\omega'(s)$, and then summing these equations, we get:

$$\mathcal{K}_3(s)\left(\omega(s)\omega'(s)+\vartheta(s)\vartheta'(s)\right)=\omega'(s)\varrho_4(s)-\vartheta'(s)\varrho_3(s).$$

Then, we can write:

$$\omega^2(s)+\vartheta^2(s)=2\int\frac{\omega'(s)\varrho_4(s)-\vartheta'(s)\varrho_3(s)}{\mathcal{K}_3(s)}ds$$

and also

$$\eta^2(s)=\left(\int\varrho_1(s)ds\right)^2.$$

Therefore, we get the desired result and complete the proof. □

Special Case 3.5. Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\varrho_1(s)=1$, $\varrho_2(s)=\varrho_3(s)=\varrho_4(s)=0$ in the last equation of equation (3.9), then we get $p^2(s)=s^2+c_1s+c_2$ where $c_1\in\mathbb{R},c_2\in\mathbb{R}_0$ (cf. [14]). This is a characterization of E_4 for Frenet frame which is examined by İlarslan and Nešović in the study [14].

Theorem 3.5. *Let $\overline{r}(s)$ be a rectifying-type curve with non-zero curvatures $\mathcal{K}_1(s),\mathcal{K}_2(s),\mathcal{K}_3(s)$ with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. Then, the followings are satisfied:*

$$\begin{cases}\left\langle\overline{r}(s),\overline{\xi}_1(s)\right\rangle=\int\varrho_1(s)ds,\\ \left\langle\overline{r}(s),\overline{\xi}_2(s)\right\rangle=0,\\ \left\langle\overline{r}(s),\overline{\xi}_3(s)\right\rangle=\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\int\varrho_1(s)ds-\frac{\varrho_2(s)}{\mathcal{K}_2(s)},\\ \left\langle\overline{r}(s),\overline{\xi}_4(s)\right\rangle=\frac{1}{\mathcal{K}_3(s)}\left(\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}\right)'\int\varrho_1(s)ds+\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)\mathcal{K}_3(s)}\varrho_1(s)-\frac{1}{\mathcal{K}_3(s)}\left(\frac{\varrho_2(s)}{\mathcal{K}_2(s)}\right)'-\frac{\varrho_3(s)}{\mathcal{K}_3(s)}.\end{cases}$$

Proof. With the help of the Eqs. (3.1) and (3.5), we can complete the proof. Hence, we omit them for the sake of brevity. □

Special Case 3.6. Let $\overline{r}(s)$ be a rectifying-type curve with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. If $\varrho_1(s)=1$, $\varrho_2(s)=\varrho_3(s)=\varrho_4(s)=0$, we have the following equations for Frenet frame in E_4:

$$\begin{cases}\left\langle\overline{r}(s),\overline{\xi}_1(s)\right\rangle=s+c,\\ \left\langle\overline{r}(s),\overline{\xi}_2(s)\right\rangle=0,\\ \left\langle\overline{r}(s),\overline{\xi}_2(s)\right\rangle=\frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)}(s+c),\\ \left\langle\overline{r}(s),\overline{\xi}_3(s)\right\rangle=\frac{\mathcal{K}_1(s)\mathcal{K}_2(s)+(s+c)\left(\mathcal{K}_1'(s)\mathcal{K}_2(s)-\mathcal{K}_1(s)\mathcal{K}_2'(s)\right)}{\mathcal{K}_2^2(s)\mathcal{K}_3(s)}\end{cases}$$

where $c\in\mathbb{R}$ (cf. [14]). This is a characterization of E_4 for Frenet frame which is examined by İlarslan and Nešović in the study [14].

By inspiring the study [26], we construct our example with respect to the rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4.

Example 3.1. Consider the following elements of the Frenet-type frame in Myller configuration for Euclidean 4-space

$$\begin{cases}\overline{\zeta}_1(s)=\left(-\frac{1}{\sqrt{3}}\sin(s),\frac{1}{\sqrt{3}}\cos(s),-\frac{\sqrt{2}}{\sqrt{3}}\sin(s),\frac{\sqrt{2}}{\sqrt{3}}\cos(s)\right),\\ \overline{\zeta}_2(s)=\left(-\frac{1}{\sqrt{3}}\cos(s),-\frac{1}{\sqrt{3}}\sin(s),-\frac{\sqrt{2}}{\sqrt{3}}\cos(s),-\frac{\sqrt{2}}{\sqrt{3}}\sin(s)\right),\\ \overline{\zeta}_3(s)=\left(\frac{\sqrt{2}}{\sqrt{3}}\sin(s),-\frac{\sqrt{2}}{\sqrt{3}}\cos(s),-\frac{1}{\sqrt{3}}\sin(s),\frac{1}{\sqrt{3}}\cos(s)\right),\\ \overline{\zeta}_4(s)=\left(\frac{\sqrt{2}}{\sqrt{3}}\cos(s),\frac{\sqrt{2}}{\sqrt{3}}\sin(s),-\frac{1}{\sqrt{3}}\cos(s),-\frac{1}{\sqrt{3}}\sin(s)\right).\end{cases}\tag{3.10}$$

• Let us choose $\varrho_1(s)=0, \varrho_2(s)=\dfrac{1}{\sqrt{2}}, \varrho_3(s)=\dfrac{1}{\sqrt{2}}$ and $\varrho_4(s)=0$. Then, we get the following curve:

$\overline{r}_1(s): I \to E_4$

$$s \mapsto \overline{r}(s)=\left(-\frac{1}{\sqrt{6}}\sin(s)-\frac{1}{\sqrt{3}}\cos(s), \frac{1}{\sqrt{6}}\cos(s)-\frac{1}{\sqrt{3}}\sin(s), -\frac{1}{\sqrt{3}}\sin(s)+\frac{1}{\sqrt{6}}\cos(s), \frac{1}{\sqrt{3}}\cos(s)+\frac{1}{\sqrt{6}}\sin(s)\right).$$

Also, we have:

$$\begin{aligned}\overline{r}_1(s)=&\left(-\frac{1}{\sqrt{6}}\sin(s)-\frac{1}{\sqrt{3}}\cos(s), \frac{1}{\sqrt{6}}\cos(s)-\frac{1}{\sqrt{3}}\sin(s), -\frac{1}{\sqrt{3}}\sin(s)+\frac{1}{\sqrt{6}}\cos(s), \frac{1}{\sqrt{3}}\cos(s)+\frac{1}{\sqrt{6}}\sin(s)\right)\\ =&\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_3(s)+\vartheta(s)\overline{\zeta}_4(s),\end{aligned}$$

where $\eta(s)=\dfrac{1}{\sqrt{2}}, \omega(s)=0$ and $\vartheta(s)=-\dfrac{1}{\sqrt{2}}$.

• Now, let us choose $\varrho_1(s)=0, \varrho_2(s)=\dfrac{1}{\sqrt{2}}, \varrho_3(s)=-\dfrac{1}{\sqrt{2}}$ and $\varrho_4(s)=0$. In that case, we have the following curve:

$\overline{r}_2(s): I \to E_4$

$$s \mapsto \overline{r}(s)=\left(-\frac{1}{\sqrt{6}}\sin(s)+\frac{1}{\sqrt{3}}\cos(s), \frac{1}{\sqrt{6}}\cos(s)+\frac{1}{\sqrt{3}}\sin(s), -\frac{1}{\sqrt{3}}\sin(s)-\frac{1}{\sqrt{6}}\cos(s), \frac{1}{\sqrt{3}}\cos(s)-\frac{1}{\sqrt{6}}\sin(s)\right).$$

Additionally, we get:

$$\begin{aligned}\overline{r}_2(s)=&\left(-\frac{1}{\sqrt{6}}\sin(s)+\frac{1}{\sqrt{3}}\cos(s), \frac{1}{\sqrt{6}}\cos(s)+\frac{1}{\sqrt{3}}\sin(s), -\frac{1}{\sqrt{3}}\sin(s)-\frac{1}{\sqrt{6}}\cos(s), \frac{1}{\sqrt{3}}\cos(s)-\frac{1}{\sqrt{6}}\sin(s)\right)\\ =&\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_3(s)+\vartheta(s)\overline{\zeta}_4(s),\end{aligned}$$

where $\eta(s)=\dfrac{1}{\sqrt{2}}, \omega(s)=0$ and $\vartheta(s)=\dfrac{1}{\sqrt{2}}$.

• Now, let us choose $\varrho_1(s)=0, \varrho_2(s)=-\dfrac{1}{\sqrt{2}}, \varrho_3(s)=\dfrac{1}{\sqrt{2}}$ and $\varrho_4(s)=0$. We obtain the following curve:

$\overline{r}_3(s): I \to E_4$

$$s \mapsto \overline{r}(s)=\left(\frac{1}{\sqrt{6}}\sin(s)-\frac{1}{\sqrt{3}}\cos(s), -\frac{1}{\sqrt{6}}\cos(s)-\frac{1}{\sqrt{3}}\sin(s), \frac{1}{\sqrt{3}}\sin(s)+\frac{1}{\sqrt{6}}\cos(s), -\frac{1}{\sqrt{3}}\cos(s)+\frac{1}{\sqrt{6}}\sin(s)\right).$$

Moreover, we attain:

$$\begin{aligned}\overline{r}_3(s)=&\left(\frac{1}{\sqrt{6}}\sin(s)-\frac{1}{\sqrt{3}}\cos(s), -\frac{1}{\sqrt{6}}\cos(s)-\frac{1}{\sqrt{3}}\sin(s), \frac{1}{\sqrt{3}}\sin(s)+\frac{1}{\sqrt{6}}\cos(s), -\frac{1}{\sqrt{3}}\cos(s)+\frac{1}{\sqrt{6}}\sin(s)\right)\\ =&\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_3(s)+\vartheta(s)\overline{\zeta}_4(s),\end{aligned}$$

where $\eta(s)=-\dfrac{1}{\sqrt{2}}, \omega(s)=0$ and $\vartheta(s)=-\dfrac{1}{\sqrt{2}}$.

• Finally, we take $\varrho_1(s)=0, \varrho_2(s)=-\dfrac{1}{\sqrt{2}}, \varrho_3(s)=-\dfrac{1}{\sqrt{2}}$ and $\varrho_4(s)=0$. We have the following curve:

$\overline{r}_4(s): I \to E_4$

$$s \mapsto \overline{r}(s)=\left(\frac{1}{\sqrt{6}}\sin(s)+\frac{1}{\sqrt{3}}\cos(s), -\frac{1}{\sqrt{6}}\cos(s)+\frac{1}{\sqrt{3}}\sin(s), \frac{1}{\sqrt{3}}\sin(s)-\frac{1}{\sqrt{6}}\cos(s), -\frac{1}{\sqrt{3}}\cos(s)-\frac{1}{\sqrt{6}}\sin(s)\right).$$

Then, we get:

$$\begin{aligned}\overline{r}_4(s)=&\left(\frac{1}{\sqrt{6}}\sin(s)+\frac{1}{\sqrt{3}}\cos(s), -\frac{1}{\sqrt{6}}\cos(s)+\frac{1}{\sqrt{3}}\sin(s), \frac{1}{\sqrt{3}}\sin(s)-\frac{1}{\sqrt{6}}\cos(s), -\frac{1}{\sqrt{3}}\cos(s)-\frac{1}{\sqrt{6}}\sin(s)\right)\\ =&\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_3(s)+\vartheta(s)\overline{\zeta}_4(s),\end{aligned}$$

where $\eta(s) = -\frac{1}{\sqrt{2}}, \omega(s) = 0$ and $\vartheta(s) = \frac{1}{\sqrt{2}}$.

Therefore, we can say that the curves $\overline{r}_1(s), \overline{r}_2(s), \overline{r}_3(s)$ and $\overline{r}_4(s)$ are a rectifying-type curve with Frenet-type frame in Myller configuration for E_4.

4. Conclusions

In this study, we determined rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space E_4. We obtained some fundamental and necessary characterizations for a curve to be a rectifying-type curve with Frenet-type frame in Myller configuration for E_4. It is noted that the rectifying curves with Frenet frame in E_4 are one of the special cases for the generalized rectifying-type curves since the geometrical theory of versor fields along a spatial curve with Myller configuration in Euclidean 4-space E_4 is a generalization of the classical theory of curves in Euclidean 4-space E_4. Then, we gave an example in order to support the given materials.

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Author's contributions

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