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A NEW MULTIDIMENSIONAL MODEL II REGRESSION BASED ON BISECTOR APPROACH

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Abstract. A new multidimensional Model II regression based on bisector point of view (BRM-II) is introduced for multivariate problems that may contain measurement error. The suggested method is constructed depending on using the bisector of the minor angle between two hyperplanes identified by linear regression. The performance of the proposed method are examined by simulations up to ten variables for different sample sizes and distribution types in terms of the Mean Square Error. Moreover, the BRM-II is applied to two real problems with two and three variables, and compared with the existing methods. The results indicate that the BRM-II is easy applicable and offers relatively better accuracy. The relevant method can be easily coded in any programming language provides convenience in its application. Thus, the proposed method provide powerful tool for prediction of relevant real life problems.

1. Introduction

Regression analysis is a statistical method used to determine the relationship between two or more variables that have a cause and effect relation and to make estimation or prediction about that subject by using this relationship [12]. The regression method, which dates back to the 1800s, was first used in astronomical events and social sciences. In classical regression, the regression model including two variables is defined as

$$\hat{Y} = \beta_0 + \beta_1 X + e_i \tag{1}$$

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while the model including more than two variables is given by

$$\hat{Y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \cdots + \beta_m X_m + e_i, \tag{2}$$

where $\hat{Y}$ represents dependent variable, e_i are uncontrollable errors and β_i are coefficients ($i = 1, 2, \ldots, m$). It is well-known that the regression analysis depends on fundamental assumptions as follows: *(i)* the relationship between the variables is linear, *(ii)* the variables have no measurement errors, *(iii)* the expected value of error terms is zero, *(iv)* the error terms display the normal distribution, *(v)* there is no relation between and the error terms, *(vi)* there is no autocorrelation between error terms, *(vii)* the variance of error terms is constant for all values of the independent variables [14]. In the analysis of classical regression, assumption *(ii)* is the important one because of losing the validity of the model in case of not providing this assumption. Besides, it is nonrealistic that the variables do not contain the measurement error stemming from the measuring equipment, observer, incorrect records, etc. in the real data sets. For this reason, the Model II regression methods have been developed in cases where assumption *(ii)* may not be provided. Model II regression methods are based on the idea of constituting a regression model that regards the fact that all variables may contain errors. In the analysis of classical regression, assumption (*ii*) is the important one because of losing the validity of the model in case of not providing this assumption. Besides, it is nonrealistic that the variables do not contain the measurement error stemming from the measuring equipment, observer, incorrect records, etc. in the real data sets. Thus, the Model II regression approach is derived, based on the idea of constructing a regression model that regards the fact that all variables may contain errors. In Model I regression, in order to determine the functional relationship between the variables and to make prediction, linear regression is used while, in Model II, linear regressions for each variable are constructed because assigning the variables as dependent or independent could not be possible and corresponding regression equations are evaluated depending on the nature of the problem.

The idea of Model II regression, which dates back to the early 1900s, has been studied theoretically by Deming (1943) [5], York (1966) [29] and Passing and Bablok (1983) [18]. Also, to estimate of the functional relationship, the line that bisects the minor angle between the two model regressions is suggested by Sprent and Dolby (1980) [23]. One of the important Model II regression called geometric mean Model II has been used in the analysis of most field data by Laws and Archie (1981) [13] while a linear regression method without particular assumptions, regarding the distribution of the samples and the measurement errors, has been investigated by Passing and Bablok (1983) [18]. In the next years, major axis and standardized major axis, which are the most known Model II methods, have been discussed by Warton *et al.* (2006) [27] to describe the key properties of line-fitting techniques in order to estimate the relationship between two variables. While it is seen that all of these mentioned studies focused on bivariate problems, a method for estimating multivariate functional relationships between sets of measured data in different fields is described for three or more variables in the studies of Stavn and Richter (2008) [24] and Richter and Stavn (2014) [20]. Besides of these theoretical developments, there are many studies focused on the application of Model II methods in the literature. A number of Model II methods have been examined in fishery studies and reviewed in biomechanics by Ricker (1973) [21] and Rayner (1985) [19], respectively. Isobe *et al.* (1990) [11] have discussed and applied five different methods to bivariate data with measurement errors in astronomical problems. Some Model II procedures have been reviewed comprehensively and compared the effectiveness of the methods on clinical and biomedical chemistry by Ludbrook (2010, 2012) [15,16]. In several research areas such as natural sciences, biological researches, environmental sciences, fisheries, osteology and microbiology, etc., different types of Model II methods have been used in the literature [1–4, 6–10, 17, 25, 28]. On the other hand, when all model II regressions in the literature are examined, it is seen that all of them are derived for two or three variable problems, except the study of Richter and Stavn (2014) [20]. In their study, however, model II regression is defined only theoretically for problems with more than three variables, but it has not been applied to any real data or simulation calculations. Therefore, the Model II regression methods available in the literature are open to development when it is desired to model a problem with four or more variables. Accordingly, the novelty of this study is to develop a new Model II regression method for the problems with any number of variables. This new method, called Bisector Regression Model II (BRM-II), is constructed on the idea of computing the bisector of hyperplanes standing for the multidimensional regression models. The BRM-II method has the flexibility to be applied to many complex problems in the natural, medical, and social sciences since real-life problems are represented by multivariate models.

In this paper, the organization is follows: in Section 2, a new multidimensional BRM-II method is introduced in detail. Then, in order to demonstrate the validity and efficiency of the method, the proposed method is applied for both simulations which are up to ten variables for different sample sizes and distribution types and real data sets with two and three variables in Section 3. Finally, the concluding remarks are presented in the last section.

2. A New Multidimensional BRM-II Method

Let we have a data set with m variables such as $X_i \;\; (i = 1, 2, \ldots, m)$. For instance, we decide X_1 as a dependent variable and set a linear regression with other $(m-1)$ variables

$$X_1 = \beta_0 + \beta_1 X_2 + \beta_2 X_3 + \cdots + \beta_{m-1} X_m.$$

From this point of view each linear regression is obtained and expressed as a system with the following form:

$$\left.\begin{array}{ll} H_1: & \beta_{0,1}+\beta_{1,1}X_1+\beta_{2,1}X_2+\beta_{3,1}X_3+\cdots+\beta_{m,1}X_m=0 \\ H_2: & \beta_{0,2}+\beta_{1,2}X_1+\beta_{2,2}X_2+\beta_{3,2}X_3+\cdots+\beta_{m,2}X_m=0 \\ H_3: & \beta_{0,3}+\beta_{1,3}X_1+\beta_{2,3}X_2+\beta_{3,3}X_3+\cdots+\beta_{m,3}X_m=0 \\ & \vdots \\ H_m: & \beta_{0,m}+\beta_{1,m}X_1+\beta_{2,m}X_2+\beta_{3,m}X_3+\cdots+\beta_{m,m}X_m=0 \end{array}\right\} \tag{3}$$

where H_i for $i=1,\ldots,m$ represent the hyperplanes geometrically, $\beta_{i,j}$ for $i=0,\ldots,m$, $j=1,\ldots,m$ are unknown regression coefficients and $\beta_{i,i}=-1$ for $i=1,\ldots,m$. A bisector-hyperplane (BH_1) is found by considering the hyperplanes H_1 and H_2 in system (3), and then a new bisector-hyperplane (BH_2) is again found by using BH_1 and H_3 and so on (see Figure 1). The geometric observations of the bisector approach with two and three variables are given in Figure 2.

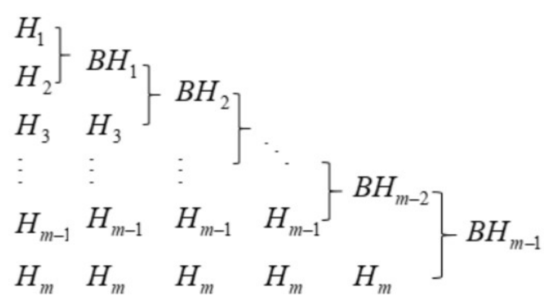
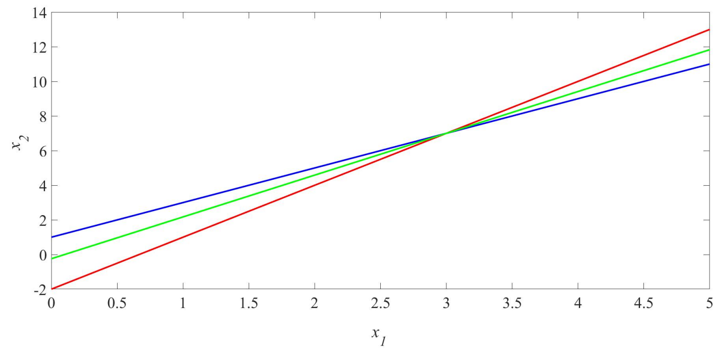


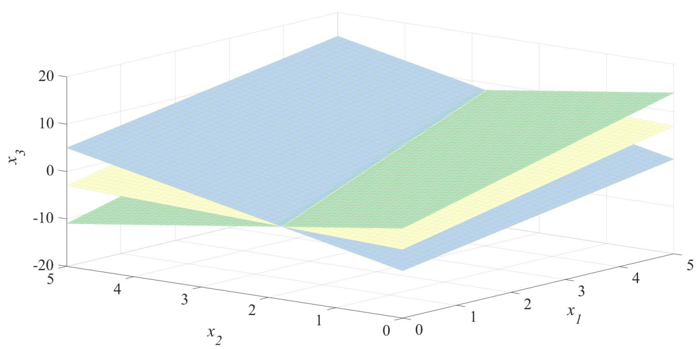
FIGURE 1. The schema of bisector hyperplanes

In order to carry out the bisector hyperplanes between each sequential hyperplanes, finding the normal vectors of the hyperplanes is needed. The matrix of normal vectors of the hyperplanes in the above system is defined by

$$\boldsymbol{A}=\begin{bmatrix} \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{m,1} \\ \beta_{1,2} & \beta_{2,2} & \beta_{3,2} & \cdots & \beta_{m,2} \\ \beta_{1,3} & \beta_{2,3} & \beta_{3,3} & \cdots & \beta_{m,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{1,m} & \beta_{2,m} & \beta_{3,m} & \cdots & \beta_{m,m} \end{bmatrix}. \tag{4}$$



(a)



(b)

FIGURE 2. The geometric observation of the bisector approach with *(a)* two variables (the red line: $\beta_{0,1} + \beta_{1,1}X_1 + \beta_{2,1}X_2 = 0$, the blue line: $\beta_{0,2} + \beta_{1,2}X_1 + \beta_{2,2}X_2 = 0$ and the green line is the bisector line) and *(b)* three variables (the blue plane: $\beta_{0,1} + \beta_{1,1}X_1 + \beta_{2,1}X_2 + \beta_{3,1}X_3 = 0$, the green plane: $\beta_{0,2} + \beta_{1,2}X_1 + \beta_{2,2}X_2 + \beta_{3,2}X_3 = 0$ and the yellow plane is the bisector plane)

Let us now obtain the first bisector hyperplane BH_1 using with the hyperplanes H_1 and H_2. It is well-known that the equation of a bisector hyperplane is computed by

$$\frac{\left|\beta_{0,1} + \beta_{1,1}X_1 + \beta_{2,1}X_2 + \cdots + \beta_{m,1}X_m\right|}{\|\tilde{n}_1\|} = \frac{\left|\beta_{0,2} + \beta_{1,2}X_1 + \beta_{2,2}X_2 + \cdots + \beta_{m,2}X_m\right|}{\|\tilde{n}_2\|} \tag{5}$$

where $\|\cdot\|$ is Euclidean norm of the corresponding vector and $\tilde{n}_1$, $\tilde{n}_2$ are the normal vectors of H_1 and H_2 as follows

$$\tilde{n}_1 = \begin{bmatrix} \beta_{1,1} & \beta_{2,1} & \beta_{3,1} & \cdots & \beta_{m,1} \end{bmatrix}, \tag{6}$$

$$\tilde{n}_2 = \begin{bmatrix} \beta_{1,2} & \beta_{2,2} & \beta_{3,2} & \cdots & \beta_{m,2} \end{bmatrix}. \tag{7}$$

Therefore, the matrix form of the normal vector of BH_1 is obtained

$$\tilde{n}_{BH_1} = \frac{\tilde{n}_1}{\|\tilde{n}_1\|} \mp \frac{\tilde{n}_2}{\|\tilde{n}_2\|}. \tag{8}$$

As can be seen from Eq. (8) there are two cases to determine the normal vector of BH_1 since there are two bisector hyperplane arising from two angles called the minor and major ones between the H_1 and H_2. The bisector hyperplane stemming from the minor angle is preferred as BH_1 because of representing the data set meaningfully. The formulae of BH_1 is written as

$$BH_1: \ \tilde{\beta}_{0,1} + \tilde{n}_{BH_1} X = 0, \tag{9}$$

where $X = \begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_m \end{bmatrix}^T$, $\tilde{n}_{BH_1} = \begin{bmatrix} \tilde{\beta}_{1,1} & \tilde{\beta}_{2,1} & \tilde{\beta}_{3,1} & \cdots & \tilde{\beta}_{m,1} \end{bmatrix}$ and $\tilde{\beta}_{0,1} = \frac{\beta_{0,1}}{\|\tilde{n}_1\|} \mp \frac{\beta_{0,2}}{\|\tilde{n}_2\|}$. In a similar manner, the normal vectors of BH_1 and H_3 are taken as

$$\tilde{n}_{BH_1} = \begin{bmatrix} \tilde{\beta}_{1,1} & \tilde{\beta}_{2,1} & \tilde{\beta}_{3,1} & \cdots & \tilde{\beta}_{m,1} \end{bmatrix}, \tag{10}$$

$$\tilde{n}_3 = \begin{bmatrix} \beta_{1,3} & \beta_{2,3} & \beta_{3,3} & \cdots & \beta_{m,3} \end{bmatrix}. \tag{11}$$

and the matrix form of the normal vector of BH_2 is obtained

$$\tilde{n}_{BH_2} = \frac{\tilde{n}_{BH_1}}{\|\tilde{n}_{BH_1}\|} \mp \frac{\tilde{n}_3}{\|\tilde{n}_3\|}. \tag{12}$$

Use the minor angle point of view,

$$BH_2: \ \tilde{\beta}_{0,2} + \tilde{n}_{BH_2} X = 0, \tag{13}$$

where $X = \begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_m \end{bmatrix}^T$, $\tilde{n}_{BH_2} = \begin{bmatrix} \tilde{\beta}_{1,2} & \tilde{\beta}_{2,2} & \tilde{\beta}_{3,2} & \cdots & \tilde{\beta}_{m,2} \end{bmatrix}$ and $\tilde{\beta}_{0,2} = \frac{\tilde{\beta}_{0,1}}{\|\tilde{n}_{BH_1}\|} \mp \frac{\beta_{0,3}}{\|\tilde{n}_3\|}$. The computation process is continued as the similar way, and the last bisector hyperplaneBH_{m-1} is obtained as

$$BH_{m-1}: \ \tilde{\beta}_{0,m-1} + \tilde{n}_{BH_{m-1}} X = 0 \tag{14}$$

where $X = \begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_m \end{bmatrix}^T$,
$\tilde{n}_{BH_{m-1}} = \begin{bmatrix} \tilde{\beta}_{1,m-1} & \tilde{\beta}_{2,m-1} & \cdots & \tilde{\beta}_{m,m-1} \end{bmatrix}$ and $\tilde{\beta}_{0,m-1} = \frac{\tilde{\beta}_{0,m-2}}{\|\tilde{n}_{BH_{m-2}}\|} \mp \frac{\beta_{0,m}}{\|\tilde{n}_m\|}$.
Thus, the process is completed, and the BH_{m-1} is called as the BRM-II model.

3. Computational Experiments

To demonstrate the applicability and efficiency of the multidimensional BRM-II, various simulations which are up to ten variables for different sample sizes and distribution types are performed in terms of the MSE

$$MSE = \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)}{n-k},$$

where y_i are real observation values, $\hat{y}_i$ are estimated values, n is the sample size and k is the number of parameters. In addition to the simulations, the BRM-II is applied to two real problems with two and three variables, and compared with the existing methods in the literature.

3.1. **Simulations.** It is well-known that simulation is defined as imitating something in an artificial environment depending on time. In this study, the performance measurement of the BRM-II method is calculated on the computer by designing the simulation conditions. The corresponding simulations are organized by randomly selecting different sample sizes n such as 20, 30, 50, 100 and 200 from the data sets with a population size N=5000 with different distributions t∼4, t∼10 and t∼30. The MSE values of the simulations were calculated for up to 10 variables in this study (the number of variables can be increased further if desired). The simulations are repeated 100000/n times, and the arithmetic means of the results obtained are calculated after the processes are completed. The produced results are listed in Table 1 in details. According to these results, when the BRM-II method is ana-

Table 1. The MSE values of the BRM-II simulations using total population N=5000

Number of Variables (nv)	**Sample Size (n)**	**The degrees of freedom of the distribution (df)**			**Number of Variables (nv)**	**Sample Size (n)**	**The degrees of freedom of the distribution (df)**		
		$t\sim 4$	$t\sim 10$	$t\sim 30$			$t\sim 4$	$t\sim 10$	$t\sim 30$
3	20	6.96	5.22	4.82	5	20	8.20	5.98	5.30
	30	6.65	5.01	4.54		30	7.15	5.32	4.85
	50	6.15	4.66	4.37		50	6.68	4.96	4.66
	100	6.08	4.60	4.18		100	6.28	4.70	4.27
	200	6.00	4.55	4.15		200	5.94	4.66	4.25
4	20	7.02	5.36	5.11	10	20	12.60	9.00	8.64
	30	6.98	5.04	4.78		30	9.54	6.60	6.38
	50	6.36	4.89	4.45		50	7.51	5.56	5.23
	100	6.13	4.66	4.32		100	6.58	5.08	4.59
	200	6.11	4.62	4.22		200	6.32	4.66	4.36

lyzed with 3 variables, the MSE decreases in all sample sizes when the degrees of freedom of the distribution are increased. For instance, the MSE equals to 6.68 if *nv, n* and *df* are taken as 5, 50 and 4, respectively, while the MSE decrease to 4.96 and 4.66 if *df* is increased to 10 and 30 without changing the other parameters.

Besides, the MSE decreases as n is increased by keeping df and nv unchanged. It can be seen that the same behaviour is valid as nv is 4, 5 and 10. Note that the MSE is expected to increase as nv increases by keeping df and n unchanged. The relationship between the sample sizes and the degrees of freedom of distribution for different number of variables are shown in Figure 3.

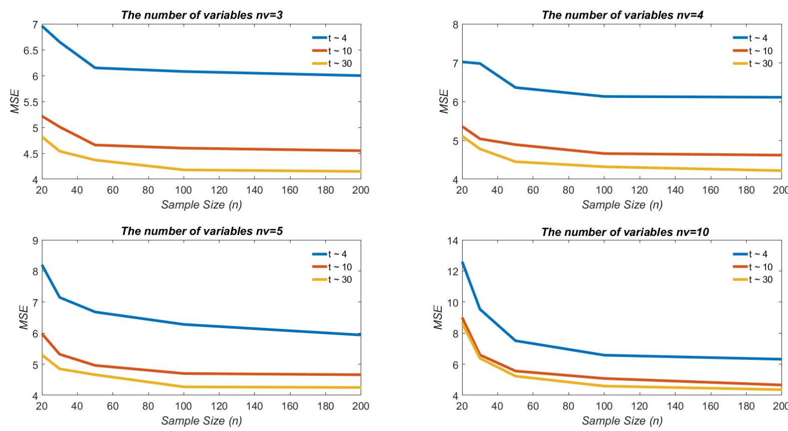


FIGURE 3. The relationship between the sample sizes and the degrees of freedom of distribution for different number of variables

Consequently, the performance of the BRM-II method draws attention in terms of supporting the theoretical expectation regarding the decreasing behaviour of MSE. Therefore, the proposed method is a powerful tool among the regression approaches.

3.2. **Application to the real data sets.** In this part, there are two real modeling processes with two and three variables including the oceanographic data sets are considered to demonstrate the performance of the BRM-II methods.

3.2.1. *Example with two variables.* As the first real application of proposed method with two variables is used the data that including weights of unspawned female cabezon (a California marine fish, Scorpaenichthys marmoratus) and the number of eggs subsequently produced for 11 fish are given as [22]:

TABLE 2. The biological oceanographic data set

Weight (to nearest 100g): X_1	14	17	24	25	27	33	34	37	40	41	42
Eggs (in thousands): X_2	61	37	65	69	54	93	87	89	100	90	97

It is desired to constitute a functional relationship between weight before spawning and number of egg produced because of both variables are subject to error.

In order to construct the BRM-II method, as the first step, the simple linear regression is applied for each variable with respect to the data in Table 2 as follows:

$$\beta_{0,1} + \beta_{1,1}X_1 + \beta_{2,1}X_2 = 0 \ , \tag{15}$$

$$\beta_{0,2} + \beta_{1,2}X_1 + \beta_{2,2}X_2 = 0 \ , \tag{16}$$

where X_1 and X_2 are dependent variables in Eqs. (15) and (16), respectively. The coefficients are computed as reported in Table 3.

TABLE 3. The coefficients of simple linear regressions

$\beta_{i,j}$	$i=0$	$i=1$	$i=2$
$j=1$	-1.5032	-1	0.4163
$j=2$	19.7668	1.8700	-1

Then, the bisector line of these system is obtained by using the Eqs. (5)-(9) in proposed method mentioned above

$$-10.7093 \quad - 1.8050 \ X_1 + 0.8559X_2 = 0. \tag{17}$$

The regression lines are illustrated in Figure 4. Moreover, in order to demonstrate the efficiency of the BRM-II, the results of MSE are compared with some previous studies in the literature, and given in Table 4. It can be seen from the results that the BRM-II method is outstanding in comparison with the study of Richter and Stavn (2014) [20]. Moreover, the MSE of our method is even better than the MSE result in their study by using the standardized data.

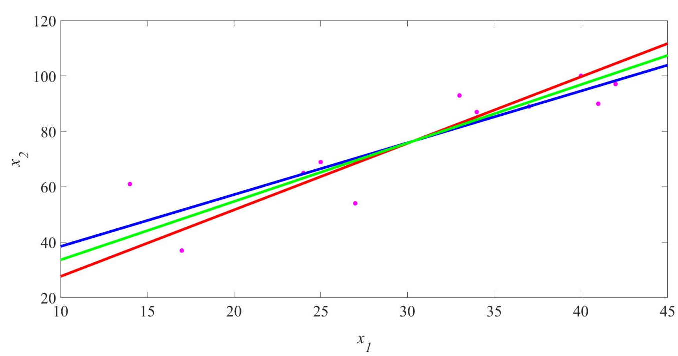


FIGURE 4. The data set and the regression lines in Example 1 (the red line: Eq. (3.1), the blue line: Eq. (16), and the green (bisector) line: Eq. (17))

TABLE 4. The results of MSE for the different regression models

Regression Model	**MSE**
BRM-II	109.01
[20]	122.62
[20] (with Standardized data)	109.47
[26]	109.54

TABLE 5. The oceanographic field data set

PIM:X_1	POM: X_2	b_{555}:X_3	PIM:X_1	POM: X_2	b_{555}:X_3
11.36	2.36	8.41	5.33	1.94	4.99
6.98	1.49	5.85	5.46	2.16	5.05
6.89	1.15	6.91	9.98	2.87	9.11
14.60	3.00	11.31	5.67	1.81	8.39
12.52	1.59	10.03	6.89	3.11	7.57
5.40	2.53	3.40	3.21	3.32	4.38
6.45	2.21	5.43	4.56	1.69	3.78
1.57	0.18	1.09	6.56	1.06	4.91
2.15	0.45	1.84	4.63	1.05	3.09
22.31	3.28	17.94	5.48	0.69	3.66
4.67	2.05	4.85	3.88	0.71	2.41
5.01	0.52	1.20	2.80	0.48	1.77

3.2.2. *Example with three variables.* The second application of proposed method with three variables is discussed the data sets that including the concentration of particulate inorganic matter [PIM (gm^{-3})], the particulate organic matter [POM (gm^{-3})] and the total scattering coefficient at a wavelength of 555 nm [b_{555} (m^{-1})] for 24 field stations at Mobile Bay, Alabama are given in Table 5 [20]:

In order to constitute a functional relationship among PIM, POM and b_{555}, the simple linear regressions for each variable are written with respect to the data in Table 4 similar to the previous example as follows:

$$\beta_{0,1} + \beta_{1,1}X_1 + \beta_{2,1}X_2 + \beta_{3,1}X_3 = 0, \tag{18}$$

$$\beta_{0,2} + \beta_{1,2}X_1 + \beta_{2,2}X_2 + \beta_{3,2}X_3 = 0, \tag{19}$$

$$\beta_{0,3} + \beta_{1,3}X_1 + \beta_{2,3}X_2 + \beta_{3,3}X_3 = 0, \tag{20}$$

where X_1, X_2 and X_3 are dependent variables in Eqs. (18), (19) and (20), respectively. The coefficients are given in the following Table 6. The bisector line of these system is obtained

$$1.0225 - 0.8603\ X_1 - 1.2936\ X_2 + 1.2434X_3 = 0. \tag{21}$$

The regression planes are illustrated in Figure 5, and the MSE result of the BRM-II is compared with some previous studies in the literature (see Table 7).

Table 6. The coefficients of simple linear regression

$\beta_{i,j}$	$i=0$	$i=1$	$i=2$	$i=3$
$j=1$	0.9066	-1	-0.6394	1.2322
$j=2$	0.8042	-0.1404	-1	0.3310
$j=3$	-0.4658	0.6897	0.8440	-1

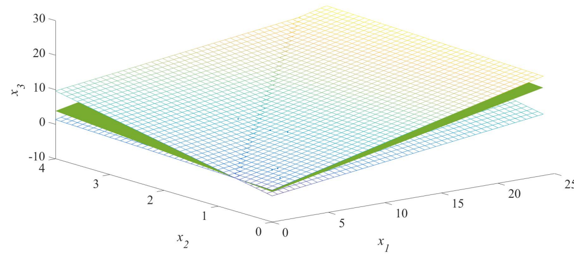
Table 7. The results of MSE for the different regression models

Regression Model	**MSE**
BRM-II	1.3336
[20]	2.2318
[20] (with standardized data)	1.3210

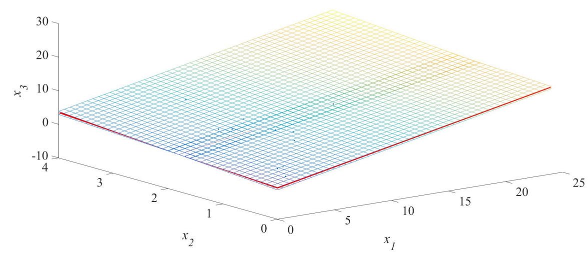
It can be said from the table that the BRM-II method is superior in comparison with the study of Richter and Stavn (2014) [20]. Also, the MSE of our method is competitive with the MSE of their study by using the standardized data.

4. Conclusion and Recommendation

In this study, a new multidimensional BRM-II method is introduced for multivariate problems that may contain measurement error. In order to demonstrate the validity and efficiency, the proposed method is applied to simulations up to ten variables for different sample sizes and distribution types in terms of the Mean Square Error (MSE), and then implemented to two real problems with two and three variables. By comparing with the methods in the literature, it is observed that the BRM-II method is outstanding in comparison with the study of Richter and Stavn (2014) for both original and standardized data. So it can be deduced that the proposed method provides relatively higher accuracy. Besides, it is easy applicable and versatile tool for prediction of relevant real life problems. For the further studies, different forms of the BRM-II can be derived for more realistic phenomena.



(a)



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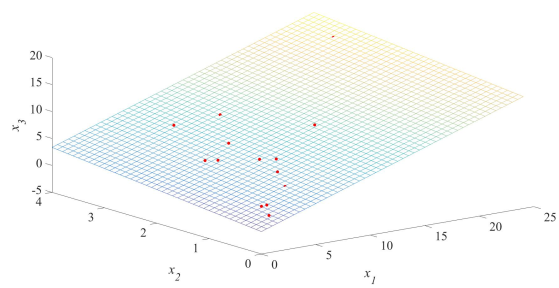

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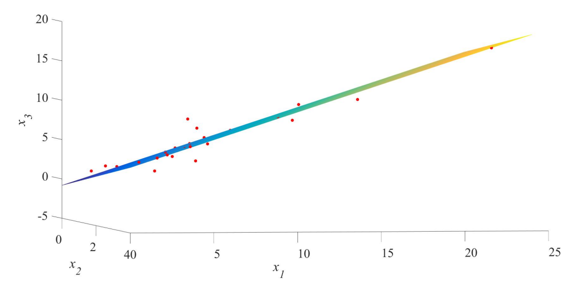

(d)

FIGURE 5. The data set and the regression planes in Example 2 *(a)* Eq. (18), Eq. (19) and the bisector-1; *(b)* Eq. (20), the bisector-1 and the bisector-2; *(c)* the final bisector plane: Eq. (21); *(d)* the final bisector plane: Eq. (21) in *(c)* with different perspective.

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