*Fundamental Journal of Mathematics and Applications*, 6 (3) (2023) 177-187
*Research Article*

FUJMA

**Fundamental Journal of Mathematics and Applications**

*Journal Homepage:* www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1292885

# Bisimplicial Commutative Algebras and Crossed Squares

**Özgün Gürmen Alansal[1*] and Erdal Ulualan[1]**

[1]*Department of Mathematics, Faculty of Arts and Sciences, Kütahya Dumlupınar University, Kütahya, Turkiye*
[*]*Corresponding author*

**Article Info**

***Keywords:*** *Bisimplicial algebras, Crossed modules, Moore complex*
***2010 AMS:*** *18G30, 55U10*
***Received:*** *5 May 2023*
***Revised:*** *19 September 2023*
***Accepted:*** *22 September 2023*
***First Online:*** *29 September 2023*
***Published:*** *30 September 2023*

**Abstract**

A simplicial commutative algebra with Moore complex of length 1 gives a crossed module structure over commutative algebras. In this study, we will give 2-dimensional version of this result by giving hypercrossed complex pairings for a bisimplicial algebra and its Moore bicomplex. We give a detailed calculation in low dimensions for these pairings to see their role in the structures of crossed squares and bisimplicial algebras. In this context, we prove that if the Moore bicomplex of a bisimplicial commutative algebra is of length 1, then it gives a crossed square structure over commutative algebras.

## 1. Introduction

The category of simplicial groups with Moore complex of length 1 is equivalent to the category of Whitehead's crossed modules [6]. This structure can be considered as an algebraic model for homotopy connected 2-types. Conduché [2] has proven 2-dimensional version of this result by giving the definition of a crossed module of length 2. He proved that the category of such objects are equivalent to the category of simplicial groups with Moore complex of length 2. The structure of a crossed square has been introduced by Guin-Walery and Loday [4]. This structure is a model for homotopy connected 3-types. The commutative algebra version of crossed modules has been defined by Porter in [11]. On the other hand crossed squares of commutative algebras has been investigated by Ellis [5]. Conduché also, [3], gave the close relationships among bisimplicial groups with crossed squares for the version of groups, and he proved that Loday's mapping cone complex of a crossed square gives a 2-crossed module.

Carrasco and Cegarra, [10], give a general version of the Dold-Kan theorem for the equivalence between simplicial groups and non-Abelian chain complexes. Porter in [12] has proven the equivalence between the category of $n$-types of simplicial groups and the category of crossed $n$-cubes. In [9], Gürmen-Alansal and Ulualan generalised these pairings for the Moore bicomplex in bisimplicial groups. It can be seen the role of these pairings for the relations among bisimplicial groups and crossed squares. Arvasi and Porter [13], using the Carrasco and Cegarra pairing operators for a Moore complex in a simplicial (commutative) algebra, and they have defined the functions $C_{\alpha,\beta}$ functions, and as an application, they proved that the category of 2-crossed modules of commutative algebras introduced by Grandjeán and Vale in [1] is equivalent to that of simplicial commutative algebras with Moore complex of length 2. Of course, this is the commutative algebra version of Conduché's result [2].

Our first aim in this work is to define the functions $C_{\alpha,\beta}$ for 2-dimensional simplicial algebras (or bisimplicial algebras) and second aim is to give the relationship between crossed squares and bisimplicial algebras by use of the functions $C_{\alpha,\beta}$.

## 2. Preliminaries

The simplicial set analogue has been studied in [8, 7, 13]. We give the following statements from [13]. Define the set $P(n)$ consisting of the pairs of elements in the form $(\alpha,\beta)$ from $S(n)$ with $\alpha\cap\beta=\emptyset$ and $\beta<\alpha$ where $\alpha=(i_r,\ldots,i_1)$, $\beta=$

***Email addresses and ORCID numbers***: **ozgun.gurmen@dpu.edu.tr, 0000-0003-2851-986X (Ö. Gürmen Alansal), e.ulualan@dpu.edu.tr, 0000-0002-4823-8267 (E. Ulualan)**
**Cite as: Ö.G. Alansal, E. Ulualan, *Bisimplicial commutative algebras and crossed squares*, Fundam. J. Math. Appl., 6(3) (2023), 177-187.**

$(j_s,\dots,j_1)\in S(n)$. The k-linear morphisms are,

$$\{C_{\alpha,\beta}: NE_{n-\#\alpha}\otimes NE_{n-\#\beta}\to NE_n|(\alpha,\beta)\in P(n),\ 0\le n\}$$

given by composing:

$$\begin{aligned}C_{\alpha,\beta}(x_\alpha\otimes y_\beta) &= p\mu(s_\alpha\otimes s_\beta)(x_\alpha\otimes y_\beta)\\ &= p(s_\alpha(x_\alpha)s_\beta(x_\beta))\\ &= (1-s_{n-1}d_{n-1})\dots(1-s_0d_0)(s_\alpha(x_\alpha)s_\beta(x_\beta))\end{aligned}$$

where

$$s_\alpha = s_{i_r}\dots s_{i_1}: NE_{n-\#\alpha}\to E_n,\ s_\beta = s_{j_s}\dots s_{j_1}: NE_{n-\#\beta}\to E_n,$$

$p: E_n\to NE_n$ is given as composite projections $p=p_{n-1}\dots p_0$ with

$$p_j = 1-s_jd_j \text{ for } j=0,1,\dots,n-1$$

and $\mu: E_n\otimes E_n\to E_n$ denotes multiplication.
Arvasi and Porter in [13] studied the truncated simplicial algebras and their properties. By using $C_{\alpha,\beta}$ functions they then proved the following result:

**Proposition 2.1.** *Suppose that $E$ is a simplicial algebra. We denote its Moore complex by $NE$. Then*

$$NE_2/\partial_3(NE_3\cap D_3)\xrightarrow{\overline{\partial_2}} NE_1\xrightarrow{\partial_1} NE_0$$

*is a 2-crossed module of algebras with the Peiffer lifting map*

$$\{-,-\}: NE_1\otimes NE_1\longrightarrow NE_2/\partial_3(NE_3\cap D_3)$$

*given by* $(x\otimes y)\mapsto\{x,y\}=C_{(0)(1)}(x\otimes y)+\partial_3(NE_3\cap D_3)=s_1(x)(s_1y-s_0y)+\partial_3(NE_3\cap D_3)$ *for all* $x,y\in NE_1$.

## 3. Hypercrossed Complex Pairings for Bisimplicial Algebras

In this section, we define the $C_{\alpha,\beta}$ functions in Moore bicomplex of a bisimplicial algebra. Let $\Delta$ be the category whose objects are the ordered sets $[n=\{0<1<2\dots<n\}]$ and whose morphisms are non decreasing maps between them. Suppose $\Delta\times\Delta$ is the product category. Its objects are the pairs $([p],[q])$, the morphisms are the pairs of increasing maps. The functor $E_{.,.}:(\Delta\times\Delta)^{op}\to Alg$ can be considered as a bisimplicial algebra. Therefore, $E_{.,.}$ is equivalent to giving for each $(p,q)$ an algebra $E_{p,q}$ and morphisms:

$$\begin{aligned}&d_i^{h(pq)}: E_{p,q}\to E_{p-1,q}; \qquad s_i^{h(pq)}: E_{p,q}\to E_{p+1,q}, \qquad p\ge i\ge 0\\ &d_j^{v(pq)}: E_{p,q}\to E_{p,q-1}; \qquad s_j^{v(pq)}: E_{p,q}\to E_{p,q+1}, \qquad q\ge j\ge 0\end{aligned}$$

where the maps $d_j^{v(pq)}, s_j^{v(pq)}$ commute with $d_i^{h(pq)}, s_i^{h(pq)}$ and that the homomorphisms $d_j^{v(pq)}, s_j^{v(pq)}$ respectively for $d_i^{h(pq)}, s_i^{h(pq)}$. These maps satisfy the simplicial identities.
We consider of $d_j^{v(pq)}, s_j^{v(pq)}$ as the vertical operators and $d_i^{h(pq)}, s_i^{h(pq)}$ as the horizontal operators. If $E_{.,.}$ is a bisimplicial algebra, an element of $E_{p,q}$ can be thought as a product of a $p$-simplex and a $q$-simplex. Let **BiSimpAlg** be the category whose objects are bisimplicial algebras given by the functors $E_{.,.}:(\Delta\times\Delta)^{op}\to Alg$ and whose morphisms are natural transformations between the functors $E_{.,.}$ and $E'_{.,.}$.
The Moore bicomplex for a bisimplicial algebra is

$$NE_{n,m}=\bigcap_{(i,j)=(0,0)}^{(n-1,m-1)}\mathrm{Ker}d_i^{h(nm)}\cap\mathrm{Ker}d_j^{v(nm)}$$

with the boundary homomorphisms

$$\partial_i^{h(nm)}: NE_{n,m}\longrightarrow NE_{n-1,m}$$

and

$$\partial_j^{v(nm)}: NE_{n,m}\longrightarrow NE_{n,m-1}$$

obtained by the maps $d_i^{h(nm)}$ and $d_j^{v(nm)}$ where $0\leqslant j\leqslant m,\ 0\leqslant i\leqslant n,\ n,m\neq 0$.

We can denote this Moore bicomplex by Figure 3.1.

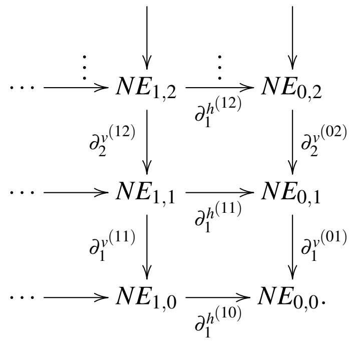


**Figure 3.1:** Moore bicomplex

Now, we give the functions $C_{\alpha,\beta}$ for bisimplicial algebras.
Given $\underline{k}=(n,m)\in\mathbb{N}\times\mathbb{N}$. Let $S(\underline{k})=S(n)\times S(m)$ with the partial product order. Take $\underline{\alpha},\underline{\beta}\in S(\underline{k})$ where $\underline{\alpha}=(\alpha_1,\alpha_2)$, $\underline{\beta}=(\beta_1,\beta_2)$ for $\alpha_1,\beta_1\in S(n)$ and $\alpha_2,\beta_2\in S(m)$. The 2-dimensional case of the $C_{\alpha,\beta}$ functions given for any simplicial algebra [13] can be obtained as follows. We will need that the Pairings

$$\left\{C_{\underline{\alpha},\underline{\beta}}:NE_{\underline{k}-\#\underline{\alpha}}\times NE_{\underline{k}-\#\underline{\beta}}\longrightarrow NE_{\underline{k}}\mid\underline{\alpha}\neq\underline{\beta},\ \underline{\alpha},\underline{\beta}\in S(\underline{k})\right\}$$

are obtained by creating of the maps given in the diagram

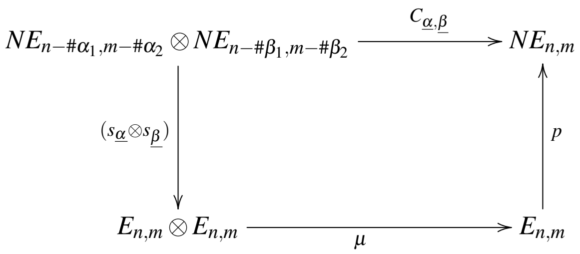


**Figure 3.2:** Construction of $C_{\underline{\alpha},\underline{\beta}}$

where $s_{\underline{\alpha}}:s^h_{\alpha_1}s^v_{\alpha_2}$, and where $s^h_{\alpha_1}=s^h_{i_r}\cdots s^h_{i_1}$ for $\alpha_1=(i_r,\cdots,i_1)\in S(n)$, similarly $s_{\underline{\beta}}:s^h_{\beta_1}s^v_{\beta_2}$, and where $s^h_{\beta_1}=s^h_{j_s}\cdots s^h_{j_1}$ for $\beta_1=(j_s,\cdots,j_1)\in S(n)$. We can define the maps similarly $s^v_{\alpha_2},s^v_{\beta_2}$ in $S(m)$. Note that $s^{(h,v)}_{\emptyset}=id$ is the identity map. By the composing the projections given below, the map $p$ is defined as

$$p=\left(p^h_{n-1}...p^h_0\right)\left(p^v_{m-1}...p^v_0\right) \tag{3.1}$$

where $p^{(h,v)}_j(x)=x-s^{(h,v)}_j d^{(h,v)}_j(x)$, and $\mu$ is given by the multiplication.
Thus for $\underline{\alpha}=(\alpha_1,\alpha_2),\underline{\beta}=(\beta_1,\beta_2)\in S(n)\times S(m)$, it is obtained from the Figure 3.2 by composing the maps that

$$\begin{aligned}C_{\underline{\alpha},\underline{\beta}}(x\otimes y) &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes y)\\ &= p\mu(s^h_{\alpha_1}s^v_{\alpha_2}(x)\otimes s^h_{\beta_1}s^v_{\beta_2}(y))\\ &= p\left(s^h_{\alpha_1}s^v_{\alpha_2}(x)\cdot s^h_{\beta_1}s^v_{\beta_2}(y)\right)\end{aligned}$$

for $x\in NE_{n-\#\alpha_1,m-\#\alpha_2}$ and $y\in NE_{n-\#\beta_1,m-\#\beta_2}$, where $p$ is given by

$$\begin{aligned}p:E_{n,m} &\to NE_{n,m}\\ a &\mapsto p^h_{n-1}\cdots p^h_1p^h_0p^v_{m-1}\cdots p^v_1p^v_0(a)=(1-s^h_{n-1}d^h_{n-1})\cdots(1-s^h_0d^h_0)(1-s^v_{m-1}d^v_{m-1})\cdots(1-s^v_0d^v_0)(a)\end{aligned}$$

for all $a\in E_{n,m}$. Note that we obtain

$$C_{\underline{\alpha},\underline{\beta}}(x\otimes y)=C_{\underline{\beta},\underline{\alpha}}(y\otimes x)$$

for $x\in NE_{n-\#\alpha_1,m-\#\alpha_2}$ and $y\in NE_{n-\#\beta_1,m-\#\beta_2}$.

## 4. Low Dimensions Cases

### 4.1. The case $(n,m)=(0,1)$ or $(1,0)$.

Take $(n,m)=(0,1)$ or $(n,m)=(1,0)$. Firstly we calculate here $C_{\underline{\alpha},\underline{\beta}}$ functions whose codomain $NE_{0,1}$ or $NE_{1,0}$. Let $(n,m)=(0,1)$. So $S(n,m)=S(0)\times S(1)=\{(\emptyset,\emptyset),(\emptyset,(0))\}$, $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. Thus the function $C_{\underline{\alpha},\underline{\beta}}$ given by

$$C_{(\emptyset,\emptyset),(\emptyset,(0))}:NE_{0,1}\otimes NE_{0,0}\longrightarrow NE_{0,1}$$

is obtained by

$$\begin{aligned}C_{(\emptyset,\emptyset),(\emptyset,(0))}(x\otimes y) &= p\mu(\overset{h}{s}_{\emptyset}\overset{v}{s}_{\emptyset}(x)\otimes\overset{h}{s}_{\emptyset}\overset{v}{s}_{(0)}(y))=p(id(x)\overset{v}{s}_{(0)}(y))\\ &= xs_0^{v(00)}(y)-s_0^{v(00)}d_0^{v(01)}(xs_0^{v(00)}(y))\\ &= xs_0^{v(00)}(y)-s_0^{v(00)}d_0^{v(01)}(x).s_0^{v(00)}(y)\\ &= xs_0^{v(00)}(y)\quad(\because x\in NE_{0,1}=\ker d_0^{v(01)})\end{aligned}$$

for $x\in NE_{0,1}$, $y\in NE_{0,0}$.
Assume that $(n,m)=(1,0)$. After this, we take $S(1)\times S(0)=\{(\emptyset,\emptyset),((0),\emptyset)\}$. Let $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=((0),\emptyset)$. Then the function

$$C_{(\emptyset,\emptyset),((0),\emptyset)}:NE_{1,0}\otimes NE_{0,0}\longrightarrow NE_{1,0}$$

is defined as

$$C_{(\emptyset,\emptyset),((0),\emptyset)}(x\otimes y)=x(s_0^{h(00)}(y))$$

for $x\in NE_{1,0}$, $y\in NE_{00}$.

### 4.2. The case $(n,m)=(1,1)$.

Let $(n,m)=(1,1)$. Define the set

$$S(1)\times S(1)=\{(\emptyset,\emptyset),\ ((0),(0)),\ (\emptyset,(0)),\ ((0),\emptyset)\}.$$

1. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=(\emptyset,(0))$, the function $C_{\underline{\alpha},\underline{\beta}}$ is from $NE_{1,1}\otimes NE_{1,0}$ to $NE_{1,1}$. The map can be defined by
$$C_{(\emptyset,\emptyset),(\emptyset,(0))}(x\otimes y)=xs_0^{v(10)}(y);\ x\in NE_{1,1},\ y\in NE_{1,0}.$$
2. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=((0),\emptyset)$. Then, the function $C_{\underline{\alpha},\underline{\beta}}$ is from $NE_{1,1}\otimes NE_{0,1}$ to $NE_{1,1}$. The map can be calculated by
$$C_{(\emptyset,\emptyset),((0),\emptyset)}(x\otimes y)=xs_0^{h(01)}(y);\ x\in NE_{1,1},\ y\in NE_{0,1}.$$
3. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=((0),(0))$, the map $C_{(\emptyset,\emptyset),((0),(0))}:NE_{1,1}\otimes NE_{0,0}\to NE_{1,1}$ is given by
$$C_{(\emptyset,\emptyset),((0),(0))}(x\otimes y)=x(s_0^{v(00)}s_0^{h(01)}(y));\ x\in NE_{1,1},\ y\in NE_{0,0}.$$
4. Take $\underline{\alpha}=((0),\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. Then the map
$$C_{((0),\emptyset),(\emptyset,(0))}:NE_{0,1}\otimes NE_{1,0}\to NE_{1,1}$$
can be calculated for any $x\in NE_{0,1}$, $y\in NE_{1,0}$ as
$$\begin{aligned}C_{((0),\emptyset)(\emptyset,(0))}(x\otimes y) &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes y)\\ &= p_0^hp_0^v(s_0^{h(01)}(x)s_0^{v(10)}(y))\\ &= p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{v(10)}d_0^{v(11)}(s_0^{h(01)}(x)s_0^{v(10)}(y))\right)\\ &= p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{v(10)}s_0^{h(00)}d_0^{v(01)}(x)s_0^{v(10)}(y)\right)\\ &= p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)\right)\quad(\because x\in\ker d_0^{v(01)})\\ &= \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}d_0^{h(11)}(s_0^{h(01)}(x)s_0^{v(10)}(y))\right)\\ &= \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}(x)s_0^{h(01)}d_0^{h(11)}s_0^{v(10)}(y)\right)\quad(\because d_0^{h(11)}s_0^{h(01)}=id)\\ &= \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}(x)s_0^{h(01)}s_0^{v(00)}d_0^{h(10)}(y)\right)\ (\because d_0^{h(11)}s_0^{v(10)}=s_0^{v(00)}d_0^{h(10)})\\ &= \left(s_0^{h(01)}(x)s_0^{v(10)}(y)\right).\quad(\because y\in\ker d_0^{h(10)}=NE_{1,0})\end{aligned}$$

5. For $\underline{\alpha} = ((0), \emptyset)$ and $\underline{\beta} = ((0),(0))$. Then the map

$$C_{((0),\emptyset),((0),(0))} : NE_{0,1} \otimes NE_{0,0} \to NE_{1,1}$$

can be given for any $x \in NE_{0,1}$, $y \in NE_{0,0}$ as

$$\begin{aligned}
C_{((0),\emptyset),((0),(0))}(x \otimes y) &= p\mu(s_{\underline{\alpha}} \otimes s_{\underline{\beta}})(x \otimes y) \\
&= p_0^h p_0^v (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \\
&= p_0^h \left( s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{v^{(10)}} d_0^{v^{(11)}} (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \right) \\
&= p_0^h \left( s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{v^{(10)}} d_0^{v^{(11)}} s_0^{h^{(01)}}(x) s_0^{v^{(10)}} d_0^{v^{(11)}} s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \\
&= p_0^h \left( s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \quad (\because x \in \ker d_0^{v^{(01)}}) \\
&= \left( s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{h^{(01)}} d_0^{h^{(11)}} (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \right) \\
&= \left( s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \quad (\because d_0^{h^{(11)}} s_0^{h^{(01)}} = id) \\
&= 0.
\end{aligned}$$

6. For $\underline{\alpha} = (\emptyset,(0))$ and $\underline{\beta} = ((0),(0))$, the map

$$C_{(\emptyset,(0)),((0),(0))} : NE_{1,0} \otimes NE_{0,0} \to NE_{1,1}$$

is the zero map as given in the previous step.

### 4.3. The case $(n,m) = (0,2)$ or $(2,0)$ and crossed modules

In this section, by considering $(n,m)$= $(2,0)$ and $(0,2)$, we can compute the possible non zero operators with codomain $NE_{2,0}$, $NE_{0,2}$ respectively. We give an application of these operators to the crossed modules.
For $(n,m) = (2,0)$. From the set

$$S(2) \times S(0) = \{((0),\emptyset),\ (\emptyset,\emptyset),\ ((1),\emptyset),\ ((1,0),\emptyset)\},$$

we can choose $\underline{\alpha} = ((1),\emptyset)$, $\underline{\beta} = ((0),\emptyset)$. Then $C_{\underline{\alpha},\underline{\beta}}$ is a map from $NE_{1,0} \otimes NE_{1,0}$ to $NE_{2,0}$. This map can be given by for $y, y' \in NE_{1,0}$

$$\begin{aligned}
C_{((1),\emptyset),((0),\emptyset)}(y \otimes y') &= p\mu(s_{\underline{\alpha}} \otimes s_{\underline{\beta}})(y \otimes y') \\
&= p_1^h p_0^h (s_1^{h^{(10)}}(y) s_0^{h^{(10)}}(y')) \\
&= s_1^{h^{(10)}}(y)(s_0^{h^{(10)}}(y') - s_1^{h^{(10)}}(y')) \in NE_{2,0}.
\end{aligned}$$

We have similarly

$$\begin{aligned}
\partial_2^{h^{(20)}}(C_{((1),\emptyset),((0),\emptyset)}(y \otimes y')) &= \partial_2^{h^{(20)}}(s_1^{h^{(10)}}(y)(s_0^{h^{(10)}}(y') - s_1^{h^{(10)}}(y'))) \\
&= y s_0^{h^{(00)}} d_1^{h^{(10)}}(y') - yy' \in NE_{1,0}.
\end{aligned}$$

Now suppose $(n,m) = (0,2)$. From the set

$$S(0) \times S(2) = \{(\emptyset,\emptyset), (\emptyset,(0)), (\emptyset,(1)), (\emptyset,(1,0))\},$$

we can find the functions $C_{\underline{\alpha},\underline{\beta}}$ with codomain $NE_{0,2}$. In this case, the only non zero operator $C_{\underline{\alpha},\underline{\beta}}$ can be calculated by choosing $\underline{\alpha} = (\emptyset,(1))$ and $\underline{\beta} = (\emptyset,(0))$. Therefore, this is a map from $NE_{0,1} \otimes NE_{0,1}$ to $NE_{0,2}$. For $x, x' \in NE_{0,1}$, we obtain

$$\begin{aligned}
C_{(\emptyset,(1)),(\emptyset,(0))}(x \otimes x') &= p\mu(s_{\underline{\alpha}} \otimes s_{\underline{\beta}})(x \otimes x') \\
&= p_1^v p_0^v (s_1^{v^{(01)}}(x) s_0^{v^{(01)}}(x')) \\
&= s_1^{v^{(01)}}(x)(s_0^{v^{(01)}}(x') - s_1^{v^{(01)}}(x')) \in NE_{0,2}.
\end{aligned}$$

We have also

$$\begin{aligned}
\partial_2^{v^{(02)}}(C_{(\emptyset,(1)),(\emptyset,(0))}(x \otimes x')) &= \partial_2^{v^{(02)}}(s_1^{v^{(01)}}(x)(s_0^{v^{(01)}}(x') - s_1^{v^{(01)}}(x'))) \\
&= x s_0^{v^{(00)}} d_1^{v^{(01)}}(x') - xx' \in NE_{0,1}.
\end{aligned}$$

**Proposition 4.1.** *Assume that $E_{*,*}$ is a bisimplicial algebra. Consider its Moore bicomplex $NE_{*,*}$. For $p+q \geqslant 2$, if $NE_{p,q}=\{0\}$, then the map*

$$\partial : NE_{0,1} \times NE_{1,0} \longrightarrow NE_{0,0}$$

*given by*

$$\partial(x,y) = d_1^{v(01)}(x) + d_1^{h(10)}(y)$$

*for $x \in NE_{0,1}$, $y \in NE_{1,0}$ is a crossed module of commutative algebras. In particular, the maps $d_1^{v(01)}$ and $d_1^{h(10)}$ are crossed modules.*

*Proof.* The action of $t \in NE_{0,0}$ on $(x,y) \in NE_{0,1} \times NE_{1,0}$ is given by

$$t \cdot (x,y) = \left((s_0^{v(00)} t)x, (s_0^{h(00)} t)y\right).$$

For this action we get

$$\begin{aligned} \partial(t \cdot (x,y)) &= \partial\left((s_0^{v(00)} t)x, (s_0^{h(00)} t)y\right) \\ &= d_1^{v(01)}((s_0^{v(00)} t)x + d_1^{h(10)}(s_0^{h(00)} t)y) \\ &= t(d_1^{v(01)}(x) + d_1^{h(10)}(y)) \\ &= t\partial(x,y) \end{aligned}$$

and this is the first axiom of the crossed module.
Now for $(x,y),(x',y') \in NE_{0,1} \times NE_{1,0}$, we obtain

$$\begin{aligned} \partial(x,y)\cdot(x',y') &= \left(s_0^{v(00)}(d_1^{v(01)}(x) + d_1^{h(10)}(y))x', s_0^{h(00)}(d_1^{v(01)}(x) + d_1^{h(10)}(y))y'\right) \\ &= \left(s_0^{v(00)} d_1^{v(01)}(x)x' + s_0^{v(00)} d_1^{h(10)}(y)x', s_0^{h(00)} d_1^{v(01)}(x)y' + s_0^{h(00)} d_1^{h(10)}(y)y'\right) \\ &= (s_0^{v(00)} d_1^{v(01)}(x)x' + d_1^{h(11)} s_0^{v(10)}(y)x', d_1^{v(11)} s_0^{h(01)}(x)y' + s_0^{h(00)} d_1^{h(10)}(y)y') \\ &\quad (\because s_0^{v(00)} d_1^{h(10)} = d_1^{h(11)} s_0^{v(10)}, s_0^{h(00)} d_1^{v(01)} = d_1^{v(11)} s_0^{h(01)}) \\ &= (s_0^{v(00)} d_1^{v(01)}(x)x', s_0^{h(00)} d_1^{h(10)}(y)y') \ (\because s_0^{h(01)}(x), s_0^{v(10)}(y) \in NE_{1,1} = \{0\}). \end{aligned}$$

Since $NE_{0,2}=\{0\}$, we obtain for $x,x' \in NE_{0,1}$

$$\partial_2^{v(02)}(C_{(\emptyset,(1)),(\emptyset,(0))}(x' \otimes x)) = x' s_0^{v(00)} d_1^{v(01)}(x) - x'x = 0$$

and therefore,

$$s_0^{v(00)} d_1^{v(01)}(x)x' = xx'.$$

Similarly, since $NE_{2,0}=\{0\}$, we obtain for $y,y' \in NE_{1,0}$

$$\partial_2^{h(20)}(C_{((1),\emptyset),((0),\emptyset)}(y' \otimes y)) = y' s_0^{h(00)} d_1^{h(10)}(y) - y'y = 0$$

and therefore,

$$s_0^{h(00)} d_1^{h(10)}(y)y' = yy'.$$

Thus, we have

$$\begin{aligned} \partial(x,y)\cdot(x',y') &= (xx',yy') \\ &= (x,y)(x',y') \end{aligned}$$

and this is the second axiom of crossed module. □

## 5. Crossed squares and Bisimplicial Algebras

If we take $(n,m)$= $(2,1)$ and $(1,2)$, we will define the possible non zero operators $C_{\underline{\alpha},\underline{\beta}}$ whose codomain $NE_{1,2}$ and $NE_{2,1}$ respectively. We give an application of these operators to the crossed squares.
Assume now that $(n,m)=(2,1)$. We think the set $S(2)\times S(1)$. We can choose appropriate pairs $\underline{\alpha},\underline{\beta}$ from the set $S(2)\times S(1)$, we can compute similarly all the non zero maps with codomain $NE_{2,1}$. To get these maps, we take the possible $\underline{\alpha},\underline{\beta}$ as follows.

1. $\underline{\alpha}=((1),\emptyset)$, $\underline{\beta}=((0),\emptyset)$
2. $\underline{\alpha}=((1),\emptyset)$, $\underline{\beta}=(\emptyset,(0))$

3. $\underline{\alpha} = ((0), \emptyset)$, $\underline{\beta} = (\emptyset, (0))$
4. $\underline{\alpha} = ((1), (0))$, $\underline{\beta} = ((0), \emptyset)$
5. $\underline{\alpha} = ((0), (0))$, $\underline{\beta} = ((1), \emptyset)$.

For $(\underline{\alpha}, \underline{\beta})$, the necessary $C_{\underline{\alpha},\underline{\beta}}$ functions can be given as follows:

1. Take $\underline{\alpha} = ((1), \emptyset)$ and $\underline{\beta} = ((0), \emptyset)$, we have that the oparetor
$$C_{((1),\emptyset),((0),\emptyset)} : NE_{1,1} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
This operator can be given by
$$C_{((1),\emptyset),((0),\emptyset)}(x \otimes y) = s_1^{h^{(11)}}(x)(s_0^{h^{(11)}}(y) - s_1^{h^{(11)}}(y)) \in NE_{2,1}$$
for $x, y \in NE_{1,1}$.
2. For $\underline{\alpha} = ((1), \emptyset), \underline{\beta} = (\emptyset, (0))$, we get the operator
$$C_{((1),\emptyset),(\emptyset,(0))} : NE_{1,1} \otimes NE_{2,0} \longrightarrow NE_{2,1}$$
defined by
$$C_{((1),\emptyset),(\emptyset,(0))}(x \otimes t) = s_1^{h^{(11)}}(x)s_0^{v^{(02)}}(t) \in NE_{2,1}$$
for $x \in NE_{1,1}$ and $t \in NE_{2,0}$.
3. For $\underline{\alpha} = ((0), \emptyset), \underline{\beta} = (\emptyset, (0))$, we get the operator
$$C_{((0),\emptyset),(\emptyset,(0))} : NE_{1,1} \otimes NE_{2,0} \longrightarrow NE_{2,1}$$
given by
$$C_{((0),\emptyset),(\emptyset,(0))}(x \otimes t) = s_0^{h^{(11)}}(x)s_0^{v^{(02)}}(t) \in NE_{2,1}$$
for $x \in NE_{1,1}$, $t \in NE_{2,0}$.
4. For $\underline{\alpha} = ((1), (0)), \underline{\beta} = ((0), \emptyset)$, we get the following operator
$$C_{((1),(0)),((0),\emptyset)} : NE_{1,0} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
It is given by
$$C_{((1),(0)),((0),\emptyset)}(x \otimes y) = s_1^{h^{(11)}}s_0^{v^{(10)}}(x)s_0^{h^{(11)}}(y) \in NE_{2,1}$$
for $x \in NE_{1,0}$, $y \in NE_{1,1}$.
5. For $\underline{\alpha} = ((0), (0)), \underline{\beta} = ((1), \emptyset)$, we obtain the following operator
$$C_{((0),(0)),((1),\emptyset)} : NE_{1,0} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
This can be defined as
$$C_{((0),(0)),((1),\emptyset)}(x \otimes y) = s_0^{h^{(11)}}s_0^{v^{(10)}}(x)s_1^{h^{(11)}}(y) \in NE_{2,1}$$
for $x \in NE_{1,0}$, $y \in NE_{1,1}$.

For $(n,m) = (1,2)$, we set
$$S(1) \times S(2) = \{(\emptyset,\emptyset), ((0),(0)), (\emptyset,(0)), (\emptyset,(1)), ((0),(1,0)), (\emptyset,(1,0)), ((0),\emptyset), ((0),(1))\}.$$

In the following calculations, if we take the appropriate pairs $\underline{\alpha}, \underline{\beta}$ from the set $S(1) \times S(2)$, we will give all the non zero maps for $NE_{1,2}$. To get these maps, we can choose the possible $\underline{\alpha}, \underline{\beta}$ from the set $S(1) \times S(2)$ as follows

1. $\underline{\alpha} = (\emptyset, (1))$, $\underline{\beta} = (\emptyset, (0))$
2. $\underline{\alpha} = (\emptyset, (1))$, $\underline{\beta} = ((0), \emptyset)$
3. $\underline{\alpha} = (\emptyset, (0))$, $\underline{\beta} = ((0), \emptyset)$
4. $\underline{\alpha} = ((0), (1))$, $\underline{\beta} = (\emptyset, (0))$
5. $\underline{\alpha} = ((0), (0))$, $\underline{\beta} = (\emptyset, (1))$.

Now we compute the functions $C_{\underline{\alpha},\underline{\beta}}$ for these pairings $(\underline{\alpha}, \underline{\beta})$.

1. For $\underline{\alpha} = (\emptyset, (1))$ and $\underline{\beta} = (\emptyset, (0))$, we obtain the operator
$$C_{(\emptyset,(1)),(\emptyset,(0))} : NE_{1,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}.$$
This operator can be calculated by
$$C_{(\emptyset,(1)),(\emptyset,(0))}(x \otimes y) = s_1^{v^{(11)}}(x)(s_0^{v^{(11)}}(y) - s_1^{v^{(11)}}(y)) \in NE_{1,2}$$
for $x, y \in NE_{1,1}$.

2. For $\underline{\alpha} = (\emptyset, (1)), \underline{\beta} = ((0), \emptyset)$, we obtain the operator
$$C_{(\emptyset,(1)),((0),\emptyset)} : NE_{1,1} \otimes NE_{0,2} \longrightarrow NE_{1,2}$$
given by
$$C_{(\emptyset,(1)),((0),\emptyset)}(x \otimes t) = s_1^{v(11)}(x) s_0^{h(02)}(t) \in NE_{1,2}$$
for $x \in NE_{1,1}$ and $t \in NE_{0,2}$.
3. For $\underline{\alpha} = (\emptyset, (0)), \underline{\beta} = ((0), \emptyset)$, we have the following operator
$$C_{(\emptyset,(0)),((0),\emptyset)} : NE_{1,1} \otimes NE_{0,2} \longrightarrow NE_{1,2}$$
given by
$$C_{(\emptyset,(0)),((0),\emptyset)}(x \otimes t) = s_0^{v(11)}(x) s_0^{h(02)}(t) \in NE_{1,2}$$
for $x \in NE_{1,1}$ and $t \in NE_{0,2}$.
4. For $\underline{\alpha} = ((0), (1)), \underline{\beta} = (\emptyset, (0))$, we get the following operator
$$C_{((0),(1)),(\emptyset,(0))} : NE_{0,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}$$
given by
$$C_{((0),(1)),(\emptyset,(0))}(x \otimes y) = s_0^{h(02)} s_1^{v(01)}(x) s_0^{v(11)}(y) \in NE_{1,2}$$
for $x \in NE_{0,1}$, $y \in NE_{1,1}$.
5. For $\underline{\alpha} = ((0), (0))$ and $\underline{\beta} = (\emptyset, (1))$, we get the following operator
$$C_{((0),(0)),(\emptyset,(1))} : NE_{0,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}$$
given by
$$C_{((0),(0)),(\emptyset,(1))}(x \otimes y) = s_0^{h(02)} s_0^{v(01)}(x) s_1^{v(11)}(y) \in NE_{1,2}$$
for $x \in NE_{0,1}$ and $y \in NE_{1,1}$.

Thus, we can give the following result.

**Proposition 5.1.** *Let $E_{*,*}$ be a bisimplicial algebra with Moore bicomplex $NE_{*,*}$. If for $p \geqslant 2$ or $q \geqslant 2$, $NE_{p,q} = \{0\}$, then, Figure 5.1*

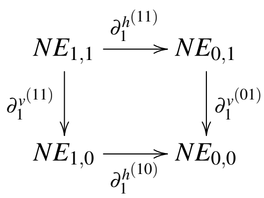


**Figure 5.1:** Crossed square of Moore bicomplex

*is a crossed square together with the h-map*
$$h : NE_{0,1} \times NE_{1,0} \longrightarrow NE_{1,1}$$
*given by*
$$h(x,y) = C_{((0),\emptyset),(\emptyset,(0))}(x \otimes y) = s_0^{h(01)}(x) s_0^{v(10)}(y)$$
*for all $x \in NE_{0,1}$ and $y \in NE_{1,0}$, where $((0), \emptyset), (\emptyset, (0)) \in S(1) \times S(1)$. (This result is the commutative algebra version of Conduché's result given in [3].)*

*Proof.* Our purpose is to see the role of the functions $C_{\underline{\alpha},\underline{\beta}}$ in the structure.
Since $NE_{1,2} = NE_{2,0} = NE_{0,2} = NE_{2,1} = \{0\}$, the maps $\partial_1^{h(10)}, \partial_1^{v(01)}, \partial_1^{h(11)}$ and $\partial_1^{v(11)}$ are crossed modules.
An action of $x \in NE_{1,0}$ on $y \in NE_{0,1}$ is given by $x \cdot y = y \cdot x = s_0^{v(00)} d_1^{h(10)}(x) y$, similarly, the action of $a \in NE_{0,1}$ on $b \in NE_{1,0}$ is given by $a \cdot b = s_0^{h(00)} d_1^{v(01)}(a) b$.

For $y \in N_{0,1}$ and $x \in NE_{1,0}$, we obtain

$$\begin{aligned} \partial_1^{h^{(11)}} h(y,x) &= \partial_1^{h^{(11)}}(s_0^{h^{(01)}}(y)s_0^{v^{(10)}}(x)) \\ &= y\partial_1^{h^{(11)}} s_0^{v^{(10)}}(x) \\ &= ys_0^{v^{(00)}} d_1^{h^{(10)}}(x) \\ &= y \cdot x. \end{aligned}$$

We obtain similarly for $a \in NE_{0,1}$ and $b \in NE_{1,0}$

$$\begin{aligned} \partial_1^{v^{(11)}} h(a,b) &= \partial_1^{v^{(11)}}(s_0^{h^{(01)}}(a)s_0^{v^{(10)}}(b)) \\ &= \partial_1^{v^{(11)}} s_0^{h^{(01)}}(a)bx \\ &= s_0^{h^{(00)}} d_1^{v^{(01)}}(a)b \\ &= a \cdot b. \end{aligned}$$

Now we show that $h(x, \partial_1^{v^{(11)}} c) = x \cdot c$ for $x \in NE_{0,1}$ and $c \in NE_{1,1}$. For $s_0^{h^{(01)}}(x), c \in NE_{1,1}$, we obtain

$$\begin{aligned} d_2^{v^{(12)}} C_{(\emptyset,(1)),(\emptyset,(0))}(s_0^{h^{(01)}}(x) \otimes c) &= d_2^{v^{(12)}}\Big((s_1^{v^{(11)}}(s_0^{h^{(01)}}(x)))s_0^{v^{(11)}}(c) - s_1^{v^{(11)}}(c)\Big) \in d_2^{v^{(12)}}(NE_{1,2}) \\ &= s_0^{h^{(01)}}(x)(s_0^{v^{(10)}} d_1^{v^{(11)}}(c) - c) \\ &= s_0^{h^{(01)}}(x)s_0^{v^{(10)}} d_1^{v^{(11)}}(c) - s_0^{h^{(01)}}(x)c = 0. \quad (\because NE_{1,2} = 0) \end{aligned}$$

Thus, we have for $x \in NE_{0,1}$ and $c \in NE_{1,1}$,

$$\begin{aligned} h(x, \partial_1^{v^{(11)}} c) &= s_0^{h^{(01)}}(x)s_0^{v^{(10)}} d_1^{v^{(11)}}(c) \\ &= s_0^{h^{(01)}}(x)c \quad (\because NE_{1,2} = 0) \\ &= x \cdot c. \end{aligned}$$

For $a \in NE_{1,1}$ and $y \in NE_{1,0}$, we obtain

$$h(\partial_1^{h^{(11)}}(a), y) = s_0^{h^{(01)}} d_1^{h^{(11)}}(a)s_0^{v^{(10)}}(y).$$

For $s_0^{v^{(10)}}(y), a \in NE_{1,1}$, we obtain

$$\begin{aligned} d_2^{h^{(21)}}\Big(C_{((1),\emptyset),((0),\emptyset)}(a \otimes s_0^{v^{(10)}}(y))\Big) &= d_2^{h^{(21)}}\Big((s_1^{h^{(11)}} s_0^{v^{(10)}}(y))(s_0^{h^{(11)}}(a) - s_1^{h^{(11)}}(a))\Big) \in d_2^{h^{(21)}}(NE_{2,1}) \\ &= s_0^{v^{(10)}}(y)(s_0^{h^{(01)}} d_1^{h^{(11)}}(a) - a) \\ &= s_0^{v^{(10)}}(y)s_0^{h^{(01)}} d_1^{h^{(11)}}(a) - s_0^{v^{(10)}}(y)a = 0. \quad (\because NE_{2,1} = 0) \end{aligned}$$

Thus, we have

$$\begin{aligned} h(\partial_1^{h^{(11)}}(a), y) &= s_0^{h^{(01)}} d_1^{h^{(11)}}(a)s_0^{v^{(10)}}(y) \\ &= as_0^{v^{(10)}}(y) \\ &= a \cdot y. \end{aligned}$$

We leave other crossed square axioms to the reader. □

Arvasi, in [14], proved that Loday's mapping cone complex

$$K \xrightarrow{(-\gamma,\gamma')} L \rtimes M \xrightarrow{\mu+\mu'} R$$

of the crossed square for commutative algebras in the Figure 5.2

$$\begin{array}{ccc} K & \xrightarrow{\gamma} & L \\ {\scriptstyle \gamma'}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow[\mu']{} & R \end{array}$$

**Figure 5.2:** Crossed square

gives a 2-crossed module analogously to that given by Conduché in the group case [3].
Thus, we obtain the following result.
Let $E_{*,*}$ be a bisimplicial algebra with Moore bicomplex $NE_{*,*}$. If for $p \geqslant 2$ or $q \geqslant 2$, $NE_{p,q} = \{0\}$, then

$$NE_{1,1} \xrightarrow{(-\partial_1^{h(11)},\partial_1^{v(11)})} NE_{0,1} \times NE_{1,0} \xrightarrow{\partial_1^{v(01)}+\partial_1^{h(10)}} NE_{0,0}$$

is a 2-crossed module together with Peiffer lifting map

$$\{-,-\} : (NE_{0,1} \times NE_{1,0}) \otimes (NE_{0,1} \times NE_{1,0}) \longrightarrow NE_{1,1}$$

given by

$$\{(x,y),(x',y')\} = C_{((0),\emptyset),(\emptyset,(0))}(x \otimes yy') = s_0^{h(01)}(x)s_0^{v(10)}(yy')$$

for all $x,x' \in NE_{0,1}$ and $y,y' \in NE_{1,0}$, where $((0),\emptyset),(\emptyset,(0)) \in S(1) \times S(1)$.

## 6. Conclusion

In this paper, we give the hypercrossed complex pairings for a Moore bicomplex of a bisimplicial algebra and we calculate in dimension 2 explicitly these pairings in the Moore bicomplex to see what the importance of these relations in the structures, for example crossed squares and 2-crossed modules. This idea can be extended to Lie algebra case. Defining these operators for bisimplicial Lie algebras and using these pairings the connection between bisimplicial Lie algebras and crossed squares over Lie algebras can be obtained similarly.

## Article Information

**Acknowledgements:** The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

**Author's Contributions:** The authors contributed equally to the writing of this paper.

**Conflict of Interest Disclosure:** No potential conflict of interest was declared by the author.



**Supporting/Supporting Organizations:** No grants were received from any public, private or non-profit organizations for this research.

**Ethical Approval and Participant Consent:** It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

**Plagiarism Statement:** This article was scanned by the plagiarism program. No plagiarism detected.

**Availability of Data and Materials:** Not applicable.

## References

[1] A.R. Grandjeán, M.J. Vale, *2-Modulos cruzados an la cohomologia de andré-quillen*, Memorias de la Real Academia de Ciencias, **22** (1986), 1–28.
[2] D. Conduché, *Modules croisés cénéralisés de longueur 2*, J. Pure Appl. Algebra, **34** (1984), 155-178.
[3] D. Conduché, *Simplicial crossed modules and mapping cones*, Georgian Math. J., **10**(4) (2003), 623-636.
[4] D. Guin-Waléry, J-L. Loday, *Obsructioná l'excision en K-theories Algébrique*, Evanston conference on Algebraic K-Theory 1980, (Lectute Notes Mathematics) 854, 179-216, Berlin Heidelberg New York, Springer 1981.
[5] G.J. Ellis, *Higher dimensional crossed modules of algebras*, J. Pure Appl. Algebra, **52** (1988), 277-282.
[6] J.H.C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc., **55** (1949), 453-496.
[7] L. Illusie, *Complex Cotangent et Deformations I, II*, Springer Lecture Notes in Mathematics 239 1971 II 283 1972.
[8] M. André, *Homologie des Algèbres Commutatives*, Die Grundlehren der Mathematischen Wissenschaften, 206 New York, Springer-Verlag, 1974.
[9] Ö. Gürmen Alansal, E. Ulualan, *Peiffer pairings in multisimplicial groups and crossed n-cubes and applications for bisimplicial groups*, Turkish J. Math., **45**(1) (2021), 360-386.
[10] P. Carrasco, A.M. Cegarra, *Group-theoretic algebraic models for homotopy types*, J. Pure Appl. Algebra, **75** (1991), 19-235.

[11] T. Porter, *Homology of commutative algebras and an invariant of simis and vasconceles*, J. Algebr., **99** (1986), 458-465.
[12] T. Porter, *n-type of simplicial groups and crossed n-cubes*, Topol., **32** (1993), 5-24.
[13] Z. Arvasi, T. Porter, *Higher dimensional Peiffer elements in simplicial commutative algebras*, Theory Appl. Categ., **3**(1) (1997), 1-23.
[14] Z. Arvasi, *Crossed squares and 2-crossed modules of commutative algebras*, Theory Appl. Categ., **3**(7) (1997), 160-181.