Konuralp Journal of Mathematics, 11 (2) (2023) 206-217

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Riemannian $\Pi-$Structure on $5-$Dimensional Nilpotent Lie Algebras

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Abstract

The object of our investigations is to classify $5-$dimensional nilpotent Lie algebras with two different Riemannian $\Pi-$structures. It is shown that the Lie groups corresponding to the Lie algebras $\mathfrak{g}_i$ equipped with two different Riemannian $\Pi-$structures are not para-Sasaki-like. Moreover, we investigate whether the considered manifolds admit Ricci-like solitons and whether they are Einstein-like manifolds.

Keywords: *Five dimensional nilpotent Lie algebras; Para-Sasaki-like manifold; Ricci-like soliton; Riemannian $\Pi-$manifold.*
2010 Mathematics Subject Classification: *53C05; 53C15; 53C25.*

1. Introduction

The notion of an almost paracontact structure on a smooth odd dimensional manifold was presented in [9, 10]. The geometry of Riemannian manifolds with an almost paracontact structure corresponding to an almost paracomplex structure has been intensively studied in [1, 2, 3, 4, 6]. These manifolds are called briefly Riemannian $\Pi-$manifolds. A classification with eleven basic classes of almost paracontact Riemannian manifolds of type (n,n) according to the covariant derivative of the $(1,1)-$ tensor of the almost paracontact structure was given in [4]. There are 2^{11} classes of Riemannian $\Pi-$structures. The investigations of Riemannian Ricci solitons carried out in [7]. Ricci solitons on manifolds such as Riemannian $\Pi-$ manifolds, Kenmotsu manifolds, paracontact manifolds have been studied in [1, 2, 3, 5, 11].
Non-isomorphic non-abelian nilpotent Lie algebras in five dimensions have six classes [8]. Our aim in this study determine the explicit classes of two different Riemannian $\Pi-$structures defined on $5-$dimensional nilpotent Lie algebras. Then, we calculate Ricci curvature tensor and scalar curvature tensor. Considering the classification obtained, we see that none of them with given structures are para-Sasaki-like. In addition, we show that the only Lie algebra $\mathfrak{g}_1$ is an $\eta-$Einstein manifold and admits Ricci-like soliton.
The present paper is structured as follows. In Section 2, we reminisce some basic facts and properties of Riemannian $\Pi-$manifolds. In Section 3, we classify five dimensional nilpotent Lie algebras with two different Riemannian $\Pi-$structure. Finally, we examine some properties of the considered manifolds.

2. Riemannian $\Pi-$manifolds

A triple (ϕ,ξ,η) on a $(2n+1)-$dimensional smooth manifold M satisfying

$$\phi^2 = Id-\eta\otimes\xi, \qquad \eta(\xi)=1, \tag{2.1}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a Reeb vector field and η is a $1-$form on M, is called an almost paracontact structure on M. In this case, M is called an almost paracontact manifold. In addition, if (M,ϕ,ξ,η) admits a Riemannian metric g with

$$g(\phi x,\phi y)=g(x,y)-\eta(x)\eta(y),$$

for all vector fields x,y, then, (M,ϕ,ξ,η,g) is called Riemannian Π-manifold. These manifolds are sometimes called by different names such as apapR manifolds, almost paracontact almost paracomplex Riemannian manifolds. Moreover, by using above basic identities, the following derived properties are valid:

$$\begin{aligned} &g(x,\xi)=\eta(x), && g(x,\phi y)=g(\phi x,y),\\ &g(\xi,\xi)=1, && \eta(\nabla_x\xi)=0, \end{aligned} \tag{2.2}$$

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where ∇ denotes the Levi-Civita connection of g. The associated metric $\widetilde{g}$ of g on (M,ϕ,ξ,η,g) determined by the equality $\widetilde{g}(x,y)=g(x,\phi y)+\eta(x)\eta(y)$ is a pseudo-Riemannian metric of signature $(n+1,n)$.
A Riemannian Π-manifold M is said to be a para-Sasaki-like manifold if the following is provided:

$$\begin{aligned}(\nabla_x\phi)y &= -g(x,y)\xi-\eta(y)x+2\eta(x)\eta(y)\xi,\\ &= -g(\phi x,\phi y)\xi-\eta(y)\phi^2x.\end{aligned} \tag{2.3}$$

In [6], it is proven that the following identities hold for any para-Sasaki-like manifold (M,g,ϕ,ξ,η):

$$\begin{array}{ll}\nabla_x\xi=\phi x, & (\nabla_x\eta)y=g(x,\phi y),\\ R(x,y)\xi=-\eta(y)x+\eta(x)y, & \mathrm{Ric}(x,\xi)=-2n\,\eta(x),\\ R(\xi,y)\xi=\phi^2y, & \mathrm{Ric}(x,\xi)(\xi,\xi)=-2n,\end{array} \tag{2.4}$$

where R and Ric denote the curvature tensor and the Ricci tensor, respectively.
In [4] the almost paracantact almost paracomplex Riemannian manifolds are classified using the tensor F of type $(0,3)$ defined by

$$F(x,y,z)=g((\nabla_x\phi)y,z),$$

where ∇ is the Levi-Civita connection of g. Moreover, the following relations are satisfied:

$$\begin{aligned}&F(x,y,z)=F(x,z,y)=-F(x,\phi y,\phi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi),\\ &(\nabla_x\eta)y=g(\nabla_x\xi,y)=-F(x,\phi y,\xi).\end{aligned} \tag{2.5}$$

Eleven basis classes of these manifolds are denoted by $\mathscr{F}_1,\dots,\mathscr{F}_{11}$. The class of $\mathscr{F}_0$ is defined by the condition $F=0$, i.e., $\nabla\phi=\nabla\xi=\nabla\eta=\nabla g=0$.
The Lie 1-forms associated with F are defined by

$$\theta(x)=g^{ij}F(e_i,e_j,x),\qquad \theta^*(x)=g^{ij}F(e_i,\phi e_j,x),\qquad \omega(x)=F(\xi,\xi,x), \tag{2.6}$$

where g^{ij}'s are the entries of the inverse matrix of g with respect to the basis $\{e_i,\xi\}$ of T_pM.
Let $\mathbb{F}$ be the set of all tensors over T_pM satisfying the properties (2.5). $\mathbb{F}$ is the direct sum of eleven subspaces $\mathbb{F}_i$, which is orthogonal and invariant with respect to the structure group of considered manifolds. If the tensor F belongs to the subspace $\mathbb{F}_i$, then the manifold is said to be in the class $\mathscr{F}_i$. It is said that M belongs to the class $\mathscr{F}_i$ if and only if the equality $F=F_i$ is valid. F_i are the components of F in the subspace $\mathbb{F}_i$ and are listed below [4].

$$\begin{aligned}F_1(x,y,z)=&\frac{1}{2n}[g(\phi x,\phi y)\theta(\phi^2z)+g(\phi x,\phi z)\theta(\phi^2y)\\ &-g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)],\end{aligned}$$

$$\begin{aligned}F_2(x,y,z)=&\frac{1}{4}[2F(\phi^2x,\phi^2y,\phi^2z)+F(\phi^2y,\phi^2z,\phi^2x)+F(\phi^2z,\phi^2x,\phi^2y)\\ &-F(\phi y,\phi z,\phi^2x)-F(\phi z,\phi y,\phi^2x)]\\ &-\frac{1}{2n}[g(\phi x,\phi y)\theta(\phi^2z)+g(\phi x,\phi z)\theta(\phi^2y)\\ &-g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)],\end{aligned}$$

$$\begin{aligned}F_3(x,y,z)=&\frac{1}{4}[2F(\phi^2x,\phi^2y,\phi^2z)-F(\phi^2y,\phi^2z,\phi^2x)-F(\phi^2z,\phi^2x,\phi^2y)\\ &+F(\phi y,\phi z,\phi^2x)+F(\phi z,\phi y,\phi^2x)],\end{aligned}$$

$$F_4(x,y,z)=\frac{\theta(\xi)}{2n}[g(\phi x,\phi y)\eta(z)+g(\phi x,\phi z)\eta(y)],$$

$$F_5(x,y,z)=\frac{\theta^*(\xi)}{2n}[g(x,\phi y)\eta(z)+g(x,\phi z)\eta(y)],$$

$$\begin{aligned}F_6(x,y,z)=&\frac{1}{4}[[F(\phi^2x,\phi^2y,\xi)+F(\phi^2y,\phi^2x,\xi)+F(\phi x,\phi y,\xi)+F(\phi y,\phi x,\xi)]\eta(z)\\ &+[F(\phi^2x,\phi^2z,\xi)+F(\phi^2z,\phi^2x,\xi)+F(\phi x,\phi z,\xi)+F(\phi z,\phi x,\xi)]\eta(y)]\\ &-\frac{\theta(\xi)}{2n}[g(\phi x,\phi y)\eta(z)+g(\phi x,\phi z)\eta(y)]\\ &-\frac{\theta^*(\xi)}{2n}[g(x,\phi y)\eta(z)+g(x,\phi z)\eta(y)],\end{aligned}$$

$$\begin{aligned}F_7(x,y,z)=&\frac{1}{4}[[F(\phi^2x,\phi^2y,\xi)-F(\phi^2y,\phi^2x,\xi)+F(\phi x,\phi y,\xi)-F(\phi y,\phi x,\xi)]\eta(z)\\ &+[F(\phi^2x,\phi^2z,\xi)-F(\phi^2z,\phi^2x,\xi)+F(\phi x,\phi z,\xi)-F(\phi z,\phi x,\xi)]\eta(y)],\end{aligned}$$

$$\begin{aligned}F_8(x,y,z)=&\frac{1}{4}[[F(\phi^2x,\phi^2y,\xi)+F(\phi^2y,\phi^2x,\xi)-F(\phi x,\phi y,\xi)-F(\phi y,\phi x,\xi)]\eta(z),\\ &+[F(\phi^2x,\phi^2z,\xi)+F(\phi^2z,\phi^2x,\xi)-F(\phi x,\phi z,\xi)-F(\phi z,\phi x,\xi)]\eta(y)],\end{aligned}$$

$$F_9(x,y,z) = \frac{1}{4}[[F(\phi^2x,\phi^2y,\xi) - F(\phi^2y,\phi^2x,\xi) - F(\phi x,\phi y,\xi) + F(\phi y,\phi x,\xi)]\eta(z) + [F(\phi^2x,\phi^2z,\xi) - F(\phi^2z,\phi^2x,\xi) - F(\phi x,\phi z,\xi) + F(\phi z,\phi x,\xi)]\eta(y)],$$

$$F_{10}(x,y,z) = \eta(x)F(\xi,\phi^2y,\phi^2z),$$

$$F_{11}(x,y,z) = \eta(x)[\eta(y)\omega(z) + \eta(z)\omega(y)].$$

A Riemannian $\Pi-$manifold belongs to a direct sum of two or more basic classes if and only if the fundamental tensor is the sum of the corresponding components F_i, F_j, ..., namely, $F = F_i + F_j + \cdots$.
The Nijenhuis torsion of ϕ is defined by

$$[\phi,\phi](x,y) = [\phi x,\phi y] + \phi^2[x,y] - \phi[\phi x,y] - \phi[x,\phi y]. \tag{2.7}$$

Normality condition of Riemannian $\Pi-$structure is equivalent to vanishing the four tensors given by

$$N^{(1)}(x,y) = [\phi,\phi](x,y) - d\eta(x,y)\xi,$$
$$N^{(2)}(x,y) = (\mathfrak{L}_{\phi x}\eta)(y) - (\mathfrak{L}_{\phi y}\eta)(x),$$
$$N^{(3)}(x,y) = (\mathfrak{L}_\xi\phi)(x),$$
$$N^{(4)}(x,y) = (\mathfrak{L}_\xi\eta)(x),$$

where $\mathfrak{L}$ denotes the Lie derivative operator.
Let us recall from [1] that the Riemannian $\Pi-$manifold (M,ϕ,ξ,η,g) is called Einstein-like with constants (a,b,c) if its Ricci tensor Ric satisfies the following formula:

$$\mathrm{Ric} = a\,g + b\,\widetilde{g} + c\;\eta\otimes\eta, \tag{2.8}$$

where a,b,c are constants. In particular, if $b=0$ and $b=c=0$, then the manifold is called an $\eta-$Einstein manifold and an Einstein manifold, respectively. If a,b,c are functions on M, the manifold M is called almost Einstein-like, almost $\eta-$Einstein-like or an almost Einstein manifold, respectively.
A Ricci-like soliton with potential vector field ξ and constants (λ,μ,ν) on a Riemannian $\Pi-$manifold (M,ϕ,ξ,η,g) is defined by

$$\frac{1}{2}\mathscr{L}_\xi g + Ric + \lambda g + \mu\tilde{g} + \nu\eta\otimes\eta = 0, \tag{2.9}$$

where the Lie derivative $\mathscr{L}$ of g along ξ is expressed by

$$\mathscr{L}_\xi g(x,y) = g(\nabla_x\xi,y) + g(x,\nabla_y\xi).$$

An almost paracontact almost paracomplex metric structure (ϕ,ξ,η,g) on a connected Lie group G is said to be left invariant if g is left invariant and the conditions

$$\phi\circ L_a = L_a\circ\phi,\; L_a(\xi) = \xi$$

are satisfied, where L_a is the left multiplication by $a\in G$ in G.
An almost paracontact almost paracomplex metric structure on G induces an almost paracontact almost paracomplex metric structure on the Lie algebra $\mathfrak{g}$ of G having the structure (ϕ,ξ,η,g).
In this study, we specify the classes of some almost paracontact almost paracomplex metric structure $5-$dimensional nilpotent Lie algebras. The non-isomorphic and non-abelian algebras $\mathfrak{g}_i$ are divided into six classes with the corresponding basis $\{e_1,\ldots,e_5\}$ and non-zero brackets in the following [8]:

$$\begin{aligned}
\mathfrak{g}_1 &: [e_1,e_2]=e_5,\ [e_3,e_4]=e_5,\\
\mathfrak{g}_2 &: [e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_2,e_4]=e_5,\\
\mathfrak{g}_3 &: [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\ [e_2,e_3]=e_5,\\
\mathfrak{g}_4 &: [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_1,e_4]=e_5,\\
\mathfrak{g}_5 &: [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\\
\mathfrak{g}_6 &: [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5.
\end{aligned} \tag{2.10}$$

3. A Riemannian $\Pi-$structures on $5-$dimensional Nilpotent Lie Algebras

Let (ϕ,ξ,η,g) be a left invariant Riemannian $\Pi-$structure on a connected Lie group G_i with corresponding Lie algebra $\mathfrak{g}_i$. We use the same notation for the corresponding Riemannian $\Pi-$structure. Now, we investigate the classes of the following Riemannian $\Pi-$structure with respect to the basis $\{e_1,\ldots,e_5\}$ on each $\mathfrak{g}_i$.

$$\begin{aligned}
&\phi(e_1)=e_3,\ \phi(e_2)=e_4,\ \phi(e_3)=e_1,\ \phi(e_4)=e_2,\ \phi(e_5)=0,\\
&\xi=e_5,\ \eta=e^5,\\
&g(e_i,e_i)=1,\ \ g(e_i,e_j)=0,\ \ i,j\in\{1,\ldots,5\},\ \ i\neq j.
\end{aligned} \tag{3.1}$$

3.1. The Lie algebra $\mathfrak{g}_1$

Theorem 3.1. *The Lie algebra $\mathfrak{g}_1$ belongs to the class $\mathscr{F}_7$ according to the structure given in (3.1).*

Proof. By using the non-zero brackets $[e_1,e_2]=e_5$, $[e_3,e_4]=e_5$ and Kozsul's formula, the covariant derivatives of the non-zero basic elements are given by

$$\nabla_{e_1}e_2=\frac{1}{2}e_5,\ \nabla_{e_1}e_5=-\frac{1}{2}e_2,\ \nabla_{e_2}e_1=-\frac{1}{2}e_5,\ \nabla_{e_2}e_5=\frac{1}{2}e_1,$$

$$\nabla_{e_3}e_4=\frac{1}{2}e_5,\ \nabla_{e_3}e_5=-\frac{1}{2}e_4,\ \nabla_{e_4}e_3=-\frac{1}{2}e_5,\ \nabla_{e_4}e_5=\frac{1}{2}e_3,$$

$$\nabla_{e_5}e_1=-\frac{1}{2}e_2,\ \nabla_{e_5}e_2=\frac{1}{2}e_1,\ \nabla_{e_5}e_3=-\frac{1}{2}e_4,\ \nabla_{e_5}e_4=\frac{1}{2}e_3.$$

Theorem 3.2. *[4] Let (M,ϕ,ξ,η,g) be a Riemannian $\Pi-$manifold. Then, we have*

a. $[\phi,\phi](x,y)=0$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_i(i=1,2,4,5,6,11)$ *or to their direct sums;*
b. $[\phi,\phi](x,y)=-2\{\phi(\nabla_{\phi x}\phi)\phi y+\phi(\nabla_{\phi^2x}\phi)\phi^2y\}$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_3$*;*
c. $[\phi,\phi](x,y)=-2(\nabla_x\eta)(y)\xi$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_7$*;*
d. $[\phi,\phi](x,y)=-2\{\eta(x)\nabla_y\xi-\eta(y)\nabla_x\xi-(\nabla_x\eta)(y)\xi\}$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_8$*;*
e. $[\phi,\phi](x,y)=-2\{\eta(x)\nabla_y\xi-\eta(y)\nabla_x\xi\}$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_9$*;*
f. $[\phi,\phi](x,y)=-\eta(x)\phi(\nabla_\xi\phi)y+\eta(y)\phi(\nabla_\xi\phi)x$ *if and only if* (M,ϕ,ξ,η,g) *belongs to* $\mathscr{F}_{10}$.

Setting $x=y=e_i$, $(i=1,2,...,5)$ in (2.7), we get

$$[\phi e_i,\phi e_i]+\phi^2[e_i,e_i]-\phi[\phi e_i,e_i]-\phi[e_i,\phi e_i]=0.$$

Moreover, we can calculate

$$(\nabla_{e_i}\eta)e_i=e_i(\eta e_i)-\eta(\nabla_{e_i}e_i)=0,$$

for every $i=1,2,\ldots,5$. In case of $i=1,\ j=2$, we obtain $[\phi,\phi](e_1,e_2)=e_5$ and $(\nabla_{e_1}\eta)e_2=-\frac{1}{2}$. Similarly, in case of $i=3,\ j=4$, we get $[\phi,\phi](e_3,e_4)=e_5$ and $(\nabla_{e_3}\eta)e_4=-\frac{1}{2}$. In other cases, we calculate $[\phi,\phi](e_i,e_j)=0$ and $(\nabla_{e_i}\eta)(e_j)=0$ for $i\neq j$. Therefore, the equality given in Theorem 3.2(c) is satisfied for the orthonormal basis $\{e_1,\ldots,e_5=\xi\}$. Hence, we conclude that $\mathfrak{g}_1$ belongs to the class $\mathscr{F}_7$. □

Now, we consider another structure (ϕ,ξ,η,g) given by

$$\begin{aligned}&\phi(e_3)=e_5,\ \phi(e_2)=e_4,\ \phi(e_5)=e_3,\ \phi(e_4)=e_2,\ \phi(e_1)=0,\\&\xi=e_1,\ \eta=e^1,\\&g(e_i,e_i)=1,\ \ g(e_i,e_j)=0,\ \ i,j\in\{1,\ldots,5\},\ \ i\neq j.\end{aligned}\tag{3.2}$$

By using above structure we compute the following non-zero components $F(e_i,e_j,e_k)=F_{ijk}$ of the structure tensor F:

$$\begin{aligned}&F_{145}=F_{154}=F_{213}=F_{231}=F_{325}=\frac{1}{2},\\&F_{352}=F_{514}=F_{523}=F_{532}=F_{541}=\frac{1}{2},\\&F_{123}=F_{132}=F_{334}=F_{343}=F_{545}=F_{554}=-\frac{1}{2},\\&F_{433}=-F_{455}=1.\end{aligned}$$

Then, we construct the following form of F for any vectors x,y,z:

$$\begin{aligned}F(x,y,z)&=F\left(\sum_i x_ie_i,\ \sum_j y_je_j,\ \sum_k z_ke_k\right)\\&=\sum_{i,j,k}x_iy_jz_kF(e_i,e_j,e_k)\\&=-\frac{1}{2}x_1y_2z_3-\frac{1}{2}x_1y_3z_2+\frac{1}{2}x_1y_4z_5+\frac{1}{2}x_1y_5z_4+\frac{1}{2}x_2y_1z_3+\frac{1}{2}x_2y_3z_1\\&\quad+\frac{1}{2}x_3y_2z_5-\frac{1}{2}x_3y_3z_4-\frac{1}{2}x_3y_4z_3+\frac{1}{2}x_3y_5z_2+x_4y_3z_3-x_4y_5z_5\\&\quad+\frac{1}{2}x_5y_1z_4+\frac{1}{2}x_5y_2z_3+\frac{1}{2}x_5y_3z_2+\frac{1}{2}x_5y_4z_1-\frac{1}{2}x_5y_4z_5-\frac{1}{2}x_5y_5z_4.\end{aligned}$$

The latter equality implies that F is represented in the form

$$F(x,y,z)=F_1(x,y,z)+F_2(x,y,z)+F_3(x,y,z)+F_6(x,y,z)+F_9(x,y,z)+F_{10}(x,y,z),$$

where

$$F_1(x,y,z) = \frac{1}{4}(-x_1y_1z_4 - x_1y_4z_1 + x_3y_2z_5 - x_3y_3z_4 - x_3y_4z_3 + x_3y_5z_2 + 2x_4y_2z_2 + x_5y_2z_3 + x_5y_3z_2 - x_5y_4z_5 - x_5y_5z_4),$$

$$F_2(x,y,z) = \frac{1}{4}(2x_3y_5z_2 + x_4y_3z_3 - 3x_4y_5z_5 + x_5y_2z_3 - x_5y_4z_5 - 2x_5y_5z_4 + x_2y_3z_5 + x_1y_1z_4 + x_1y_4z_1 + x_3y_4z_3 - 2x_4y_4z_2),$$

$$F_3(x,y,z) = \frac{1}{4}(2x_3y_2z_5 - x_3y_5z_2 - x_3y_3z_4 - 2x_3y_4z_3 + 3x_4y_3z_3 - x_4y_5z_5 + x_5y_3z_2 + x_5y_5z_4 - x_2y_3z_5),$$

$$F_6(x,y,z) = \frac{1}{4}(x_2y_3z_1 + x_5y_4z_1 + x_3y_2z_1 + x_4y_5z_1 + x_2y_1z_3 + x_5y_1z_4 + x_3y_1z_2 + x_4y_1z_5),$$

$$F_9(x,y,z) = \frac{1}{4}(x_2y_3z_1 + x_5y_4z_1 - x_3y_2z_1 - x_4y_5z_1 + x_2y_1z_3 + x_5y_1z_4 - x_3y_1z_2 - x_4y_1z_5),$$

$$F_{10}(x,y,z) = -\frac{1}{2}x_1y_2z_3 - \frac{1}{2}x_1y_3z_2 + \frac{1}{2}x_1y_4z_5 + \frac{1}{2}x_1y_5z_4.$$

Therefore, $\mathfrak{g}_1$ with the structure (3.2) is in the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_6 \oplus \mathscr{F}_9 \oplus \mathscr{F}_{10}$.
The Ricci tensor Ric and the scalar curvature scal according to the basis $\{e_1,\dots,e_4,e_5=\xi\}$ are presented by

$$\mathrm{Ric}(x,y) = \sum_{i=1}^{5} g(R(e_i,x)y,e_i) \text{ and } \mathrm{scal} = \sum_{i=1}^{5} \mathrm{Ric}(e_i,e_i), \tag{3.3}$$

respectively. The non-zero components of Ricci tensor Ric corresponding to the Lie algebra $\mathfrak{g}_1$ are calculated according to the basis $\{e_1,\dots,e_4,e_5=\xi\}$ as follows:

$$Ric_{11} = -\frac{1}{2},\ Ric_{22} = -\frac{1}{2},$$
$$Ric_{33} = -\frac{1}{2},\ Ric_{44} = -\frac{1}{2},$$
$$Ric_{55} = 1,$$

where $Ric_{ij} = Ric(e_i,e_j)$ for $i,j \in \{1,2,...,5\}$. The scalar curvature scal of $\mathfrak{g}_1$ is evaluated by $scal = -1$. (G_1,ϕ,ξ,η,g) is a $\eta-$Einstein manifold with constants $(a,b,c) = (-\frac{1}{2},0,\frac{3}{2})$.
The nonzero components of $\mathscr{L}_\xi g$ for the structure (3.2) are the following:

$(\mathscr{L}_\xi g)_{25} = (\mathscr{L}_\xi g)_{52} = -1.$

(G_1,ϕ,ξ,η,g) is neither Einstein-like nor Ricci-like soliton for the structure (3.2).

3.2. The Lie algebra $\mathfrak{g}_2$

Theorem 3.3. *The Lie algebra $\mathfrak{g}_2$ belongs to the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$ with regard to the structure given in (3.1).*

Proof. By using the relations $[e_1,e_2] = e_3, [e_1,e_3] = e_5, [e_2,e_4] = e_5$ and Kozsul's formula we get

$$\nabla_{e_1}e_2 = \frac{1}{2}e_3,\ \nabla_{e_1}e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_5,\ \nabla_{e_1}e_5 = -\frac{1}{2}e_3,\ \nabla_{e_2}e_1 = -\frac{1}{2}e_3,$$

$$\nabla_{e_2}e_3 = \frac{1}{2}e_1,\ \nabla_{e_2}e_4 = \frac{1}{2}e_5,\ \nabla_{e_2}e_5 = -\frac{1}{2}e_4,\ \nabla_{e_3}e_1 = -\frac{1}{2}e_2 - \frac{1}{2}e_5,$$

$$\nabla_{e_3}e_2 = \frac{1}{2}e_1,\ \nabla_{e_3}e_5 = \frac{1}{2}e_1,\ \nabla_{e_4}e_2 = -\frac{1}{2}e_5,\ \nabla_{e_4}e_5 = \frac{1}{2}e_2,$$

$$\nabla_{e_5}e_1 = -\frac{1}{2}e_3,\ \nabla_{e_5}e_2 = -\frac{1}{2}e_4,\ \nabla_{e_5}e_3 = \frac{1}{2}e_1,\ \nabla_{e_5}e_4 = \frac{1}{2}e_2.$$

We evaluate the projections and determine the class of the structure. The nonzero structure constants F_{ijk} are given in the following:

$$F_{115} = F_{134} = F_{143} = F_{151} = \frac{1}{2},$$
$$F_{225} = F_{252} = F_{314} = F_{341} = \frac{1}{2},$$
$$F_{112} = F_{121} = F_{323} = F_{332} = -\frac{1}{2},$$
$$F_{335} = F_{353} = F_{445} = F_{454} = -\frac{1}{2},$$
$$F_{211} = F_{511} = F_{522} = 1,$$
$$F_{233} = F_{533} = F_{544} = -1.$$

For any x,y,z, by using above relations, the tensor F can be calculated in the following way:

$$\begin{aligned}
F(x,y,z) &= F\left(\sum_i x_i e_i, \sum_j y_j e_j, \sum_k z_k e_k\right) \\
&= \sum_{i,j,k} x_i y_j z_k F\left(e_i, e_j, e_k\right) \\
&= -\frac{1}{2}x_1y_1z_2 - \frac{1}{2}x_1y_2z_1 + \frac{1}{2}x_1y_3z_4 + \frac{1}{2}x_1y_4z_3 + \frac{1}{2}x_3y_4z_1 + \frac{1}{2}x_3y_1z_4 \\
&\quad + \frac{1}{2}x_1y_1z_5 + \frac{1}{2}x_2y_2z_5 + x_2y_1z_1 - x_2y_3z_3 - \frac{1}{2}x_3y_3z_5 - \frac{1}{2}x_4y_4z_5 \\
&\quad + \frac{1}{2}x_1y_5z_1 + \frac{1}{2}x_2y_5z_2 - \frac{1}{2}x_3y_2z_3 - \frac{1}{2}x_3y_5z_3 - \frac{1}{2}x_4y_5z_4 - \frac{1}{2}x_3y_3z_2 \\
&\quad + x_5y_1z_1 + x_5y_2z_2 - x_5y_3z_3 - x_5y_4z_4.
\end{aligned}$$

Since

$$\begin{aligned}
F_1(x,y,z) &= \frac{1}{4}(-x_1y_1z_2 - 2x_2y_2z_2 - x_3y_3z_2 - x_5y_5z_2 - x_1y_2z_1 - x_3y_2z_3 \\
&\quad - x_5y_2z_5 + x_1y_4z_3 + 2x_2y_4z_4 + x_3y_4z_1 + x_1y_3z_4 + x_3y_1z_4), \\
F_2(x,y,z) &= \frac{1}{4}(2x_2y_1z_1 + x_2y_2z_2 - 2x_2y_3z_3 - 2x_2y_4z_4 + 2x_3y_1z_4 \\
&\quad + 2x_3y_4z_1 - 2x_3y_2z_3 - 2x_3y_3z_2 + x_5y_2z_5 + x_5y_5z_2), \\
F_3(x,y,z) &= \frac{1}{4}(-x_1y_1z_2 - x_1y_2z_1 + x_1y_3z_4 + x_1y_4z_3 + 2x_2y_1z_1 \\
&\quad - 2x_2y_3z_3 - x_3y_1z_4 + x_3y_2z_3 + x_3y_3z_2 - x_3y_4z_1), \\
F_8(x,y,z) &= \frac{1}{2}x_1y_1z_5 + \frac{1}{2}x_2y_2z_5 - \frac{1}{2}x_3y_3z_5 - \frac{1}{2}x_4y_4z_5 + \frac{1}{2}x_1y_5z_1 \\
&\quad + \frac{1}{2}x_2y_5z_2 - \frac{1}{2}x_3y_5z_3 - \frac{1}{2}x_4y_5z_4, \\
F_{10}(x,y,z) &= x_5y_1z_1 + x_5y_2z_2 - x_5y_3z_3 - x_5y_4z_4,
\end{aligned}$$

the tensor F can be written as $F = F_1 + F_2 + F_3 + F_8 + F_{10}$. The only nonzero projections are $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_8, \mathscr{F}_{10}$. Therefore, the Lie algebra $\mathfrak{g}_2$ is in the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$. □

For the structure (3.2), the non-zero components F_{ijk} can be found as

$$\begin{aligned}
&F_{134} = F_{215} = F_{225} = F_{251} = F_{252} = F_{313} = F_{314} = \frac{1}{2}, \\
&F_{331} = F_{341} = F_{423} = F_{432} = F_{143} = F_{515} = \frac{1}{2}, \\
&F_{125} = F_{152} = F_{234} = F_{243} = F_{445} = F_{454} = -\frac{1}{2}, \\
&F_{155} = F_{522} = -F_{133} = -F_{544} = 1.
\end{aligned}$$

The only nonzero projections of the tensor F are calculated by

$$\begin{aligned}
F_2(x,y,z) &= \frac{1}{2}x_2y_2z_5 - \frac{1}{2}x_2y_3z_4 - \frac{1}{2}x_2y_4z_3 + \frac{1}{2}x_2y_5z_2 + \frac{1}{2}x_4y_2z_3 \\
&\quad + \frac{1}{2}x_4y_3z_2 - \frac{1}{2}x_4y_4z_5 - \frac{1}{2}x_4y_5z_4 + x_5y_2z_2 - x_5y_4z_5, \\
F_4(x,y,z) &= \frac{1}{4}(x_2y_2z_1 + x_3y_3z_1 + x_4y_4z_1 + x_5y_5z_1 + x_2y_1z_2 + x_3y_1z_3 + x_4y_1z_4 + x_5y_1z_5), \\
F_6(x,y,z) &= \frac{1}{4}(x_5y_5z_1 + x_2y_5z_1 + x_3y_3z_1 + x_3y_4z_1 + x_5y_2z_1 + x_4y_3z_1 + x_5y_1z_5 + x_2y_1z_5 \\
&\quad + x_3y_1z_3 + x_3y_1z_4 - x_5y_1z_2 + x_4y_1z_3 - x_2y_2z_1 - x_4y_4z_1 - x_2y_1z_2 - x_4y_1z_4),
\end{aligned}$$

$$F_9(x,y,z) = \frac{1}{4}(x_2y_5z_1 + x_3y_4z_1 - x_5y_2z_1 - x_4y_3z_1 + x_2y_1z_5 + x_3y_1z_4 - x_5y_1z_2 - x_4y_1z_3),$$

$$F_{10}(x,y,z) = -\frac{1}{2}x_1y_2z_5 - x_1y_3z_3 + \frac{1}{2}x_1y_3z_4. + \frac{1}{2}x_1y_4z_3 - \frac{1}{2}x_1y_5z_2 + x_1y_5z_5.$$

Hence, in similiar way, it can be easily seen that the structure (3.2) on $\mathfrak{g}_2$ is of type $\mathscr{F}_2 \oplus \mathscr{F}_4 \oplus \mathscr{F}_6 \oplus \mathscr{F}_9 \oplus \mathscr{F}_{10}$.
The nonzero components $Ric_{ij} = Ric(e_i, e_j)$ of Ricci curvature tensor are given by

$$Ric_{11} = -1, \ \ Ric_{22} = -1,$$
$$Ric_{33} = 0, \ \ Ric_{44} = -\frac{1}{2},$$
$$Ric_{55} = 1.$$

With the aid of the above relations, we can compute the scalar curvature scal as follows:

$$\begin{aligned} \text{scal} &= \sum_{i=1}^{5} Ric_{ii} \\ &= Ric_{11} + Ric_{22} + Ric_{33} + Ric_{44} + Ric_{55} \\ &= -1 - 1 + 0 - \frac{1}{2} + 1 \\ &= -\frac{3}{2} \end{aligned}$$

By direct computation it can be easily shown that the Lie algebra $\mathfrak{g}_2$ is not Einstein - like manifold. Moreover, the nonzero components of the Lie derivative $\mathscr{L}_\xi g$ for the structure (3.2) are as follows:

$$(\mathscr{L}_\xi g)_{23} = (\mathscr{L}_\xi g)_{35} = (\mathscr{L}_\xi g)_{32} = (\mathscr{L}_\xi g)_{53} = -1.$$

Hence, $\mathfrak{g}_2$ is not Ricci-like soliton for both structures.

3.3. The Lie algebra $\mathfrak{g}_3$

Theorem 3.4. *The Lie algebra $\mathfrak{g}_3$ belongs to the class $\mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$ with respect to the structure given in (3.1).*

Proof. With the aid of the relations given in $\mathfrak{g}_3$, the basic components of the Levi-Civita connection ∇ can be found as

$$\nabla_{e_1}e_2 = \frac{1}{2}e_3, \ \nabla_{e_1}e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_4, \ \nabla_{e_1}e_4 = -\frac{1}{2}e_3 + \frac{1}{2}e_5, \ \nabla_{e_1}e_5 = -\frac{1}{2}e_4,$$

$$\nabla_{e_2}e_1 = -\frac{1}{2}e_3, \ \nabla_{e_2}e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_5, \ \nabla_{e_2}e_5 = -\frac{1}{2}e_3, \ \nabla_{e_3}e_1 = -\frac{1}{2}e_2 - \frac{1}{2}e_4,$$

$$\nabla_{e_3}e_2 = \frac{1}{2}e_1 - \frac{1}{2}e_5, \ \nabla_{e_3}e_4 = \frac{1}{2}e_1, \ \nabla_{e_3}e_5 = \frac{1}{2}e_2, \ \nabla_{e_4}e_1 = -\frac{1}{2}e_3 - \frac{1}{2}e_5, \ \nabla_{e_4}e_3 = \frac{1}{2}e_1,$$

$$\nabla_{e_4}e_5 = \frac{1}{2}e_1, \ \nabla_{e_5}e_1 = -\frac{1}{2}e_4, \ \nabla_{e_5}e_2 = -\frac{1}{2}e_3, \ \nabla_{e_5}e_3 = \frac{1}{2}e_2, \ \nabla_{e_5}e_4 = \frac{1}{2}e_1.$$

By direct computation we get nonzero basic components F_{ijk} of the tensor F as follows:

$$F_{114} = F_{125} = F_{134} = F_{141} = F_{143} = F_{251} = \frac{1}{2},$$
$$F_{152} = F_{215} = F_{312} = F_{314} = F_{321} = F_{341} = \frac{1}{2},$$
$$F_{112} = F_{121} = F_{123} = F_{132} = F_{323} = F_{332} = -\frac{1}{2},$$
$$F_{334} = F_{343} = F_{345} = F_{354} = F_{435} = F_{453} = -\frac{1}{2},$$
$$F_{211} = F_{411} = F_{512} = F_{521} = 1,$$
$$F_{233} = F_{433} = F_{534} = F_{543} = -1.$$

The nonzero projections F_i are in the following:

$$\begin{aligned} F_2(x,y,z) = \frac{1}{4}(&-x_1y_1z_2 + 2x_1y_1z_4 - x_1y_2z_1 - 2x_1y_2z_3 - 2x_1y_3z_2 \\ &+ 2x_1y_4z_1 + x_1y_4z_3 + 2x_2y_1z_1 - 2x_2y_3z_3 + 2x_3y_1z_2 + 3x_3y_1z_4 + 2x_3y_2z_1 \\ &- 3x_3y_2z_3 - 3x_3y_3z_2 - 2x_3y_3z_4 + 3x_3y_4z_1 - 2x_3y_4z_3 + 4x_4y_1z_1 - 4x_4y_3z_3), \end{aligned}$$

$$F_3(x,y,z) = \frac{1}{4}(-x_1y_1z_2 - x_1y_2z_1 + 2x_1y_3z_4 + x_1y_4z_3 + 2x_2y_1z_1$$
$$- 2x_2y_3z_3 - x_3y_1z_4 + x_3y_2z_3 - x_3y_1z_4 + x_3y_2z_3 + x_3y_3z_2 - x_3y_4z_1),$$
$$F_8(x,y,z) = \frac{1}{2}x_1y_2z_5 + \frac{1}{2}x_1y_5z_2 + \frac{1}{2}x_2y_1z_5 + \frac{1}{2}x_2y_5z_1 - \frac{1}{2}x_3y_4z_5$$
$$- \frac{1}{2}x_3y_5z_4 - \frac{1}{2}x_4y_3z_5 - \frac{1}{2}x_4y_5z_3,$$
$$F_{10}(x,y,z) = x_5y_1z_2 + x_5y_2z_1 - x_5y_3z_4 - x_5y_4z_3.$$

Then, the tensor F can be written as

$$F(x,y,z) = F_2(x,y,z) + F_3(x,y,z) + F_8(x,y,z) + F_{10}(x,y,z).$$

The class of $\mathfrak{g}_3$ according to the structure (3.1) is $\mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$. □

By the structure (3.2), the nonzero basic components F_{ijk} are calculated by

$$F_{145} = F_{154} = F_{215} = F_{251} = F_{312} = F_{314} = \frac{1}{2},$$
$$F_{321} = F_{323} = F_{332} = F_{341} = F_{413} = F_{415} = \frac{1}{2},$$
$$F_{431} = F_{451} = F_{512} = F_{521} = F_{525} = F_{552} = \frac{1}{2},$$
$$F_{123} = F_{132} = F_{345} = F_{354} = F_{534} = F_{543} = -\frac{1}{2},$$
$$F_{255} = -F_{233} = 1.$$

By using above basic components of the tensor F, we obtain the following projections:

$$F_1(x,y,z) = \frac{1}{4}(2x_2y_2z_2 + x_3y_3z_2 + x_5y_5z_2 + x_3y_2z_3 + x_5y_2z_5$$
$$- 2x_2y_4z_4 - x_3y_5z_4 - x_5y_3z_4 - x_5y_4z_3 - x_3y_4z_5),$$
$$F_2(x,y,z) = \frac{1}{2}(-x_2y_3z_3 + x_2y_5z_5 - x_5y_3z_4 - x_5y_4z_3 + x_5y_2z_5 + x_5y_5z_2 - x_2y_2z_2 - x_2y_4z_4)$$
$$F_3(x,y,z) = \frac{1}{4}(-2x_2y_3z_3 + 2x_2y_5z_5 + x_3y_2z_3 + x_3y_3z_2 - x_3y_4z_5$$
$$- x_3y_5z_4 + x_5y_3z_4 + x_5y_4z_3 - x_5y_2z_5 - x_5y_5z_2)$$
$$F_6(x,y,z) = \frac{1}{4}(2x_2y_5z_1 + x_3y_2z_1 + 2x_3y_4z_1 + 2x_4y_3z_1 + x_4y_5z_1 + 2x_5y_2z_1 + x_2y_3z_1 + x_5y_4z_1$$
$$+ 2x_2y_1z_5 + x_3y_1z_2 + 2x_3y_1z_4 + 2x_4y_1z_3 + x_4y_1z_5 + 2x_5y_1z_2 + x_2y_1z_3 + x_5y_1z_4),$$
$$F_9(x,y,z) = \frac{1}{4}(x_3y_2z_1 + x_4y_5z_1 - x_2y_3z_1 - x_5y_4z_1 + x_3y_1z_2 + x_4y_1z_5 - x_2y_1z_3 - x_5y_1z_4),$$
$$F_{10}(x,y,z) = -\frac{1}{2}x_1y_2z_3 - \frac{1}{2}x_1y_3z_2 + \frac{1}{2}x_1y_4z_5 + \frac{1}{2}x_1y_5z_4.$$

Namely, $\mathfrak{g}_3$ belongs to $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_6 \oplus \mathscr{F}_9 \oplus \mathscr{F}_{10}$.
The nonzero components of Ricci curvature tensor for $\mathfrak{g}_3$ are given below.

$$Ric_{11} = -\frac{3}{2}, \quad Ric_{22} = -1,$$
$$Ric_{33} = -\frac{1}{2}, \quad Ric_{44} = 0,$$
$$Ric_{55} = 1.$$

Using above equations, we compute $scal = -2$. The nonzero components of $\mathscr{L}_\xi g$ for the structure (3.2) are the following:

$$(\mathscr{L}_\xi g)_{23} = (\mathscr{L}_\xi g)_{32} = (\mathscr{L}_\xi g)_{34} = (\mathscr{L}_\xi g)_{43} = (\mathscr{L}_\xi g)_{45} = (\mathscr{L}_\xi g)_{54} = -1.$$

By direct calculation, it is easily checked that $\mathfrak{g}_3$ is not Einstein - like and Ricci-like soliton for given two structure.

3.4. The Lie algebra $\mathfrak{g}_4$

Theorem 3.5. *The Lie algebras $\mathfrak{g}_4$ belongs to the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_7 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$ according to the structure given in (3.1).*

Proof. Similarly, by using the relations $[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5$ and Kozsul's formula, the basic components of ∇ are calculated by

$$\nabla_{e_1}e_2=\frac{1}{2}e_3,\ \nabla_{e_1}e_3=-\frac{1}{2}e_2+\frac{1}{2}e_4,\ \nabla_{e_1}e_4=-\frac{1}{2}e_3+\frac{1}{2}e_5,\ \nabla_{e_1}e_5=-\frac{1}{2}e_4,$$

$$\nabla_{e_2}e_1=-\frac{1}{2}e_3,\ \nabla_{e_2}e_3=\frac{1}{2}e_1,\ \nabla_{e_3}e_1=-\frac{1}{2}e_2-\frac{1}{2}e_4,$$

$$\nabla_{e_3}e_2=\frac{1}{2}e_1,\ \nabla_{e_3}e_4=\frac{1}{2}e_1,\ \nabla_{e_4}e_1=-\frac{1}{2}e_3-\frac{1}{2}e_5,\ \nabla_{e_4}e_3=\frac{1}{2}e_1,$$

$$\nabla_{e_4}e_5=\frac{1}{2}e_1,\ \nabla_{e_5}e_1=-\frac{1}{2}e_4,\ \nabla_{e_5}e_4=\frac{1}{2}e_1.$$

The basic components F_{ijk} are given by

$$\begin{aligned}
&F_{114}=F_{125}=F_{134}=F_{141}=F_{143}=F_{512}=\frac{1}{2},\\
&F_{152}=F_{521}=F_{312}=F_{314}=F_{321}=F_{341}=\frac{1}{2},\\
&F_{112}=F_{121}=F_{123}=F_{132}=F_{323}=F_{332}=-\frac{1}{2},\\
&F_{334}=F_{343}=F_{435}=F_{453}=F_{534}=F_{543}=-\frac{1}{2},\\
&F_{211}=F_{411}=-F_{233}=-F_{433}=1.
\end{aligned}$$

Since the projections F_i are too long, they are not written explicitly. It can be seen that the class of $\mathfrak{g}_4$ is in $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_7 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$. □

Using the structure given in (3.2), the nonzero structure constants F_{ijk} are given below.

$$\begin{aligned}
&F_{145}=F_{154}=F_{215}=F_{251}=F_{312}=F_{314}=\frac{1}{2},\\
&F_{321}=F_{341}=F_{413}=F_{415}=\frac{1}{2},\\
&F_{431}=F_{451}=F_{512}=F_{521}=\frac{1}{2},\\
&F_{123}=F_{132}=-\frac{1}{2}.
\end{aligned}$$

Using above relations, we have

$$\begin{aligned}
F_6(x,y,z)=\frac{1}{4}(&2x_2y_5z_1+x_3y_2z_1+2x_3y_4z_1+2x_4y_3z_1+x_4y_5z_1+2x_5y_2z_1+x_2y_3z_1+x_5y_4z_1\\
&+2x_2y_1z_5+x_3y_1z_2+2x_3y_1z_4+2x_4y_1z_3+x_4y_1z_5+2x_5y_1z_2+x_2y_1z_3+x_5y_1z_4),
\end{aligned}$$

$$F_9(x,y,z)=\frac{1}{4}(x_3y_2z_1+x_4y_5z_1-x_2y_3z_1-x_5y_4z_1+x_3y_1z_2+x_4y_1z_5-x_2y_1z_3-x_5y_1z_4),$$

$$F_{10}(x,y,z)=-\frac{1}{2}x_1y_2z_3-\frac{1}{2}x_1y_3z_2+\frac{1}{2}x_1y_4z_5+\frac{1}{2}x_1y_5z_4.$$

Therefore, we acquire that $\mathfrak{g}_4$ is in the class $\mathscr{F}_6 \oplus \mathscr{F}_9 \oplus \mathscr{F}_{10}$.
The nonzero components Ric_{ij} of Ricci curvature tensor are determined by the following equations:

$$\begin{aligned}
&Ric_{11}=-\frac{3}{2},\ \ Ric_{22}=-\frac{1}{2},\\
&Ric_{55}=\frac{1}{2}.
\end{aligned}\tag{3.4}$$

Taking into account (3.4), we obtain $scal=-\frac{3}{2}$. The nonzero components of $\mathscr{L}_\xi g$ for the structure (3.2) are the following:

$$(\mathscr{L}_\xi g)_{23}=(\mathscr{L}_\xi g)_{32}=(\mathscr{L}_\xi g)_{34}=(\mathscr{L}_\xi g)_{43}=(\mathscr{L}_\xi g)_{45}=(\mathscr{L}_\xi g)_{54}=-1.$$

It can be easily checked that $\mathfrak{g}_4$ is neither $\eta-$Einstein-like nor Ricci-like soliton for given two structures.

3.5. The Lie algebra $\mathfrak{g}_5$

Theorem 3.6. *The class of the Lie algebra $\mathfrak{g}_5$ is $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$ considering the structure given in (3.1).*

Proof. The basic terms of ∇ are computed as follows:

$$\nabla_{e_1}e_2 = \frac{1}{2}e_4,\ \nabla_{e_1}e_3 = \frac{1}{2}e_5,\ \nabla_{e_1}e_4 = -\frac{1}{2}e_2,\ \nabla_{e_1}e_5 = -\frac{1}{2}e_3,\ \nabla_{e_2}e_1 = -\frac{1}{2}e_4,$$

$$\nabla_{e_2}e_4 = \frac{1}{2}e_1,\ \nabla_{e_3}e_1 = -\frac{1}{2}e_5,\ \nabla_{e_3}e_5 = \frac{1}{2}e_1,$$

$$\nabla_{e_4}e_1 = -\frac{1}{2}e_2,\ \nabla_{e_4}e_2 = \frac{1}{2}e_1,\ \nabla_{e_5}e_1 = -\frac{1}{2}e_3,\ \nabla_{e_5}e_3 = \frac{1}{2}e_1.$$

The nonzero projections F_i are given as follows:

$$\begin{aligned}
F_1(x,y,z) = \frac{1}{4}(&2x_1y_1z_1 + x_2y_2z_1 + x_3y_3z_1 + x_4y_4z_1 + x_5y_5z_1 + x_2y_1z_2 \\
&+ x_3y_1z_3 + x_4y_1z_4 + x_5y_1z_5 - x_2y_4z_3 - x_3y_1z_3 - x_4y_2z_3) \\
&- 2x_1y_3z_3 - x_2y_3z_4 - x_3y_3z_1 - x_4y_3z_2),
\end{aligned}$$

$$\begin{aligned}
F_2(x,y,z) = \frac{1}{4}(&-2x_1y_2z_2 + 2x_1y_4z_4 - 2x_4y_3z_2 + 2x_4y_4z_1 + 2x_4y_1z_4 - 2x_4y_2z_3 \\
&- 2x_1y_1z_1 - x_5y_5z_1 - x_5y_1z_5 + 2x_1y_3z_3),
\end{aligned}$$

$$\begin{aligned}
F_3(x,y,z) = \frac{1}{4}(&-2x_1y_2z_2 + 2x_1y_4z_4 + x_2y_1z_2 + x_2y_2z_1 - x_2y_3z_4 \\
&- x_2y_4z_3 + x_4y_3z_2 - x_4y_4z_1 - x_4y_1z_4 + x_4y_2z_3),
\end{aligned}$$

$$F_8(x,y,z) = \frac{1}{2}x_1y_1z_5 - \frac{1}{2}x_3y_3z_5 - \frac{1}{2}x_3y_5z_3 + \frac{1}{2}x_1y_5z_1,$$

$$F_{10}(x,y,z) = x_5y_1z_1 - x_5y_3z_3.$$

The basic components of F are calculated by

$$\begin{aligned}
&F_{115} = F_{151} = F_{212} = F_{221} = F_{414} = F_{441} = \frac{1}{2}, \\
&F_{234} = F_{243} = F_{335} = F_{353} = F_{423} = F_{432} = -\frac{1}{2}, \\
&F_{144} = F_{511} = -F_{122} = -F_{533} = 1.
\end{aligned}$$

If the tensor F is written explicitly for any vectors x,y,z, then we obtain that $\mathfrak{g}_5$ is in the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$. □

Moreover, using the structure (3.2), we have

$$\begin{aligned}
&F_{212} = F_{221} = F_{313} = F_{331} = \frac{1}{2}, \\
&F_{414} = F_{441} = F_{515} = F_{551} = \frac{1}{2}, \\
&F_{144} = F_{155} = -F_{122} = -F_{133} = 1.
\end{aligned}$$

By direct computation, we get the nonzero projections F_6 and F_{10} in the following way:

$$\begin{aligned}
F_6(x,y,z) = &\frac{1}{2}x_2y_1z_2 + \frac{1}{2}x_2y_2z_1 + \frac{1}{2}x_3y_1z_3 + \frac{1}{2}x_3y_3z_1 \\
&+ \frac{1}{2}x_4y_1z_4 + \frac{1}{2}x_4y_4z_1 + \frac{1}{2}x_5y_1z_5 + \frac{1}{2}x_5y_5z_1,
\end{aligned}$$

$$F_{10}(x,y,z) = -x_1y_2z_2 - x_1y_3z_3 + x_1y_4z_4 + x_1y_5z_5.$$

Hence, we obtain that $\mathfrak{g}_5$ is in $\mathscr{F}_6 \oplus \mathscr{F}_{10}$.
The non-zero components Ric_{ij} for $\mathfrak{g}_5$ are

$$\begin{aligned}
&Ric_{11} = -1,\ \ Ric_{22} = -\frac{1}{2}, \\
&Ric_{33} = -\frac{1}{2},\ \ Ric_{44} = \frac{1}{2}, \\
&Ric_{55} = \frac{1}{2}.
\end{aligned}$$

Using above equations, the scalar curvature is -1. The nonzero components of $\mathscr{L}_\xi g$ for the structure (3.2) are the following:

$(\mathscr{L}_\xi g)_{24} = (\mathscr{L}_\xi g)_{42} = (\mathscr{L}_\xi g)_{35} = (\mathscr{L}_\xi g)_{53} = -1.$

It can be easily checked that $\mathfrak{g}_5$ is neither Einstein-like nor Ricci-like soliton for structures (3.1) and (3.2).

3.6. The Lie algebra $\mathfrak{g}_6$

Theorem 3.7. *The Lie algebra $\mathfrak{g}_6$ belongs to the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_7 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$ according to the structure given in (3.1).*

Proof. Similarly, the basic components of ∇ are computed as follows:

$$\nabla_{e_1} e_2 = \frac{1}{2}e_3,\ \nabla_{e_1} e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_4,\ \nabla_{e_1} e_4 = -\frac{1}{2}e_3,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2}e_3,\ \nabla_{e_2} e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_5,\ \nabla_{e_2} e_5 = -\frac{1}{2}e_3,\ \nabla_{e_3} e_1 = -\frac{1}{2}e_2 - \frac{1}{2}e_4,$$

$$\nabla_{e_3} e_2 = \frac{1}{2}e_1 - \frac{1}{2}e_5,\ \nabla_{e_3} e_4 = \frac{1}{2}e_1,\ \nabla_{e_3} e_5 = \frac{1}{2}e_2,\ \nabla_{e_4} e_1 = -\frac{1}{2}e_3,\ \nabla_{e_4} e_3 = \frac{1}{2}e_1,$$

$$\nabla_{e_5} e_2 = -\frac{1}{2}e_3,\ \nabla_{e_5} e_3 = \frac{1}{2}e_2.$$

The nonzero components of the structure tensor F are as follows:

$$\begin{aligned}
&F_{114} = F_{134} = F_{141} = F_{143} = F_{215} = F_{251} = \frac{1}{2},\\
&F_{312} = F_{314} = F_{321} = F_{341} = F_{512} = F_{521} = \frac{1}{2},\\
&F_{112} = F_{121} = F_{123} = F_{132} = F_{323} = F_{332} = -\frac{1}{2},\\
&F_{334} = F_{343} = F_{345} = F_{354} = F_{534} = F_{543} = -\frac{1}{2},\\
&F_{211} = F_{411} = -F_{233} = -F_{433} = 1.
\end{aligned}$$

We omit the nonzero projections F_i since they are very long. Hence, it is not hard to verify that $\mathfrak{g}_6$ is in the class $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_7 \oplus \mathscr{F}_8 \oplus \mathscr{F}_{10}$. □

For the structure (3.2) on $\mathfrak{g}_6$, the nonzero components F_{ijk} are in the following:

$$\begin{aligned}
&F_{134} = F_{143} = F_{145} = F_{154} = F_{215} = F_{251} = \frac{1}{2},\\
&F_{312} = F_{314} = F_{321} = F_{323} = F_{332} = \frac{1}{2},\\
&F_{341} = F_{415} = F_{451} = F_{525} = F_{552} = \frac{1}{2},\\
&F_{123} = F_{125} = F_{132} = F_{152} = -\frac{1}{2},\\
&F_{345} = F_{354} = F_{534} = F_{543} = -\frac{1}{2},\\
&F_{255} = -F_{233} = 1.
\end{aligned}$$

Using the general form of F, it can be seen that $\mathfrak{g}_6$ is of type $\mathscr{F}_1 \oplus \mathscr{F}_2 \oplus \mathscr{F}_3 \oplus \mathscr{F}_6 \oplus \mathscr{F}_9 \oplus \mathscr{F}_{10}$.
The nonzero components of Ricci curvature tensor are in the following:

$$\begin{aligned}
Ric_{11} &= -1,\ Ric_{22} = -1,\\
Ric_{33} &= -\frac{1}{2},\ Ric_{44} = \frac{1}{2},\\
Ric_{55} &= \frac{1}{2}.
\end{aligned}$$

With the aid of above relations, the scalar curvature tensor is $scal = -\frac{3}{2}$. The nonzero components of $\mathscr{L}_\xi g$ for the structure (3.2) are the following:

$(\mathscr{L}_\xi g)_{23} = (\mathscr{L}_\xi g)_{32} = (\mathscr{L}_\xi g)_{34} = (\mathscr{L}_\xi g)_{43} = -1.$

It is not hard to check that $\mathfrak{g}_5$ is neither Einstein-like nor Ricci-like soliton for structures (3.1) and (3.2).
Note that all components $\mathscr{L}_\xi g$ for the structure (3.1) are zero.
As a result, we get the followings.

Corollary 3.8. *The vector field ξ defined on the Lie algebras $\mathfrak{g}_i$ for the structure (3.1) for $i = 1,\ldots,6$ is a Killing vector field.*

Corollary 3.9. *The structures given in (3.1) and (3.2) on five dimensional nilpotent Lie algebras $\mathfrak{g}_i$ are not para-Sasaki-like.*

4. Conclusion

In this manuscript, we give two different Riemannian $\Pi-$structure on $5-$dimensional nilpotent Lie algebras. The classes of given structures on $5-$dimensional nilpotent Lie algebras are determined. We obtain the examples from certain classes. Only the $\mathfrak{g}_1$ Lie algebra among $5-$dimensional nilpotent Lie algebras is Einstein-like and admit Ricci-like soliton according to the structure given in (3.1). $5-$dimensional nilpotent Lie algebras $\mathfrak{g}_i$ for the structure (3.2) are neither Einstein-like nor Ricci-like soliton.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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