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On the simplicity crossed polymodules

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Abstract — The conception of crossed modules was first expressed by Whitehead when he was working on combinatorial theory. This concept has many uses in various fields, such as category theory, algebra, and k-theory. Moreover, one of the equations that plays a very important role in mathematical problems is the Yang-Baxter equation. In fact, although it doesn't seem like it, these equations play an effective role in studies such as particle physics, statistical mechanics, quantum field theory, and quantum groups. We use crossed modules to solve them. In crossed modules, the Actor was defined by Alp. Nilpotent, Solvable, n-Complete, and Representations of crossed modules were studied by Dehghanizadeh and Davvaz. In studies of group theory, a simplicial group is an object. Davvaz and Alp studied simplicial polygroups, and the generalized Moore complexes. They proved the existence of a functor from the category cat^1-polygroups to the category of groups and, furthermore, conversely. In this paper, we provide simplicity-crossed polymodules and some of their properties. We have also presented a simple crossed polymodules theorem.

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1. Introduction

We remind you that Yang-Baxter equations play a very important role in various fields of applied mathematics. We have already mentioned some of these fields. Among its solutions, which are made in the name of braidings, the following can be mentioned:

i. from Yetter-Drinfel'd modules over a Hopf algebra,

ii. from self-distributive structures,

iii. from crossed modules of groups.

Furthermore, in the abstract, we have mentioned a number of fields in which crossed modules are used in their study. Therefore, studying crossed modules and all kinds of automorphisms at least through this, is very important. This is one of the motivations of recent half-century studies in this field. Crossed modules were defined by Whitehead [1].

There are many interesting applications of crossed modules, such as Actor, Pullback, Pushout, and Induced crossed modules [2–4]. Nilpotent, Solvable, n-Complete, and Representations of crossed

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modules were studied by Dehghanizadeh and Davvaz [5–9]. Polygroups were studied by Comer [10], also see in [11]. In fact, Comer and Davvaz extended the algebraic theory to polygroups. Alp and Davvaz [12], expressed the concept of crossed polymodule of polygroups along with some properties and characteristics of it. Moreover, they introduced new important classes by the fundamental relations. The pushout and pullback in crossed polymodules theory, have been introduced by Alp and Davvaz, and they described the structure of these two concepts in crossed polymodules [13]. Arvasi et al. [14–17], introduced the notion of a 2-crossed module, which is a generalization of crossed modules, in addition to, was defined by Brown et al. [18, 19]. In [20, 21], Dehghanizadeh et al. introduce the notion of crossed polysquare. In fact, after studying the simplicity of groups and crossed modules, we have studied and researched the simplicity of crossed polymodules.

2. Preliminaries

To continue the study, we state some definitions and necessary theorems of crossed modules [22–24]. A crossed module (T, G, ∂) consist of a group homomorphism $\partial : T \to G$ called the boundary map, together with an action $(g, t) \to^g t$ of G on T satisfying

$$\partial({}^g t) = g\partial(t)g^{-1} \tag{2.1}$$

$$^{\partial(s)}t = sts^{-1} \tag{2.2}$$

for all $g \in G$ and $s, t \in T$. Here are some popular examples of crossed modules:

i. $N \to G$, where is N the normal subgroup G,

ii. $M \to G$, where is a G-module M with the zero homomorphism,

iii. $E \to G$, where is epimorphism with the central kernel.

Moreover, we have the following results from the crossed module definition:

i. The kernel of ∂, ker∂, is subset of $Z(T)$.

ii. The image of ∂, $\partial(T)$, is a normal subgroup of G.

iii. The action of G on T, induces a natural $(G/\partial(T))$-module structure on $Z(T)$, and ker∂ is a submodule of $Z(T)$.

In addition to, (S, H, ∂') is a subcrossed module of the crossed module (T, G, ∂) if

i. S is a subgroup of T, and H is a subgroup of G.

ii. ∂' is the restriction of ∂ to S.

iii. the action of H on S is included by the action of G on T.

Furthermore, a subcrossed module (S, H, ∂) of (T, G, ∂) is normal if

i. H is a normal subgroup of G.

ii. ${}^g s \in S$, for all $g \in G$ and $s \in S$

iii. ${}^h t t^{-1} \in S$, for all $h \in H$ and $t \in T$

In this case, we consider the triple $(T/S, G/H, \bar{\partial})$, where $\bar{\partial} : T/S \to G/H$ is induced by ∂, and the new action is given by ${}^{gH}(tS) = ({}^g t)S$. This is the quotient crossed module of (T, G, ∂) by (S, H, ∂).

A crossed module morphism $<\alpha, \phi>: (T, G, \partial) \to (T', G', \partial')$ is a commutative diagram of homomorphisms of groups

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\partial"'] & T' \arrow[d, "\partial'"] \\ G \arrow[r, "\phi"'] & G' \end{tikzcd} \tag{2.3}$$

such that for all $x \in G$ and $t \in T$, we have $\alpha({}^{x}t) = {}^{\phi(x)}\alpha(t)$. We say that $<\alpha, \phi>$ is an isomorphism if α and ϕ are both isomorphisms. We denote the group of automorphisms of (T, G, ∂) by $Aut(T, G, \partial)$. The kernel of the crossed module morphism $<\alpha, \phi>$ is the normal subcrossed module $(\ker\alpha, \ker\phi, \partial)$ of (T, G, ∂), denoted by $\ker <\alpha, \phi>$. The image $im <\alpha, \phi>$ of $<\alpha, \phi>$ is the subcrossed module $(im\alpha, im\phi, \partial')$ of (T', G', ∂'). For a crossed module (T, G, ∂), denoted by $\mathrm{Der}(G, T)$, set of all derivations from G to T, i.e., all maps $\chi : G \to T$ such that for all $x, y \in G$,

$$\chi(xy) = \chi(x)\, {}^{x}\chi(y) \tag{2.4}$$

Each such derivation χ defines endomorphisms $\sigma = (\sigma_x)$ and $\theta(= \theta_x)$ of G and T, respectively, given by

$$\sigma(x)) = \partial\chi(x)x, \quad \theta(t) = \chi\partial(t)t \tag{2.5}$$

and $\sigma\partial(t) = \partial\theta(t)$, $\theta\chi(x) = \chi\partial(x)$, $\theta({}^{x}t) = {}^{\sigma(x)}\theta(t)$. If we define in $\mathrm{Der}(G, T)$ a multiplication by the formula $\chi_1 \circ \chi_2 = \chi$, where

$$\chi(x) = \chi_1\sigma_2(x)\chi_2(x) \qquad (= \theta_1\, \chi_2(x)\chi_1(x)) \tag{2.6}$$

then $\mathrm{Der}(G, T)$ is a semigroup, also the identity element into in semigroup is the derivation which maps each element of G into the identity element of T. Moreover, if $\chi = \chi_1 \circ \chi_2$, then $\sigma = \sigma_1\sigma_2$. The group of units of $\mathrm{Der}(G, T)$, called is the whitehead group $D(G, T)$, and regular derivations are the elements of $D(G, T)$.

Proposition 2.1. The following are equivalent in crossed modules:

i. $\chi \in D(G, T)$

ii. $\sigma \in \mathrm{Aut}(G)$

iii. $\theta \in \mathrm{Aut}(T)$

3. Polygroups and Crossed Polymodules

We remind you that one of several natural generalizations of group theory, which is studied, is the theory of polygroups. Regarding the action on their elements, in any group, the combination of two elements is one element, but in any polygroup, that is a set. In addition, we point out that polygroups have important uses in many fields, such as lattices, geometry, color scheme, and combinatorics. As a good source for study, including definition, suitable examples, and actually studying polygroups as a subclass of supergroups, it can be referred to [11]. Applications of hypergroups studied by Comer [10], also see [11,25]. In fact, they extended the algebraic theory to polygroups. According [10], a *polygroup* is a multi-valued system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, and $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold, for all $r, s, t \in P$:

i. $(r \circ s) \circ t = r \circ (s \circ t)$

ii. $e \circ r = r \circ e = r$

iii. $r \in s \circ t$ implies $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

$\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and also if $x \in P$ and A, B are non-empty subsets of P, then we have $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

The following are the facts that are clearly concluded from the principles of the polygroups: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$, and $(r^{-1})^{-1} = r$.

Example 3.1. If we consider the set P as $P = \{e, r, s\}$, then $P =< P, \circ, e, ^{-1} >$ along with polyaction, according to the table below

$\circ$	e	r	s
e	e	r	s
r	r	$\{e, s\}$	$\{r, s\}$
s	s	$\{r, s\}$	$\{e, r\}$

is a polygroup.

Example 3.2. In every polygroup, the set containing only the identity member is always a subpolygroup, and this subpolygroup is normal in the polygroup. Therefore, we have crossed polymodule $(1, P) = (1, P, c_1, id_c|_1)$.

Example 3.3. Every polygroup P contains the whole polygroup P as a normal subpolygroup. Therefore, we always have crossed polymodule $(G, G) = (G, G, c, id_G)$.

Example 3.4. Consider the following polygroup morphisms of an abelian polygroup P, written multiplicatively,

$$l : 1 \to Aut(P) \qquad i \to id_P \qquad k : P \to 1 \qquad p \to 1 \tag{3.1}$$

Then, we have a crossed polymodule $(P, 1) = (P, 1, l, k)$.

Definition 3.5. [24] A crossed polymodule $\chi = (C, P, \partial, \alpha)$ consists of polygroups $< C, *, e, ^{-1} >$ and $< P, \circ, e, ^{-1} >$ together with a strong homomorphism $\partial : C \longrightarrow P$ and a (left) action $\alpha : P \times C \longrightarrow \mathcal{P}^*(C)$ on C, satisfying the conditions:

i. $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$

ii. ${}^{\partial(c)}c' = c * c' * c^{-1}$, for all $c, c' \in C$

Example 3.6. [24] A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In fact, for any polygroup P, the identity map $\mathrm{id}_P : P \longrightarrow P$ is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways a polygroup P may be regarded as a crossed polymodule: via the identity map or the inclusion of the trivial subpolygroup.

Example 3.7. [24] If C is a P-polymodule, in this case there is a well defined action α of P on C. This, together with the zero homomorphisms, creates a crossed polymodule (C, P, ∂, α).

Example 3.8. Let P be a polygroup and $N \trianglelefteq P$ be a normal subpolygroup. Consider the polygroup morphism

$$\begin{aligned} C_N : \quad & P \to Aut(N) \\ & p \to (c_p|_N^N : n \to n^p) \end{aligned} \tag{3.2}$$

Then, the following crossed polymodule exists:

$$(N, P) = (N, P, C_N, id_P|_N) \tag{3.3}$$

Definition 3.9. Consider the crossed polymodules $\chi = (C, P, \partial, \alpha)$ and $\chi' = (C', P', \partial', \alpha')$. A crossed polymodule morphism $f = (\lambda, \Gamma) : \chi \to \chi'$ is a tuple of strong homomorphism, such that the diagram

$$\begin{tikzcd} C \arrow[r, "\lambda"] \arrow[d, "\partial"'] & C' \arrow[d, "\partial'"] \\ P \arrow[r, "\Gamma"'] & P' \end{tikzcd} \tag{3.4}$$

commutes, and $\lambda(p\alpha c) = \Gamma(p)\alpha'\lambda(c)$, for all $p \in P, c \in C$.

4. Simplicity Crossed Polymodule

In this part, we express the concept of the simplicity of crossed polymodules of polygroups, and we will examine some interesting properties of its. In fact, results extend the classical results of crossed modules to crossed polymodules of polygroups.

Definition 4.1. Suppose given crossed polymodules

$$(C_i, P_i, \partial_i, \alpha_i),\ \ for\ i = 1, 2, 3 \tag{4.1}$$

with crossed polymodule morphisms as following

$$(\lambda_i, \gamma_i) : (C_i, P_i, \partial_i, \alpha_i) \to (C_{i+1}, P_{i+1}, \partial_{i+1}, \alpha_{i+1}),\ \ for\ i = 1, 2 \tag{4.2}$$

such that

$$1 \longrightarrow C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} C_3 \longrightarrow 1 \tag{4.3}$$

and

$$1 \longrightarrow P_1 \xrightarrow{\gamma_1} P_2 \xrightarrow{\gamma_2} P_3 \longrightarrow 1 \tag{4.4}$$

be short exact sequences. We call

$$1 \longrightarrow (C_1, P_1, \partial_1, \alpha_1) \xrightarrow{(\lambda_1,\gamma_1)} (C_2, P_2, \partial_2, \alpha_2) \xrightarrow{(\lambda_2,\gamma_2)} (C_3, P_3, \partial_3, \alpha_3) \longrightarrow 1 \tag{4.5}$$

a short exact sequence of crossed polymodules.

Lemma 4.2. Consider a crossed polymodule of (C, P, ∂, α). If (N, H, ∂, α), be a normal subcrossed polymodule of (C, P, ∂, α) and

$$\frac{(C, P, \partial, \alpha)}{(N, P, \partial, \alpha)} \tag{4.6}$$

be the factor crossed polymodule, then there is the residue class morphism

$$f : C \to \frac{C}{N} \qquad\qquad g : P \to \frac{P}{H} \tag{4.7}$$

and if $i = idC|_N$ and $j = idp|_H$ be the inclusion maps, then we have a short exact sequence

$$1 \longrightarrow (N, H, \partial, \alpha) \xrightarrow{(i,j)} (C, P, \partial, \alpha) \xrightarrow{(f,g)} \frac{(C,P,\partial,\alpha)}{(N,H,\partial,\alpha)} \longrightarrow 1 \tag{4.8}$$

Proof.

There are short exact sequences

$$1 \longrightarrow N \xrightarrow{i} C \xrightarrow{f} \frac{C}{N} \longrightarrow 1 \tag{4.9}$$

$$1 \longrightarrow H \xrightarrow{j} P \xrightarrow{g} \frac{P}{H} \longrightarrow 1 \tag{4.10}$$

and (i, j) and (f, g) are crossed polymodule morphisms. $\square$

Lemma 4.3. Consider crossed polymodules of (N, H, β, k) and (L, F, r, s). If there is a short exact sequence,

$$1 \longrightarrow (N, H, \beta, k) \xrightarrow{(\phi,\psi)} (C, P, \partial, \alpha) \xrightarrow{(\lambda,\gamma)} (L, F, r, s) \longrightarrow 1 \tag{4.11}$$

then the map $(\phi|^{ker\lambda}, \psi|^{ker\gamma}) : (N, H, \beta, k) \to \ker(\lambda, \gamma)$ is an isomorphism of crossed polymodule.

Proof.

There are short exact sequences,

$$1 \longrightarrow N \xrightarrow{\phi} C \xrightarrow{\lambda} L \longrightarrow 1 \tag{4.12}$$

$$1 \longrightarrow H \xrightarrow{\psi} P \xrightarrow{\gamma} F \longrightarrow 1 \tag{4.13}$$

Thus, we have bijective polygroup morphisms $\phi|^{\ker\lambda} : N \to \ker\lambda$ and $\psi|^{\ker\gamma} : H \to \ker\gamma$ and $(\phi|^{\ker\lambda}, \psi|^{\ker\gamma})$ is a crossed polymodule morphism and also is an isomorphism of crossed polymodules. $\square$

Definition 4.4. A simple crossed polymodule (C, P, ∂, α) is a crossed polymodule, where it is not isomorphic with $(1, 1, id)$ and not have normal crossed subpolymodules apart from its trivial crossed subpolymodule $(1, 1, id)$ and the crossed polymodule (C, P, ∂, α) itself.

Proposition 4.5. If (G, G, c, id_G) be a crossed module, where G is a group, then exist a crossed submodule $(1, G, c, id_G) \leq (G, G, c, id_G)$, and $(1, G, c, id_G) \trianglelefteq (G, G, c, id_G)$, if G is abelian and vice versa.

Proof.

There are $(1)id_G = 1$ and $(1)gc = 1^g = g^{-1}1g = 1$, for $g \in G$. As a result, $(1, G, c, id_G) \leq (G, G, c, id_G)$. If $(1, G, c, id_G) \trianglelefteq (G, G, c, id_G)$, then we find that $g^{-1}g^h = 1 \leftrightarrow hg = gh$, for $g, h \in G$. Therefore, G is abelian.

On the contrary, we assume that G is an abelian group. For $g, h \in G\, g^{-1}g^h = 1$ and $1^g = 1$. This proves that $(1, G, c, id_G) \trianglelefteq (G, G, c, id_G)$. $\square$

Lemma 4.6. Consider a crossed polymodule with form (C, P, ∂, α). Then, there is a crossed polymodule $(C\alpha, P, c_{c\alpha}, id_P|_{c\alpha})$, and a surjective crossed polymodule morphism

$$(\alpha|^{C\alpha, id_P}) : (C, P, \partial, \alpha) \to (C\alpha, P, \partial, \alpha) \tag{4.14}$$

Proof.

There are $C\alpha \trianglelefteq P$. Thus, (C, P, ∂, α) is a crossed polymodule. If $\bar{\alpha} = \alpha|^{C\alpha}$ and $\bar{k} = id_P|_{C\alpha}$, then, for $\phi \in P, c \in C$, we get that

$$(p)\bar{\alpha}\bar{k} = (p)\alpha\bar{k} = (p)\alpha = (p)\alpha id_P \tag{4.15}$$

Moreover, $(p^c)\bar{\alpha} = (p^c)\alpha = (p)\alpha = (p\alpha)^c = p\bar{\alpha}^c$. Thus, $(\alpha|^{C\alpha}, id_P)$ is a crossed polymodule morphism and by construction, it is surjective. $\square$

Lemma 4.7. Let (C, P, ∂, α) be a crossed polymodule. Then, there is a short exact sequence given as follows,

$$1 \longrightarrow (\ker\alpha, 1, \partial, \alpha) \xrightarrow{\trianglelefteq} (C, P, \partial, \alpha) \xrightarrow{(\alpha|^{C\alpha}, id_P)} (C\alpha, P, \partial, \alpha) \longrightarrow 1 \tag{4.16}$$

Proof.

The kernel of

$$(\alpha|^{C\alpha}, id_P) : (C, P, \partial, \alpha) \to (C\alpha, P, \partial, \alpha) \tag{4.17}$$

is given by $(\ker \alpha, 1)$. But $(\ker \alpha, 1) \leq (C, P, \partial, \alpha)$ is a crossed subpolymodule. Therefore, we have the following inclusion morphism:

$$(id_C|_{\ker \alpha}, id_P|_1) : (\ker \alpha, 1) \to (C, P, \partial, \alpha) \tag{4.18}$$

Hence, the sequence in the lemma exists and is short and exact. □

Proposition 4.8. Suppose that we have two polygroups, P and C such that P be a normal subpolygroup of C, $P \trianglelefteq C$. Consider the crossed polymodule $(C, P, \alpha|_C, id_P|_C)$,

i. Then, we get a crossed polymodule $(1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1)$ and a surjective crossed polymodule morphism

$$(k, r) : (C, P, \alpha|_C, id_P|_C) \to (1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1) \tag{4.19}$$

where $k : C \to 1$, $c \to 1 = 1_{\frac{P}{C}}$, $r : P \to \frac{P}{C}$, and $p \to pC$.

ii. There is a short exact sequence given by

$$\begin{tikzcd} 1 \arrow[r] & (C, C) \arrow[r, "\trianglelefteq"] & (C, P, \alpha|_C, id_P|_C) \arrow[r, "(k,r)"] & (1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1) \arrow[r] & 1 \end{tikzcd} \tag{4.20}$$

In addition, the sequence exists and this sequence is both exact and short.

Proof.

i. We have a crossed polymodule $(1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1)$. If $m \in C$ and let $p \in P$, then we get that

$$(m)id_P|_C r = (m)r = mC = 1C = (1)id_{\frac{P}{C}}|_1 = (m)kid_{\frac{P}{C}}|_1 \tag{4.21}$$

and

$$(m^p)k = 1C = 1^pC = (1m)^{pc} = (mk)^{pr} \tag{4.22}$$

Thus, (k, r) is a crossed polymodule morphism. Moreover, is surjective with how to construction.

ii. For $(k, r) : (C, P, \alpha|_C, id_P|_C) \to (1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1)$, kernel is given by (C, C). But $(C, P, \alpha|_C, id_P|_C) \leq (1, \frac{P}{C}, \alpha_1, id_{\frac{P}{C}}|_1)$ is a crossed subpolymodule. Therefore, the inclusion morphism will exist as follows

$$(id_c, id_{\frac{P}{C}}) : (C, C) \to (C, P) \tag{4.23}$$

□

Theorem 4.9. Let P be a polygroup given $C \trianglelefteq P$. Consider the crossed polymodule $(C, P, C_c, id_{\frac{P}{C}})$. Then, a crossed subpolymodule $(N, H) \leq (C, P, C_c, id_{\frac{P}{C}})$ is normal in $(C, P, C_c, id_{\frac{P}{C}})$ if and only if

$$N \trianglelefteq C \quad H \trianglelefteq P \quad N \trianglelefteq H \quad N \trianglelefteq P \quad [C, H] \leq P \tag{4.24}$$

In fact, we will have the following diagram:

$$\begin{tikzcd} {[C, H]} \arrow[r] & N \arrow[r, "\trianglelefteq"] \arrow[d, "\trianglelefteq"'] \arrow[dr] & C \arrow[d, "\trianglelefteq"] \\ & H \arrow[r, "\trianglelefteq"'] & P \end{tikzcd} \tag{4.25}$$

Proof.

For proof, consider that $(N,H)\trianglelefteq(C,P)$. Thus, we have $N\trianglelefteq C$, $H\trianglelefteq P$. But, for $n\in N, p\in P$, we have $n^p=p^{-1}np\subseteq N, N\trianglelefteq P$ since (N,H) carries the morphism $id_P|_C|_N^H$, it results that $N\trianglelefteq H$. Thus, $N\trianglelefteq C$. For $p\in P, h\in H$, $[p,h]\subseteq N$. Hence $[C,H]\leq N$.

Contrariwise, $N\trianglelefteq C$, $H\trianglelefteq P$, for $n\in N, c\in C$ and $h\in H, p\in P$, will have $p^{-1}p^h\subseteq[C,H]\leq N, n^p\subseteq N$. Hence, we get that $(N,H)\trianglelefteq(C,P)$. □

Theorem 4.10. Consider χ to be a crossed polymodule. In this case, χ is simple, if and only if *i.* or *ii.* or *iii.* holds,

i. $\chi=(C,P)\cong(N,N,c,id_N)$, for some non-abelian and simple polygroup N.

ii. $\chi\cong(1,K,c_1,id_K|_1)$, for some simple polygroup K.

iii. $\chi\cong(M,1,l,k)$, for some of polygroup M from the prime order and also cyclic.

Proof.

If *i.* be true, then we may assume that $\chi=(N,N,c,id_N)$, with N a simple and non-abelian polygroup. The crossed polymodule $(1,1)$ is not simple. Thus, we may assume that $N\neq 1$.

Suppose given a normal crossed subpolymodule $(M,H)\trianglelefteq(N,N)$. Then, we get that $M\trianglelefteq N$ and $H\trianglelefteq N$. But we know that $M=1$ and $H=N$. As a result $(l,N)\ntrianglelefteq(N,N)$. Because N is assumed non-abelian, so we conclude that $M=N$ and $H=1$, thus $(N,1)\ntrianglelefteq(N,N)$, since we do not have $N\trianglelefteq 1$.

On the only normal crossed subpolymodules. Hence, (N,N) is simple.

If *ii.* be true, then we may assume $(1,K)=(1,K,c,id_K|_1)$, when we consider K a simple ploygroup. If a normal crossed polymodule is given $(M,H)\trianglelefteq(1,K)$, Then, we conclude that $M\trianglelefteq 1$ and $H\trianglelefteq K$. Therefore, $M=1$ concludes that $H=1$ or $H=K$. Hence, we conclude that $(1,K)$ is simple.

If *iii.* be true, then we consider that $(P,1)=(P,1,l,k)$. Suppose we have a normal crossed subpolymodule $(M,H)\trianglelefteq(P,1)$. Then, we will have $M\trianglelefteq P$ and $H\trianglelefteq 1$. Thus, $M=1$ or $M=P$. Therefore. it concludes that $H=1$, and subsequently we conclude that, $(P,1)$ is simple.

For contrariwise, suppose there is a simple crossed polymodule $(P,N)=(P,N,\alpha,f)$, by the short exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \ker f \arrow[r, "\trianglelefteq"] \arrow[d] & P \arrow[r, "f|^{Pf}"] \arrow[d, "f"'] & Pf \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] & N \arrow[r, "id_N"'] & N \arrow[r] & 1
\end{tikzcd} \tag{4.26}$$

$(\ker f,1)\trianglelefteq(P,N)$. But (P,N) is simple, so it concludes that $(\ker f,1)=(1,1)$ or $(\ker f,1)=(P,N)$.

(a) If $(\ker f,1)=(P,N)$, $\ker f$ and P are abelian. In the following, we prove that the polygroup P is simple.

For this, assume a normal subpolygroup exists, which is also not obvious $1\neq M\trianglelefteq P$. But a subpolygroup of the abelian polygroup P, the polygroup M is abelian.

Hence, there is a normal crossed subpolymodule and non-trivial, $1\neq(M,1)\trianglelefteq(P,1)$, of course, this is a contradiction to the simplicity of $(P,1)$. Hence, P is both abelian and simple; so, $(P,N)=(P,1)$. Thus, *iii.* is holds.

(b) If $(\ker f,1)=(1,1)$, then $\ker f=1$, which gives the result that the mapping f is injective. Thus, $\bar{f}=f|^{Pf}:P\to Pf$ is bijective. Therefore, $(f|^{Pf},id_N)$ is an isomorphism. Hence, it suffices to prove that (Pf, N) satisfies *i.* or *ii.*. Therefore, we may assume that $P \trianglelefteq N$, $\alpha = C_P, f = id_N|_P$.

Therefore, the following is the exact sequence,

$$\begin{tikzcd}
1 \arrow[r] & P \arrow[r, "id_P"] \arrow[d, "id_P"'] & P \arrow[r, "k"] \arrow[d, "\trianglelefteq"'] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P \arrow[r] & N \arrow[r, "r"'] & \frac{N}{P} \arrow[r] & 1
\end{tikzcd} \tag{4.27}$$

Hence, $(P, P) \trianglelefteq (P, N)$. But (P, N) is simple, we get $(P, P) = 1$ or $(P, P) = (P, N)$.

(a) If $(P, P) = 1$, the $(P, r) : (P, N) \to (1, \frac{N}{P})$ is a isomorphism of crossed polymodule and $(P, N) \simeq (1, k)$ with $K = \frac{N}{P}$. In the following, we prove that the Polygroup K is simple.

For this, if we have a normal subpolygroup which is non-trivial, $1 \neq M \trianglelefteq K$, then there is a normal crossed subpolymodule $(1, M) \triangle (1, K)$ such that is non-trivial.

This contradiction is clear, because, $(P, N) \cong (1, K)$ is simple. Thus, we have $(P, N) \cong (1, K)$.

(b) If $(P, P) = (P, N)$, therefore, we prove that P is a simple polygroup and non-abelian.

Assuming that P is abelian, there exists a normal crossed subpolymodule $(1, P) \trianglelefteq (P, P)$, which is a contradiction to the simplicity of (P, P).

Suppose P is not simple. Therefore, there is a non-trivial normal subpolygroup $M \trianglelefteq P$. This gives us the result of a normal crossed subpolymodule $(M, M) \trianglelefteq (P, P)$. In this case, it obviously contradicts the simplicity of (P, P). Hence, $(P, N) = (P, P)$ where P is a simple, non-abelian polygroup. Thus, *i.* is also held. □

5. Conclusion

In this article, we studied the simplicity of crossed polymodules and came to state and prove the simplicity theorem of crossed polymodules. Therefore, the simplicity of crossed polysquares and theorems related to intersecting polysquares can be studied in the future. Moreover, analyzing the simplicity of 2-crossed modules, obtaining their properties and theorems, and researching and investigating it regarding 2-crossed polymodules, while being interesting and practical, can be the research subject for those interested.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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