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# Weakly Prime Radical of Submodules

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## Abstract

In this paper, some properties of weakly prime radical are stated. The characterization of weakly prime radical for finitely generated modules is given. Also, the relationship between the weakly prime radical of a submodule and the ideals of the ring $T$ is considered.

***Keywords:*** *Prime radical; Prime submodule; weakly prime radical; weakly prime submodule.*
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## 1. Introduction

All rings will be commutative and all modules will be unitary throughout this work. Let $T$ be a ring and $H$ be a $T$-module and $K$ be a proper submodule. Then $K$ is called prime if $rm \in K$ implies $m \in K$ or $rH \subseteq K$ where $r \in T, m \in H$. Also $K$ is called weakly prime submodule whenever $rsm \in K$ either $rm \in K$ or $sm \in K$ for some $r,s \in T, m \in H$. Weakly prime submodules were introduced in [4]. It can be easily shown that every prime submodule is weakly prime. But not every weakly prime submodule is prime as the following example shows [4].

Let $T = \mathbb{Q}[x,y]$, $P = \langle x \rangle$ be a non-zero prime ideal of $T$, $H$ be a free $T$-module $T \oplus T$. Then $Q = 0 \oplus P$ is a weakly prime submodule of $H$ but it is not a prime submodule.

The concept of the radical of an ideal was generalized to modules over commutative rings [5]. As a result of this generalization, the definition of a prime submodule has shown up. If $K$ is a proper submodule of a $T$-module $H$, then the prime radical of $K$, $rad_H(K)$, is the intersection of all prime submodules containing $K$. If there is no prime submodule contains $K$, then $rad_H(K) = H$. In 2006, Behboodi gave the definition of weakly prime radical of submodule [3]. The weakly prime radical of $K$ in $H$, $wrad_H(K)$, is the intersection of all weakly prime submodules of $H$ containing $K$. If there is no weakly prime submodule containing $K$, then $wrad_H(K) = H$. By the definitions, $wrad_H(K) \subseteq rad_H(K)$.

In this work, we dealt with the weakly prime radical of submodules, its properties, and its relationship with the ideals of the ring $T$.

## 2. Weakly Prime Submodules

It is well-known that if $K$ is a prime submodule of $H$, then $(K : H) = \{r \in T : rH \subseteq K\}$ is a prime ideal. If $K$ is weakly prime, then $(K : H)$ is a radical ideal [6]. Also, the following can easily be shown if $I$ is an ideal of $T$.

**Proposition 2.1.** *Let $W$ be any weakly prime submodule of $H$ and $I$ be an ideal of $T$. Then $(W : I) = \{m \in H : Im \subseteq W\}$ is a weakly prime submodule of $H$.*

*Proof.* Let $r,s \in T$ and $m \in H$ such that $rsm \in (W : I)$. Then $(ars)m \in W$ for every $a \in I$. Since $W$ is weakly prime submodule, $(ar)m \in W$ or $sm \in W$. If $(ar)m \in W$, then either $am \in W$ or $rm \in W$. $am \in W$ implies that $m \in (W : I)$. Since $W \subseteq (W : I)$, $sm \in W$ or $rm \in W$ also gives that $sm \in (W : I)$ or $rm \in (W : I)$. Hence in each cases, either $rm \in (W : I)$ or $sm \in (W : I)$. □

The radical of an ideal $I$ of a ring is known as the intersection of all minimal prime ideals of $I$. We can consider a similar characterization for the weakly prime radical of a submodule.

**Definition 2.2.** *A weakly prime submodule $W$ of $H$ is called a minimal weakly prime submodule of a submodule $K$ if $K \subseteq W$ and there is no weakly prime submodule of $H$ containing $K$ which is smaller than $W$. A minimal weakly prime submodule of $\langle 0 \rangle$ is minimal weakly prime submodule of $H$.*

**Proposition 2.3.** *Let $\{W_i : i \in I\}$ be a non-empty family of weakly prime submodules of a $T$-module $H$. If the family is totally ordered by inclusion, then $\cap_{i\in I} W_i$ is a weakly prime submodule of $H$.*

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*Proof.* Let $xym \in \cap_{i\in I} W_i$ for $x,y \in T$ and $m \in H$. Then $xym \in W_i$ for all $i \in I$. Since $W_i$ is weakly prime, $xm \in W_i$ or $ym \in W_i$. Since the family is totally ordered, $xm \in \cap_{i\in I} W_i$ or $ym \in \cap_{i\in I} W_i$. Thus $\cap_{i\in I} W_i$ is a weakly prime submodule. □

**Proposition 2.4.** *If a submodule $K$ of a $T$-module $H$ is contained in a weakly prime submodule $N$, then $N$ contains a minimal weakly prime submodule of $K$.*

*Proof.* Let $M = \{L : L \text{is weakly prime submodule of } H \text{and} K \subseteq L \subseteq N\}$. It is clear that $M$ is non-empty. If $L_1, L_2 \in A$, define a relation $\leq$ on $M$ as $L_1 \leq L_2$ if $L_2 \subseteq L_1$. It is a partial order on $M$. For any non-empty totally ordered subset $M_1$ of $M$, if $\bar{L} = \cap_{L_i \in M_1} L_i$, then $\bar{L}$ is a weakly prime submodule since $M_1$ is totally ordered and also by the above proposition. Then $K \subseteq \bar{L} \subseteq N$. Hence $\bar{L} \in M$ and thus it is an upper bound for $M_1$. $M$ contains a maximal element $Y$ by Zorn's Lemma. Since $Y \in M$, it is a weakly prime submodule of $H$.
To complete the proof it is enough to show that $Y$ is a minimal prime submodule of $K$. Suppose that $\bar{Y}$ is a weakly prime submodule of $H$ where $K \subseteq \bar{Y} \subseteq N$ and $\bar{Y} \subseteq Y$.Then $\bar{Y} \in M$ and hence $Y \leq \bar{Y}$. Thus $Y = \bar{Y}$. Therefore $Y$ is a minimal weakly prime submodule of $K$. □

**Corollary 2.5.** *Every proper submodule of a finitely generated module possesses at least one minimal weakly prime submodule.*

*Proof.* Let $H$ be a finitely generated module, and $Q$ be a proper submodule of $H$. Then there exists a submodule $K$ of $H$ such that $Q \subseteq K$ and $K$ is maximal. Since $K$ is maximal, it is a prime submodule [7]. Hence $K$ is a weakly prime submodule of $H$. Then by Proposition 2.4, $K$ will contain a minimal weakly prime submodule of $Q$. Thus $Q$ has at least one minimal weakly prime submodule. □

If $H$ is a finitely generated module, then we can redefine the weakly prime radical of submodules of $H$ by the above corollary.

**Theorem 2.6.** *Let $H$ be a finitely generated $T$-module and $K$ be a proper submodule of $H$. Then weakly prime radical of $K$ is the intersection of its minimal weakly prime submodules.*

*Proof.* Let $K$ be a proper submodule of $H$. By Corollary 2.5, $K$ has at least one minimal weakly prime submodule, say $W_i$. Let $L$ be the intersection of all minimal weakly prime submodules of $H$ containing $K$. By the definition of weakly prime radical, $wrad_H(K) \subseteq \cap_{i\in I} W_i = L$. On the other hand, if $W$ is any weakly prime submodule containing $K$, then $W$ contains some minimal weakly prime submodule $Q_i$ of $K$ by Proposition 2.4. Hence $L = \cap_{i\in I} W_i \subseteq wrad_H(K)$. □

Some properties of weakly prime radical is given in the following proposition.

**Proposition 2.7.** *Let $H$ be a $T$-module, $J$ be an index set and let $N, N_j$ be submodules of $H$ for $j \in J$ and $I$ be an ideal of $T$. Then*

*(i)* $N \subseteq wrad_H(N)$,

*(ii)* $wrad_H(wrad_H(N)) = wrad_H(N)$,

*(iii)* $wrad_H(\bigcap\limits_{j\in J} N_j) \subseteq \bigcap_{j\in J} wrad_H(N_j) = wrad_H(\bigcap_{j\in J} wrad_H(N_j))$,

*(iv)* $\sum_{j\in J} wrad_H(N_j) \subseteq wrad_H(\sum\limits_{j\in J} N_j) = wrad_H(\sum_{j\in J} wrad_H(N_j))$,

*(v)* $wrad_H(IH) = wrad_H(\sqrt{I}H) = wrad_H(I^n H)$ *for every positive integer n,*

*(vi)* $\sqrt{(N:H)} \subseteq (wrad_H(N):H)$

*Proof.* (i) If $x \in N$, then $x \in W$ for every weakly prime submodule of $H$ containing $N$. Thus $x \in wrad_H(N)$.

(ii) Since $N \subseteq wrad_H(N) \subseteq W_k$ for any weakly prime submodule $W_k$ containing $wrad_H(N)$, $\bigcap_{k\in K} W_k \subseteq P_j$ for all weakly prime submodules $P_j$ containing $N$. Hence $wrad_H(wrad_M(N)) = \bigcap_{k\in K} W_k \subseteq wrad_H(N)$. The other side is clear by part (i).

(iii) Let $wrad_H(\bigcap_{j\in J} N_j) = \bigcap_{k\in K} W_k$ where $W_k$ is weakly prime submodule containing $\cap_{j\in J} N_j$ for every $k \in K$, and let $\{Q_{kj}\}$ be the set of weakly prime submodules containing $N_j$. Since $\bigcap_{j\in J} N_j \subseteq N_j \subseteq Q_{kj}$ for all $k$ and $j$, $wrad_H(\bigcap_{j\in J} N_j) \subseteq wrad_H(N_j)$.

Since $\bigcap_{j\in J} wrad_H(N_j) \subseteq wrad_H(N_j)$ for all $j \in J$,

$$wrad_H(\textstyle\bigcap_{j\in J} wrad_H(N_j)) \subseteq wrad_H(wrad_M(N_j)) = wrad_H(N_j) \text{ for all } j \in J.$$

Hence $wrad_H(\bigcap_{j\in J} wrad_H(N_j)) \subseteq \bigcap_{j\in J} wrad_H(N_j)$. By part (i), $\bigcap_{j\in J} wrad_H(N_j) = wrad_M(\bigcap_{j\in J} wrad_H(N_j))$ is clear.

(iv) Since $N_j \subseteq \sum\limits_{j\in J} N_j$ for all $j$, $wrad_H(N_j) \subseteq wrad_H(\sum\limits_{j\in J} N_j)$. Then $\sum\limits_{j\in J} wrad_H(N_j) \subseteq wrad_H(\sum\limits_{j\in J} N_j)$. Since $\sum\limits_{j\in J} N_j \subseteq \sum\limits_{j\in J} wrad_H(N_j)$, it can easily be shown that $wrad_H(\sum\limits_{j\in J} N_j) = wrad_H(\sum_{j\in J} wrad_H(N_j))$.

(v) This is trivially true if $wrad_H(IH) = H$. If $wrad_H(IH) \neq H$, then there exists a weakly prime submodule $W$ of $H$ where $IH \subseteq W$. Since $I \subseteq (IH:H) \subseteq (W:H)$, $\sqrt{I} \subseteq \sqrt{(W:H)} = (W:H)$. So that $\sqrt{I}H \subseteq (W:H)H \subseteq W$. Thus $wrad_H(\sqrt{I}H) \subseteq wrad_H(IH)$. The other side is clear since $I \subseteq \sqrt{I}$ is always true.
Since $\sqrt{I^n} = \sqrt{I}$ for any positive integer $n$,

$$wrad_H(I^n H) = wrad_H(\sqrt{I^n}H) = wrad_H(\sqrt{I}H) = wrad_H(IH).$$

(vi) Let $0 \neq a \in \sqrt{(N:H)}$. Then there exists $k \in \mathbb{Z}^+$ such that $a^k H \subseteq N \subseteq W$ for every weakly prime submodules $W$ which contains $N$. Therefore $a \in (W:H)$ since $(W:H)$ is a prime ideal. So $aH \subseteq wrad_H(N)$ and thus $a \in (wrad_H(N):H)$.

□

**Lemma 2.8.** *If $H$ is finitely generated $T$-module and $N$ is a submodule of $H$, then $wrad_H(N) = H$ if and only if $N = H$.*

*Proof.* Assume that $wrad_H(N) = H$ and $N \neq H$. By Corollary 2.5, $N$ has at least one minimal weakly prime submodule $W$. Hence $H = wrad_H(N) \subseteq W$. Since $W$ is weakly prime, $N = H$. The other side is clear if $N = H$. □

**Corollary 2.9.** *Let $H$ be a finitely generated module, $K$, and $L$ be submodules of $H$. Then $wrad_H(K) + wrad_H(L) = H$ if and only if $K + L = H$.*

*Proof.* Assume that $wrad_H(K) + wrad_H(L) = H$. Then by Lemma 2.8 and Proposition 2.7, $wrad_H(K + L) = H$ and hence $K + L = H$. Conversely if $K + L = H$, then $H = wrad_H(K + L) = wrad_H(wrad_H(K) + wrad_H(L))$. Hence $wrad_H(K) + wrad_H(K) = H$ by Lemma 2.8. □

**Proposition 2.10.** *If $K$ is a proper submodule of a $T$-module $H$ , then $wrad_H(K) = wrad_H(K + pH)$ where $p = \sqrt{(K : H)}$ is a prime ideal.*

*Proof.* It is clear that $wrad_H(K) \subseteq wrad_H(K + pH)$, since $K \subseteq K + pH$. Assume $wrad_H(K) = \bigcap_{i\in I} W_i$ where $W_i$ is weakly prime submodule of $H$ containing $K$. Since $(W_i : H)$ is a prime ideal, $p \subseteq (W_i : H)$ which implies that $K + pH \subseteq W_i$. So, $wrad_H(K + pH) \subseteq \bigcap_{i\in I} W_i = wrad_H(K)$. □

**Corollary 2.11.** *Let $H$ be a finitely generated module, $K$ be a proper submodule of $H$, and $p = (K : H)$ be a maximal ideal of $T$. Then $wrad_H(K)$ is weakly prime submodule and $wrad_H(K) = K + pH$.*

*Proof.* $K + pH \subseteq wrad_H(K + pH) = wrad_H(K)$ by Proposition 2.10. Since $p = (K : H) \subseteq ((K + pH) : H)$, $((K + pH) : H) = p$ or $((K + pH) : H) = T$. If $((K + pH) : H) = T$, then $TH = H \subseteq K + pH \subseteq wrad_H(K)$ which implies that $H = wrad_H(K)$. Since $H$ is finitely generated, $H = K$ by Lemma 2.8. So $((K + pH) : H) = p$. Hence $K + pH$ is a prime submodule by [7] and thus it is a weakly prime submodule containing $K$. Therefore $wrad_H(K) = K + pH$. □

For the rest of this section, the relationship between the weakly prime radical of a submodule and the ideals of the ring $T$ will be examined.

**Lemma 2.12.** *Let $N$ be a proper submodule of a $T$-module $H$ and $I$ be an ideal of $T$. Then $wrad_H(wrad_H(N) : I) = (wrad_H(N) : I)$.*

*Proof.* Assume that $wrad_H(N) = \bigcap_{j\in J} W_j$ for all weakly prime submodules $W_j$ of $H$ containing $N$. Then

$$wrad_H(wrad_H(N) : I) = wrad_H(\textstyle\bigcap_{j\in J} W_j : I) = wrad_H(\bigcap_{j\in J}(W_j : I)) \subseteq \bigcap_{j\in J} wrad_H(W_j : I)$$

by Proposition 2.7. Also Proposition 2.1 implies that

$$\textstyle\bigcap_{j\in J} wrad_H(W_j : I) = \bigcap_{j\in J}(W_j : I) \subseteq (W_j : I)$$

for all weakly prime submodules $W_j$ of $H$ containing $N$. So,

$$I(wrad_H(wrad_H(N) : I)) \subseteq I(W_j : I) \subseteq W_j$$

and it implies that $I(wrad_H(wrad_H(N) : I)) \subseteq wrad_H(N)$. Hence $wrad_H(wrad_H(N) : I) \subseteq (wrad_H(N) : I)$. The other side is clear by Proposition 2.7. □

**Corollary 2.13.** *Let $N$ be a proper submodule of a $T$-module $H$ and $I$ be an ideal of $T$. Then $wrad_H(N : I) = (wrad_H(N) : I)$.*

*Proof.* Since $N \subseteq wrad_H(N)$, $(N : I) \subseteq (wrad_H(N) : I)$ which means that $wrad_H(N : I) \subseteq wrad_H(wrad_H(N) : I) = (wrad_H(N) : I)$ by Lemma 2.12. □

**Proposition 2.14.** *Let $N$ be a proper submodule of a $T$-module $H$, $I$ and $J$ be ideals of $T$. Then*

$$wrad_H(IJN) = wrad_H(IN) \cap wrad_H(JN).$$

*Proof.* Consider the sets $X_1 = \{W : IJN \subseteq W\}$, $X_2 = \{W' : IN \subseteq W'\}$, and $X_3 = \{\bar{W} : JN \subseteq \bar{W}\}$ where $W, W'$ and $\bar{W}$ are weakly prime submodules of $H$. Since each $W \in X_1$ is weakly prime, $IJN \subseteq W$ implies $IN \subseteq W$ or $IJ \subseteq W$. Hence $X_1 = X_2 \cup X_3$ and

$$wrad_H(IJN) = \textstyle\bigcap_{W\in X_1} W = (\bigcap_{W'\in X_2} W') \cap (\bigcap_{\bar{W}\in X_3} \bar{W}).$$

Therefore $wrad_H(IJN) = wrad_H(IN) \cap wrad_H(JN)$.

□

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