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Research Article

A note on nonlinear mixed type product $[E \diamond K, D]_*$ on $*$-algebras

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Abstract

Let $\mathfrak{B}$ be a $*$-algebra with the unity and a nontrivial projection. In the present paper, we show under certain restrictions that if a map $\Psi : \mathfrak{B} \to \mathfrak{B}$ satisfies $\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$ for all $E, K, D \in \mathfrak{B}$, then Ψ is an additive $*$-derivation.

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1. Introduction

Let $\mathfrak{B}$ be a $*$-algebra with unity over the complex field $\mathbb{C}$. For $E, K \in \mathfrak{B}$, let $E \circ K = EK + KE$, $[E, K] = EK - KE$, $E \bullet K = EK + KE^*$, $E \diamond K = E^*K + KE^*$ and $[E, K]_* = EK - KE^*$ denote Jordan product, Lie product, Jordan $*$-product, bi-skew Jordan product and skew Lie product of E and K respectively. An additive map $\Psi : \mathfrak{B} \to \mathfrak{B}$ is known as an additive derivation if $\Psi(EK) = \Psi(E)K + E\Psi(K)$ for all $E, K \in \mathfrak{B}$. Moreover, if $\Psi(E^*) = \Psi(E)^*$ holds for all $E \in \mathfrak{B}$, then Ψ is termed as an additive $*$-derivation. Let $\Psi : \mathfrak{B} \to \mathfrak{B}$ be a mapping (not necessarily additive). Then Ψ is called a nonlinear skew Lie derivation if

$$\Psi([E, K]_*) = [\Psi(E), K]_* + [E, \Psi(K)]_*$$

holds for all $E, K \in \mathfrak{B}$. A map (not necessarily additive) $\Psi : \mathfrak{B} \to \mathfrak{B}$ is said to be a nonlinear mixed Lie triple derivation if

$$\Psi([[E, K]_*, D]) = [[\Psi(E), K]_*, D] + [[E, \Psi(K)]_*, D] + [[E, K]_*, \Psi(D)]$$

holds for all $E, K, D \in \mathfrak{B}$ (for details see [15]). Throughout the text, a map (not necessarily additive) $\Psi : \mathfrak{B} \to \mathfrak{B}$ is called a nonlinear mixed skew triple derivation if

$$\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$$

holds for all $E, K, D \in \mathfrak{B}$.

From the past few years, the evaluation of Jordan product, Jordan $*$-product, skew Lie product, bi-skew Jordan product, mixed Lie product in $*$-algebras have attracted the

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attention of many algebraists (see [1, 2, 4–8, 10, 11, 13, 14]). Darvish et al. [3] showed that if $\mathfrak{B}$ is a prime $*$-algebra and $\Psi : \mathfrak{B} \to \mathfrak{B}$ is a map such that

$$\Psi(E \triangle K \triangle D) = \Psi(E) \triangle K \triangle D + E \triangle \Psi(K) \triangle D + E \triangle K \triangle \Psi(D),$$

for all $E, K, D \in \mathfrak{B}$, where $E \triangle K = E^*K + K^*E$, then Ψ is an additive $*$-derivation. Taghavi et al. [9] showed that if $\mathfrak{B}$ is a prime $*$-algebra and $\Psi : \mathfrak{B} \to \mathfrak{B}$ is a map satisfying

$$\Psi(E \triangleleft_\lambda K \triangleleft_\lambda D) = \Psi(E) \triangleleft_\lambda K \triangleleft_\lambda D + E \triangleleft_\lambda \Psi(K) \triangleleft_\lambda D + E \triangleleft_\lambda K \triangleleft_\lambda \Psi(D),$$

for all $E, K, D \in \mathfrak{B}$ and for all $\lambda \in \mathbb{C}$, where $E \triangleleft_\lambda K = EK + \lambda KE^*$ with $|\lambda| \neq 0, 1$, then Ψ is additive. Moreover, if $\Psi(I)$ is self-adjoint, then Ψ is an additive $*$-derivation. Yaoxian et al. [12] studied the structure of nonlinear mixed Lie triple derivation on factor von Neumann algebras and proved that every nonlinear mixed Lie triple derivation on factor von Neumann algebra is an additive $*$-derivation. Zhou et al. [15], extended their result to prime $*$-algebras and obtained the same conclusion.

Inspired by the results mentioned above, in this paper we characterize the form of nonlinear mixed skew triple derivations on $*$-algebras. Precisely, we show that under certain conditions, every nonlinear mixed skew triple derivation on $*$-algebra is an additive $*$-derivation.

2. Main result

We begin with our main result.

Theorem 2.1. *Let $\mathfrak{B}$ be a unital $*$-algebra with a non trivial projection P_1 satisfying*

$$K\mathfrak{B}P_1 = 0 \ \text{ implies } K = 0 \tag{2.1}$$

and

$$K\mathfrak{B}(I - P_1) = 0 \ \text{ implies } K = 0, \tag{2.2}$$

where $K \in \mathfrak{B}$. Suppose that a map $\Psi : \mathfrak{B} \to \mathfrak{B}$ satisfies

$$\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$$

for all $E, K, D \in \mathfrak{B}$. Then Ψ is additive. Moreover, if $\Psi(I)$ is self-adjoint, then Ψ is a $*$-derivation.

Proof. Let $P_2 = I - P_1$ and $\mathfrak{B}_{ij} = P_i\mathfrak{B}P_j$ for $i, j = 1, 2$. By Peirce decomposition of $\mathfrak{B}$, we have $\mathfrak{B} = \mathfrak{B}_{11} \oplus \mathfrak{B}_{12} \oplus \mathfrak{B}_{21} \oplus \mathfrak{B}_{22}$. Note that any $E \in \mathfrak{B}$ can be written as $E = E_{11} + E_{12} + E_{21} + E_{22}$, where $E_{ij} \in \mathfrak{B}_{ij}$ for $i, j = 1, 2$. Now to show the additivity of Ψ on $\mathfrak{B}$, we use the above partition on $\mathfrak{B}$ and establish some lemmas that will show that Ψ is additive on each $\mathfrak{B}_{ij}$ for $i, j = 1, 2$. Also the following multiplicative relations are satisfied:

(i) $\mathfrak{B}_{ij}\mathfrak{B}_{jl} \subseteq \mathfrak{B}_{il} \quad (i, j, l = 1, 2)$.
(ii) $\mathfrak{B}_{ij}\mathfrak{B}_{kl} = 0 \ (k = 1, 2) \ \ if \ j \neq k$.

□

So our main Theorem 2.1 is a consequence of the following lemmas.

Lemma 2.2. $\Psi(0) = 0$.

Proof. It is trivial that
$\Psi(0) = \Psi([0 \diamond 0, \ 0]_*) = [\Psi(0) \diamond 0, \ 0]_* + [0 \diamond \Psi(0), \ 0]_* + [0 \diamond 0, \ \Psi(0)]_* = 0.$ □

Lemma 2.3. *Let $E_{12} \in \mathfrak{B}_{12}$ and $E_{21} \in \mathfrak{B}_{21}$. Then $\Psi(E_{12} + E_{21}) = \Psi(E_{12}) + \Psi(E_{21})$.*

Proof. Let $K = \Psi(E_{12}+E_{21})-\Psi(E_{12})-\Psi(E_{21})$. Since $[I\diamond E_{21}, P_2]_* = 0$, utilizing Lemma 2.2, we have

$$\begin{aligned}\Psi([I\diamond(E_{12}+E_{21}), P_2]_*) &= \Psi([I\diamond E_{12}, P_2]_*) + \Psi([I\diamond E_{21}, P_2]_*)\\ &= [\Psi(I)\diamond E_{12}, P_2]_* + [I\diamond \Psi(E_{12}), P_2]_* + [I\diamond E_{12}, \Psi(P_2)]_*\\ &\quad +[\Psi(I)\diamond E_{21}, P_2]_* + [I\diamond \Psi(E_{21}), P_2]_* + [I\diamond E_{21}, \Psi(P_2)]_*.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Psi([I\diamond(E_{12}+E_{21}), P_2]_*) &= [\Psi(I)\diamond(E_{12}+E_{21}), P_2]_* + [I\diamond\Psi(E_{12}+E_{21}), P_2]_*\\ &\quad +[I\diamond(E_{12}+E_{21}), \Psi(P_2)]_*.\end{aligned}$$

From the last two relations, we infer that $[I\diamond K, P_2]_* = 0$, i.e., $KP_2 - P_2K^* = 0$. Multiplying the previous relation by P_1 from left, we get $P_1KP_2 = 0$. Analogously, we can show $P_2KP_1 = 0$.

Now, again since $[I\diamond i(P_1-P_2),\ E_{21}]_* = 0$, where i is the imaginary unit, invoking Lemma 2.2, we have

$$\begin{aligned}\Psi([I\diamond i(P_1-P_2), E_{12}+E_{21}]_*) &= \Psi([I\diamond i(P_1-P_2), E_{12}]_*) + \Psi([I\diamond i(P_1-P_2), E_{21}]_*)\\ &= [\Psi(I)\diamond i(P_1-P_2), E_{12}]_* + [I\diamond\Psi(i(P_1-P_2)), E_{12}]_*\\ &\quad +[I\diamond i(P_1-P_2), \Psi(E_{12})]_* + [\Psi(I)\diamond i(P_1-P_2), E_{21}]_*\\ &\quad +[I\diamond\Psi(i(P_1-P_2)), E_{21}]_* + [I\diamond i(P_1-P_2), \Psi(E_{21})]_*.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Psi([I\diamond i(P_1-P_2), E_{12}+E_{21}]_*) &= [\Psi(I)\diamond i(P_1-P_2), E_{12}+E_{21}]_*\\ &\quad +[I\diamond\Psi(i(P_1-P_2)), E_{12}+E_{21}]_*\\ &\quad +[I\diamond i(P_1-P_2), \Psi(E_{12}+E_{21})]_*.\end{aligned}$$

From the previous two relations, we get $[I\diamond i(P_1-P_2), K]_* = 0$, i.e., $2iP_1K - 2iP_2K + 2iKP_1 - 2iKP_2 = 0$. Multiplying the previous relation by P_1 from both left and right, we get $P_1KP_1 = 0$. Analogously, multiplying the previous relation by P_2 from both left and right, we get $P_2KP_2 = 0$. Hence, $K = 0$, i.e., $\Psi(E_{12}+E_{21}) = \Psi(E_{12}) + \Psi(E_{21})$. □

Lemma 2.4. *For every* $E_{11}\in\mathfrak{B}_{11}, E_{12}\in\mathfrak{B}_{12}$, $E_{21}\in\mathfrak{B}_{21}$ *and* $E_{22}\in\mathfrak{B}_{22}$, *we have*

(i) $\Psi(E_{11}+E_{12}+E_{21}) = \Psi(E_{11})+\Psi(E_{12})+\Psi(E_{21})$.

(ii) $\Psi(E_{12}+E_{21}+E_{22}) = \Psi(E_{12})+\Psi(E_{21})+\Psi(E_{22})$.

Proof. Let $K = \Psi(E_{11}+E_{12}+E_{21}) - \Psi(E_{11}) - \Psi(E_{12}) - \Psi(E_{21})$. On one hand, we have

$$\begin{aligned}\Psi([iI\diamond P_2,\ E_{11}+E_{12}+E_{21}]_*) &= [\Psi(iI)\diamond P_2,\ E_{11}+E_{12}+E_{21}]_*\\ &\quad +[iI\diamond\Psi(P_2),\ E_{11}+E_{12}+E_{21}]_*\\ &\quad +[iI\diamond P_2,\ \Psi(E_{11}+E_{12}+E_{21})]_*.\end{aligned}$$

On the other hand, invoking Lemma 2.3 and using $[iI\diamond P_2, E_{11}]_* = 0$, we have

$$\begin{aligned}\Psi([iI\diamond P_2,\ E_{11}+E_{12}+E_{21}]_*) &= \Psi([iI\diamond P_2,\ E_{11}]_*) + \Psi([iI\diamond P_2,\ E_{12}]_*)\\ &\quad +\Psi([iI\diamond P_2,\ E_{21}]_*)\\ &= [\Psi(iI)\diamond P_2,\ E_{11}]_* + [iI\diamond\Psi(P_2),\ E_{11}]_*\\ &\quad +[iI\diamond P_2,\ \Psi(E_{11})]_* + [\Psi(iI)\diamond P_2, E_{12}]_*\\ &\quad +[iI\diamond\Psi(P_2), E_{12}]_* + [iI\diamond P_2, \Psi(E_{12})]_*\\ &\quad +[\Psi(iI)\diamond P_2,\ E_{21}]_* + [iI\diamond\Psi(P_2),\ E_{21}]_*\\ &\quad +[iI\diamond P_2,\ \Psi(E_{21})]_*.\end{aligned}$$

From the last two relations, we infer that $[iI\diamond P_2,\ K]_* = 0$, i.e., $2iP_2K + 2iKP_2 = 0$. Solving this, we obtain $P_2KP_1 = P_2KP_2 = P_1KP_2 = 0$. Now, again since $[I\diamond i(P_1-P_2), E_{21}]_*$

$= 0 = [I \diamond i(P_1 - P_2), E_{12}]_*$, where i is the imaginary unit, invoking Lemma 2.2, we have

$$\begin{aligned}\Psi([I \diamond i(P_1 - P_2), E_{11} + E_{12} + E_{21}]_*) &= \Psi([I \diamond i(P_1 - P_2), E_{11}]_*) + \Psi([I \diamond i(P_1 \\ &\quad -P_2), E_{12}]_*) + \Psi([I \diamond i(P_1 - P_2), E_{21}]_*) \\ &= [\Psi(I) \diamond i(P_1 - P_2), E_{11}]_* + [I \diamond \Psi(i(P_1 \\ &\quad -P_2)), E_{11}]_* + [I \diamond i(P_1 - P_2), \Psi(E_{11})]_* \\ &\quad +[\Psi(I) \diamond i(P_1 - P_2), E_{12}]_* + [I \diamond \Psi(i(P_1 \\ &\quad -P_2)), E_{12}]_* + [I \diamond i(P_1 - P_2), \Psi(E_{12})]_* \\ &\quad +[\Psi(I) \diamond i(P_1 - P_2), E_{21}]_* + [I \diamond \Psi(i(P_1 \\ &\quad -P_2)), E_{21}]_* + [I \diamond i(P_1 - P_2), \Psi(E_{21})]_*.\end{aligned}$$

On the other way, we have

$$\begin{aligned}\Psi([I \diamond i(P_1 - P_2), E_{11} + E_{12} + E_{21}]_*) &= [\Psi(I) \diamond i(P_1 - P_2), E_{11} + E_{12} + E_{21}]_* \\ &\quad +[I \diamond \Psi(i(P_1 - P_2)), E_{11} + E_{12} + E_{21}]_* \\ &\quad +[I \diamond i(P_1 - P_2), \Psi(E_{11} + E_{12} + E_{21})]_*.\end{aligned}$$

From the last two relations, we obtain $[I \diamond i(P_1 - P_2), K]_* = 0$, i.e., $2iP_1K - 2iP_2K + 2iKP_1 - 2iKP_2 = 0$. Multiplying the previous relation by P_1 from both left and right, we get $P_1KP_1 = 0$. Hence, $K = 0$, i.e., $\Psi(E_{11} + E_{12} + E_{21}) = \Psi(E_{11}) + \Psi(E_{12}) + \Psi(E_{21})$. In the similar way, we can prove other part also. □

Lemma 2.5. *For any $E_{ij} \in \mathfrak{B}_{ij}, 1 \leq i, j \leq 2$, we have*

$$\Psi(\sum_{i,j=1}^{2} E_{ij}) = \sum_{i,j=1}^{2} \Psi(E_{ij}).$$

Proof. Let $K = \Psi(E_{11} + E_{12} + E_{21} + E_{22}) - \Psi(E_{11}) - \Psi(E_{12}) - \Psi(E_{21}) - \Psi(E_{22})$. On one hand, we have

$$\begin{aligned}\Psi([I \diamond iP_2,\ E_{11} + E_{12} + E_{21} + E_{22}]_*) &= [\Psi(I) \diamond iP_2,\ E_{11} + E_{12} + E_{21} + E_{22}]_* \\ &\quad +[I \diamond \Psi(iP_2),\ E_{11} + E_{12} + E_{21} + E_{22}]_* \\ &\quad +[I \diamond iP_2,\ \Psi(E_{11} + E_{12} + E_{21} + E_{22})]_*.\end{aligned}$$

On the other hand, since $[I \diamond iP_2, E_{11}]_* = 0$, invoking Lemmas 2.2 and 2.4, we have

$$\begin{aligned}\Psi([I \diamond iP_2, E_{11} + E_{12} + E_{21} + E_{22}]_*) &= \Psi([I \diamond iP_2, E_{11}]_*) + \Psi([I \diamond iP_2, E_{12}]_*) \\ &\quad +\Psi([I \diamond iP_2, E_{21}]_*) + \Psi([I \diamond iP_2, E_{22}]_*) \\ &= [\Psi(I) \diamond iP_2, E_{11}]_* + [I \diamond \Psi(iP_2), E_{11}]_* \\ &\quad +[I \diamond iP_2, \Psi(E_{11})]_* + [\Psi(I) \diamond iP_2, E_{12}]_* \\ &\quad +[I \diamond \Psi(iP_2), E_{12}]_* + [I \diamond iP_2, \Psi(E_{12})]_* \\ &\quad +[\Psi(I) \diamond iP_2, E_{21}]_* + [I \diamond \Psi(iP_2), E_{21}]_* \\ &\quad +[I \diamond iP_2, \Psi(E_{21})]_* + [\Psi(I) \diamond iP_2, E_{22}]_* \\ &\quad +[I \diamond \Psi(iP_2), E_{22}]_* + [I \diamond iP_2, \Psi(E_{22})]_*.\end{aligned}$$

From the last two relations, we get $[I \diamond iP_2, K]_* = 0$. Hence $P_1KP_2 = P_2KP_1 = P_2KP_2 = 0$. Analogously, we can show that $P_1KP_1 = 0$. Thus $K = 0$, i.e., $\Psi(E_{11} + E_{12} + E_{21} + E_{22}) = \Psi(E_{11}) + \Psi(E_{12}) + \Psi(E_{21}) + \Psi(E_{22})$. □

Lemma 2.6. *For any $E_{ij}, N_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j, 1 \leq i, j \leq 2, \Psi(E_{ij} + N_{ij}) = \Psi(E_{ij}) + \Psi(N_{ij})$.*

Proof. Let $N = \Psi(E_{ij} + N_{ij}) - \Psi(E_{ij}) - \Psi(N_{ij})$. Since

$$[\frac{I}{2} \diamond (P_i + E_{ij}), P_j + N_{ij}]_* = N_{ij} + E_{ij} - E_{ij}^* - N_{ij}E_{ij}^*$$

Invoking Lemma 2.5, we get

$$\begin{aligned}
\Psi(N_{ij} + E_{ij}) &+ \Psi(-E_{ij}^*) + \Psi(-N_{ij}E_{ij}^*) \\
&= \Psi([\frac{I}{2} \diamond (P_i + E_{ij}),\ P_j + N_{ij}]_*) \\
&= [\Psi(\frac{I}{2}) \diamond (P_i + E_{ij}),\ P_j + N_{ij}]_* \\
&\quad + [\frac{I}{2} \diamond \Psi(P_i + E_{ij}),\ P_j + N_{ij}]_* \\
&\quad + [\frac{I}{2} \diamond (P_i + E_{ij}),\ \Psi(P_j + N_{ij})]_* \\
&= [\Psi(\frac{I}{2}) \diamond (P_i + E_{ij}),\ P_j + N_{ij}]_* \\
&\quad + [\frac{I}{2} \diamond (\Psi(P_i) + \Psi(E_{ij})),\ P_j + N_{ij}]_* \\
&\quad + [\frac{I}{2} \diamond (P_i + E_{ij}),\ \Psi(P_j) + \Psi(N_{ij})]_* \\
&= \Psi([\frac{I}{2} \diamond P_i,\ P_j]_*) + \Psi([\frac{I}{2} \diamond P_i,\ N_{ij}]_*) \\
&\quad + \Psi([\frac{I}{2} \diamond E_{ij},\ P_j]_*) + \Psi([\frac{I}{2} \diamond E_{ij},\ N_{ij}]_*) \\
&= \Psi(N_{ij}) + \Psi(E_{ij} - E_{ij}^*) + \Psi(-N_{ij}E_{ij}^*) \\
&= \Psi(N_{ij}) + \Psi(E_{ij}) + \Psi(-E_{ij}^*) + \Psi(-N_{ij}E_{ij}^*).
\end{aligned}$$

Solving this, we arrive at $\Psi(E_{ij} + N_{ij}) = \Psi(E_{ij}) + \Psi(N_{ij})$.

Lemma 2.7. *For any $E_{ii}, N_{ii} \in \mathfrak{B}_{ii}, 1 \leq i \leq 2$, we have*

$$\Psi(E_{ii} + N_{ii}) = \Psi(E_{ii}) + \Psi(N_{ii}).$$

Proof. Let $T = \Psi(E_{ii}+N_{ii})-\Psi(E_{ii})-\Psi(N_{ii})$. Since $[iP_j \diamond I,\ E_{ii}]_* = 0$ for $i \neq j$, invoking Lemma 2.2, we have

$$\begin{aligned}
\Psi([iP_j \diamond I,\ (E_{ii} + N_{ii})]_*) &= \Psi([iP_j \diamond I, E_{ii}]_*) + \Psi([iP_j \diamond I,\ N_{ii}]_*) \\
&= [\Psi(iP_j) \diamond I,\ E_{ii}]_* + [iP_j \diamond \Psi(I), E_{ii}]_* + [iP_j \diamond I, \Psi(E_{ii})]_* \\
&\quad + [\Psi(iP_j) \diamond I, N_{ii}]_* + [iP_j \diamond \Psi(I), N_{ii}]_* + [iP_j \diamond I, \Psi(N_{ii})]_*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Psi([iP_j \diamond I,\ (E_{ii} + N_{ii})]_*) &= [\Psi(iP_j) \diamond I,\ (E_{ii} + N_{ii})]_* + [iP_j \diamond \Psi(I),\ (E_{ii} + N_{ii})]_* \\
&\quad +[iP_j \diamond I,\ \Psi(E_{ii} + N_{ii})]_*.
\end{aligned}$$

From the last two relations, we conclude that $[iP_j \diamond I, T]_* = 0$. It follows that $P_jTP_j = P_jTP_i = P_iTP_j = 0$.
Next, for any $X_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j$, we have

$$\begin{aligned}
\Psi([I \diamond (E_{ii} + N_{ii}), X_{ij}]_*) &= [\Psi(I) \diamond (E_{ii} + N_{ii}), X_{ij}]_* + [I \diamond \Psi(E_{ii} + N_{ii}), X_{ij}]_* \\
&\quad +[I \diamond (E_{ii} + N_{ii}), \Psi(X_{ij})]_*.
\end{aligned}$$

On the other hand, using Lemma 2.6, we have

$$\begin{aligned}\Psi([I \diamond (E_{ii} + N_{ii}),\ X_{ij}]_*) &= \Psi([2(E_{ii} + N_{ii}),\ X_{ij}]_*)\\ &= \Psi([2E_{ii} + 2N_{ii},\ X_{ij}]_*)\\ &= \Psi(2E_{ii}X_{ij} + 2N_{ii}X_{ij})\\ &= \Psi([2E_{ii},\ X_{ij}]_* + [2N_{ii},\ X_{ij}]_*)\\ &= \Psi([2E_{ii},\ X_{ij}]_*) + \Psi([2N_{ii},\ X_{ij}]_*)\\ &= \Psi([I \diamond E_{ii},\ X_{ij}]_*) + \Psi([I \diamond N_{ii}, X_{ij}]_*)\\ &= [\Psi(I) \diamond E_{ii}, X_{ij}]_* + [I \diamond \Psi(E_{ii}), X_{ij}]_* + [I \diamond E_{ii},\ \Psi(X_{ij})]_*\\ &\quad +[\Psi(I) \diamond N_{ii}, X_{ij}]_* + [I \diamond \Psi(N_{ii}), X_{ij}]_* + [I \diamond N_{ii}, \Psi(X_{ij})]_*.\end{aligned}$$

From the last two relations, we obtain that $[I \diamond T, X_{ij}]_* = 0$. Now solving this, we get $TX_{ij} - X_{ij}T^* = 0$, which implies $T_{ii}X_{ij} = 0$ and it follows from conditions (2.1) and (2.2) that $T_{ii} = 0$. Thus $T = 0$. □

Lemma 2.8. Ψ *is additive .*

Proof. For any $E, N \in \mathfrak{B}$, we write $E = E_{11} + E_{12} + E_{21} + E_{22}$ and $N = N_{11} + N_{12} + N_{21} + N_{22}$. Invoking Lemmas 2.5 - 2.7, we get

$$\begin{aligned}\Psi(E + N) &= \Psi(E_{11} + E_{12} + E_{21} + E_{22} + N_{11} + N_{12} + N_{21} + N_{22})\\ &= \Psi(E_{11} + N_{11}) + \Psi(E_{12} + N_{12}) + \Psi(E_{21} + N_{21}) + \Psi(E_{22} + N_{22})\\ &= \Psi(E_{11}) + \Psi(N_{11}) + \Psi(E_{12}) + \Psi(N_{12}) + \Psi(E_{21}) + \Psi(N_{21})\\ &\quad +\Psi(E_{22}) + \Psi(N_{22})\\ &= \Psi(E_{11} + E_{12} + E_{21} + E_{22}) + \Psi(N_{11} + N_{12} + N_{21} + N_{22})\\ &= \Psi(E) + \Psi(N).\end{aligned}$$

Hence the additivity of Ψ follows from the above lemmas. □

Now in the rest of the paper, we show that Ψ is a $*$-derivation.

Lemma 2.9. $\Psi(I)$ *is a central element of* $\mathfrak{B}$*, i.e.,* $\Psi(I)L = L\Psi(I)$ *for all* $L \in \mathfrak{B}$.

Proof. We have $[I \diamond I, L]_* = 0$. Now applying Lemma 2.2, we have

$$\begin{aligned}0 &= \Psi([I \diamond I, L]_*)\\ &= [\Psi(I) \diamond I, L]_* + [I \diamond \Psi(I), L]_* + [I \diamond I, \Psi(L)]_*\\ &= [2\Psi(I)^*, L]_* + [2\Psi(I), L]_* + [2I, \Psi(L)]_*\\ &= 2\Psi(I)^*L - 2L\Psi(I) + 2\Psi(I)L - 2L\Psi(I)^*.\end{aligned}$$

Using given hypothesis, we get

$$0 = 2\Psi(I)L - 2L\Psi(I) + 2\Psi(I)L - 2L\Psi(I).$$

Which implies that $\Psi(I)L = L\Psi(I)$ for all $L \in \mathfrak{B}$. □

Lemma 2.10. *(i)* $P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2$.
(ii) $P_2\ \Psi(P_1)P_1 = -P_2\Psi(P_2)P_1$.
(iii) $P_1\Psi(P_2)P_1 = P_2\Psi(P_1)P_2 = 0$.

Proof. (i) We have $[I \diamond P_1, P_2]_* = 0$. Using given hypothesis, Lemmas 2.2 and 2.9, we get

$$\begin{aligned}0 &= \Psi([I \diamond P_1, P_2]_*)\\ &= [\Psi(I) \diamond P_1, P_2]_* + [I \diamond \Psi(P_1), P_2]_* + [I \diamond P_1, \Psi(P_2)]_*\\ &= [2\Psi(I)P_1, P_2]_* + [2\Psi(P_1), P_2]_* + [2P_1, \Psi(P_2)]_*\\ &= 2\Psi(P_1)P_2 - 2P_2\Psi(P_1)^* + 2P_1\Psi(P_2) - 2\Psi(P_2)P_1.\end{aligned}$$

Multiplying the previous relation by P_1 from left and by P_2 from right, we get

$$P_1\Psi(P_1)P_2 = -P_1\Psi(P_2)P_2.$$

(*ii*) Since $[P_1 \diamond I, P_2]_* = 0$, using given hypothesis and applying Lemmas 2.2 and 2.9, we get

$$\begin{aligned} 0 &= \Psi([P_1 \diamond I, P_2]_*) \\ &= [\Psi(P_1) \diamond I, P_2]_* + [P_1 \diamond \Psi(I), P_2]_* + [P_1 \diamond I, \Psi(P_2)]_* \\ &= 2\Psi(P_1)^* P_2 - 2P_2\Psi(P_1) + 2P_1\Psi(P_2) - 2\Psi(P_2)P_1. \end{aligned}$$

Multiplying the previous relation by P_2 from left and by P_1 from right, we get

$$P_2\Psi(P_2)P_1 = -P_2\Psi(P_1)P_1.$$

(*iii*) For $1 \leq i \neq j \leq 2$, we have $[iP_i \diamond I, P_j]_* = 0$. Now utilizing given hypothesis and Lemmas 2.2, 2.9, we have

$$\begin{aligned} 0 &= \Psi([iP_i \diamond I, P_j]_*) \\ &= [\Psi(iP_i) \diamond I, P_j]_* + [iP_i \diamond \Psi(I), P_j]_* + [iP_i \diamond I, \Psi(P_j)]_* \\ &= [2\Psi(iP_i)^*,\ P_j]_* + [-2(iP_i), \Psi(P_j)]_* \\ &= 2\Psi(iP_i)^* P_j - 2P_j\Psi(iP_i) - 2iP_i\Psi(P_j) - 2i\Psi(P_j)P_i. \end{aligned}$$

Multiplying above relation by P_i from both right and left, we get $P_i\Psi(P_j)P_i = 0$. Thus $P_1\Psi(P_2)P_1 = P_2\Psi(P_1)P_2 = 0$. □

Lemma 2.11. $P_1\Psi(P_1)P_1 = P_2\Psi(P_2)P_2 = 0$.

Proof. For every $E_{21} \in \mathfrak{B}_{21}$, applying Lemma 2.8, we have

$$\Psi([I \diamond P_2,\ E_{21}]_*) = 2\Psi(E_{21}).$$

On the other hand from given hypothesis and Lemma 2.9, we have

$$\begin{aligned} \Psi([I \diamond P_2,\ E_{21}]_*) &= [\Psi(I) \diamond P_2,\ E_{21}]_* + [I \diamond \Psi(P_2),\ E_{21}]_* + [I \diamond P_2,\ \Psi(E_{21})]_* \\ &= [2\Psi(I)P_2,\ E_{21}]_* + [2\Psi(P_2),\ E_{21}]_* + [2P_2,\ \Psi(E_{21})]_* \\ &= 2\Psi(I)E_{21} + 2\Psi(P_2)E_{21} - 2E_{21}\Psi(P_2)^* + 2P_2\Psi(E_{21}) - 2\Psi(E_{21})P_2. \end{aligned}$$

Using the last two relations, we infer that

$$2\Psi(I)E_{21} + 2\Psi(P_2)E_{21} - 2E_{21}\Psi(P_2)^* + 2P_2\Psi(E_{21}) - 2\Psi(E_{21})P_2 - 2\Psi(E_{21}) = 0.$$

Multiplying above relation by P_1 from the right and by P_2 from the left, we get

$$2P_2\Psi(I)E_{21} + 2P_2\Psi(P_2)E_{21} - 2E_{21}\Psi(P_2)^* P_1 = 0.$$

Using Lemmas 2.9 and 2.10, we get

$$\Psi(I)E_{21} + P_2\Psi(P_2)E_{21} = 0. \tag{2.3}$$

Similarly, for every $E_{21} \in \mathfrak{B}_{21}$, we have

$$[P_2 \diamond I,\ E_{21}]_* = 2E_{21}.$$

Applying Lemma 2.8, we get

$$\Psi([P_2 \diamond I,\ E_{21}]_*) = 2\Psi(E_{21}).$$

Similarly as above, invoking Lemmas 2.9 and 2.10, we have

$$\Psi(I)E_{21} + P_2\Psi(P_2)^* E_{21} = 0. \tag{2.4}$$

Also for any $E_{21} \in \mathfrak{B}_{21}$, we have

$$[P_2 \diamond P_2, E_{21}]_* = 2E_{21}.$$

Applying Lemma 2.8, we get

$$\Psi([P_2 \diamond P_2, E_{21}]_*) = 2\Psi(E_{21}).$$

Similarly as above, using Lemma 2.10, we get

$$P_2\Psi(P_2)^*E_{21} + P_2\Psi(P_2)E_{21} = 0. \tag{2.5}$$

Solving (2.3), (2.4) and (2.5), we get $P_2\Psi(P_2)E_{21} = 0$. Now using (2.1) and (2.2), we get $P_2\Psi(P_2)P_2 = 0$. Similarly, we can show that $P_1\Psi(P_1)P_1 = 0$. □

Lemma 2.12. *(i)* $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1,\ \Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1$.
(ii) $\Psi(I) = 0$.

Proof. *(i)* By Peirce decomposition, we have

$$\Psi(P_1) = P_1\Psi(P_1)P_1 + P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1 + P_2\Psi(P_1)P_2.$$

In view of Lemmas 2.10 and 2.11, it follows that $\Psi(P_1) = P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1$. Analogously, we can show that $\Psi(P_2) = P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1$.
(ii) Invoking Lemmas 2.8, 2.10 and 2.11, we have

$$\begin{aligned}\Psi(I) = \Psi(P_2 + P_1) &= \Psi(P_2) + \Psi(P_1)\\ &= P_1\Psi(P_2)P_2 + P_2\Psi(P_2)P_1 + P_1\Psi(P_1)P_2 + P_2\Psi(P_1)P_1\\ &= 0.\end{aligned}$$

□

Lemma 2.13. *Ψ preserves $'*'$, i.e., $\Psi(E^*) = \Psi(E)^*$ for all $E \in \mathfrak{B}$.*

Proof. By Lemma 2.8, we have

$$\Psi([I \diamond E, I]_*) = \Psi(2E - 2E^*) = 2\Psi(E) - 2\Psi(E^*).$$

On the other hand, using Lemma 2.12, we have

$$\Psi([I \diamond E, I]_*) = [I \diamond \Psi(E), I]_* = 2\Psi(E) - 2\Psi(E)^*.$$

Comparing the above two relations, we get
$\Psi(E^*) = \Psi(E)^*$ for all $E \in \mathfrak{B}$.

□

Lemma 2.14. *(i)$\Psi(iI) = 0$.*
(ii)$\Psi(-iI) = 0$, where i is the imaginary unit.

Proof. *(i)* Since $[I \diamond iI, iI]_* = -4I$, applying Lemmas 2.8 and 2.12, we have

$$\Psi([I \diamond iI, iI]_*) = 0,$$

which implies that

$$[I \diamond \Psi(iI), iI]_* + [I \diamond iI, \Psi(iI)]_* = 0.$$

Now using Lemmas 2.8 and 2.13, we have $8i\Psi(iI) = 0$, thus $\Psi(iI) = 0$.
(ii) Analogously, we can show that $\Psi(-iI) = 0$. □

Lemma 2.15. *(i)* $\Psi(-iE) = -i\Psi(E)$
(ii) $\Psi(iE) = i\Psi(E)$*, where i is the imaginary unit.*

Proof. *(i)* Since $[(-iE) \diamond I,\ I]_* = [E \diamond iI,\ I]_*$. Therefore,

$$\Psi([(-iE) \diamond I,\ I]_*) = \Psi([E \diamond iI,\ I]_*).$$

Invoking Lemmas 2.12 and 2.14, we have

$$\Psi(-iE)^* - \Psi(-iE) = i\Psi(E)^* + i\Psi(E).$$

Also, since $[-iE \diamond -iI,\ I]_* = [-I \diamond E,\ I]_*$. Therefore,

$$\Psi([(-iE) \diamond -iI,\ I]_*) = \Psi([-I \diamond E,\ I]_*).$$

Now by Lemmas 2.12 and 2.14, we have

$$-i\Psi(-iE)^* - i\Psi(-iE) = \Psi(E)^* - \Psi(E).$$

Multiplying both sides of the above relation by iI, we get

$$\Psi(-iE)^* + \Psi(-iE) = i\Psi(E)^* - i\Psi(E).$$

Solving the above two relations, we get $\Psi(-iE) = -i\Psi(E)$ for all $E \in \mathfrak{B}$.
(ii) Analogously, we can show that $\Psi(iE) = i\Psi(E)$. □

Lemma 2.16. Ψ *is a derivation.*

Proof. For every E, K $\in \mathfrak{B}$, we have $[I \diamond E, K]_* = 2(EK - KE^*)$. So, applying Lemmas 2.8 and 2.12, we have

$$\begin{aligned} 2\Psi(EK - KE^*) &= \Psi([I \diamond E, K]_*) \\ &= [I \diamond \Psi(E), K]_* + [I \diamond E, \Psi(K)]_* \\ &= 2\Psi(E)K - 2K\Psi(E)^* + 2E\Psi(K) - 2\Psi(K)E^*. \end{aligned}$$

Therefore,

$$\Psi(EK - KE^*) = \Psi(E)K - K\Psi(E)^* + E\Psi(K) - \Psi(K)E^*. \tag{2.6}$$

Also, we have $[I \diamond (-iE), iK]_* = 2(EK + KE^*)$. So, invoking Lemmas 2.8, 2.12, 2.13 and 2.15, we have

$$\begin{aligned} 2\Psi(EK + KE^*) &= \Psi([I \diamond (-iE), iK]_*) \\ &= [I \diamond \Psi(-iE), iK]_* + [I \diamond (-iE), \Psi(iK)]_* \\ &= 2\Psi(E)K + 2K\Psi(E)^* + 2\Psi(K)E^* + 2E\Psi(K). \end{aligned}$$

Therefore

$$\Psi(EK + KE^*) = \Psi(E)K + K\Psi(E)^* + \Psi(K)E^* + E\Psi(K). \tag{2.7}$$

Adding (2.6) and (2.7), we get

$$\Psi(EK) = \Psi(E)K + E\Psi(K).$$

Hence Ψ is a derivation. This completes the proof of Theorem 2.1. □

3. Corollaries

Let $\mathfrak{B}$ be an algebra, we say that it is prime if for each E, K $\in \mathfrak{B}$, $E\mathfrak{B}K = 0$, implies either $E = 0$ or $K = 0$. So, it is very simple to see that every prime ∗-algebra satisfies conditions (2.1) and (2.2) in the Theorem 2.1. So we have the following corollary.

Corollary 3.1. *Suppose* $\mathfrak{B}$ *is a unital prime ∗-algebra with a non-trivial projection. If* $\Psi : \mathfrak{B} \to \mathfrak{B}$ *satisfies*

$$\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$$

for all $E, K, D \in \mathfrak{B}$*. Then* Ψ *is additive. Moreover, if* $\Psi(I)$ *is self-adjoint, then* Ψ *is a ∗-derivation.*

Consider $\mathcal{H}$, as a complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators and let $\mathbb{T}(\mathcal{H})$ be its subalgebra consisting of finite rank operators. It is well known that $\mathbb{T}(\mathcal{H})$ forms a ∗-closed ideal of $\mathfrak{B}(\mathcal{H})$. A subalgebra $\mathcal{F}$ of $\mathfrak{B}(\mathcal{H})$ is called a standard operator algebra if $\mathbb{T}(\mathcal{H}) \subseteq \mathcal{F}$. As a result, we have the following immediate corollary.

Corollary 3.2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{F}$ be a unital standard operator algebra on $\mathcal{H}$ such that $\mathcal{F}$ is closed under adjoint operation. Suppose that $\Psi : \mathcal{F} \to \mathcal{F}$ is a map satisfying*

$$\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$$

for all $E, K, D \in \mathcal{F}$. Then Ψ is additive. Moreover, if $\Psi(I)$ is self-adjoint, then Ψ is a $*$-derivation.

A von Neumann algebra $\mathcal{Z}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator. Also, it is well known that if a von Neumann algebra $\mathcal{Z}$ has no central summands of type I_1, then $\mathcal{Z}$ satisfies conditions (2.1) and (2.2) of Theorem 2.1. As a result, we have the following immediate corollary.

Corollary 3.3. *Let $\mathcal{Z}$ be a von Neumann algebra with no central summands of type I_1. If the map $\Psi : \mathcal{Z} \to \mathcal{Z}$ satisfies*

$$\Psi([E \diamond K, D]_*) = [\Psi(E) \diamond K, D]_* + [E \diamond \Psi(K), D]_* + [E \diamond K, \Psi(D)]_*$$

for all $E, K, D \in \mathcal{Z}$, then Ψ is additive. Moreover, if $\Psi(I)$ is self-adjoint, then Ψ is a $*$-derivation.

□

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