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Research Article

Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups

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Abstract

In this paper, we completely classify Ricci bi-conformal vector fields on simply-connected five-dimensional two-step nilpotent Lie groups which are also connected and we show which of them are the Killing vector fields and gradient vector fields.

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1. Introduction

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. A vector field X on a Riemannian manifold (M, g) is said to be a Killing field [7] if $\mathcal{L}_X g = 0$ where $\mathcal{L}_X$ is the Lie derivative in the direction of X. Recently, various generalizations of Killing vector fields have been studied. For instance, conformal vector fields [10, 17] are generalized of Killing vector fields and a conformal vector field X on a Riemannian manifold (M, g) is defined by $\mathcal{L}_X g = 2\psi g$ for some smooth function ψ. If the potential function $\psi = 0$ then X is a Killing vector field. A vector field X on M is called a Kerr-Schild vector field if $\mathcal{L}_X g = \alpha l \otimes l$, $\mathcal{L}_X l = \beta l$, where l is a null 1-form field and α, β are smooth functions over M. Also, the generalized Kerr-Schild vector field is determined by

$$\mathcal{L}_X g = \alpha g + \beta l \otimes l, \qquad \mathcal{L}_X l = \gamma l,$$

where α, β, γ are smooth functions. Coll et al. [8] studied the generalized Kerr-Schild vector field. A symmetric tensor field h on M is said to be a square root of g if $h_{ik} h^k_j = g_{ij}$. Garcia-Parrado and Senovilla [11] introduced bi-conformal vector fields by using the concept of square root of g. A vector field X is called a bi-conformal vector field if it satisfies the following equations:

$$\mathcal{L}_X g = \alpha g + \beta h, \qquad \mathcal{L}_X h = \alpha h + \beta g,$$

where h is a symmetric square root of g and α, β are smooth functions. The functions α and β are called gauges [8, 11] of the symmetry and they play a role analogous to the

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factor ψ appearing in the definition of the conformal vector fields. Also, Ricci soliton is introduced by Hamilton [12] as follows

$$\mathcal{L}_X g + S = \lambda g, \quad \lambda \in \mathbb{R},$$

which is a natural generalization of Einstein metric. Wears in [16] studied Lorentzian Ricci solitons on simply-connected five-dimensional two-step nilpotent Lie groups which are also connected. For more details, see [1–6, 13–15]. Next, De et al. in [9] applying the metric tensor field g and the Ricci tensor field S introduced Ricci bi-conformal vector fields as follows:

Definition 1.1. A vector field X on a Riemannian manifold (M, g) is said to be Ricci bi-conformal vector field if it satisfies the following equations

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1.1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{1.2}$$

for any vector fields Y, Z and some smooth functions α and β, where S is the Ricci tensor of M with respect to the metric g.

Motivated by [9, 16], we study the Ricci bi-conformal vector fields on simply-connected five-dimensional two-step nilpotent Lie groups (G, g) with Lorentzian left invariant metric g which are also connected.
The paper is organized as follows. In Section 2, we recall some necessary concepts on simply-connected five-dimensional two-step nilpotent Lie groups with Lorentzian left invariant metric which are also connected and will be used throughout this paper. In Section 3, we give the main results and their proofs.

2. Preliminaries

Let $\mathfrak{g}$ be a five-dimensional Lie algebra with basis vector fields $e_1, \cdots, e_4$ and e_5 with the Lie algebra structure generated by the non-trivial Lie brackets $[e_1, e_5] = e_3$ and $[e_2, e_3] = e_5$. The Lie algebra $\mathfrak{g}$ is a two-step nilpotent with center $\mathfrak{Z} = span\{e_3, e_4\}$ and contains a four-dimensional maximal abelian subalgebra $\mathfrak{h} = \{e_1, e_2, e_3, e_4\}$. Suppose that G is the simply-connected five-dimensional two-step nilpotent Lie group with corresponding Lie algebra $\mathfrak{g}$ which is also connected. We will identify G with $\mathbb{R}^5$ equipped with coordinates (x, y, u, v, z). The group operation $\circ$ on G in coordinates is defined by

$$(x_1, y_1, u_1, v_1, z_1) \circ (x_2, y_2, u_2, v_2, z_2) = (x_1 + x_2, y_1 + y_2, u_1 + u_2 + x_1 z_2, v_1 + v_2 + y_1 z_2, z_1 + z_2).$$

We will identity the Lie algebra $\mathfrak{g}$ of G with the left invariant vector fields on G by considering the following basis

$$e_1 = \frac{\partial}{\partial x}, \ e_2 = \frac{\partial}{\partial y}, \ e_3 = \frac{\partial}{\partial u}, \ e_4 = \frac{\partial}{\partial v}, \ e_5 = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v} + \frac{\partial}{\partial z}. \tag{2.1}$$

The co-frame dual to the left invariant frame (2.1) is determined by

$$\omega^1 = dx, \ \omega^2 = dy, \ \omega^3 = du - xdz, \ \omega^4 = dv - ydz, \ \omega^5 = dz.$$

Identifying T_eG with $\mathfrak{g}$, the action of $Aut\mathfrak{g}$ on the set of left invariant metrics is described by

$$(g, H) \to g.H \tag{2.2}$$

where $H \in Aut\mathfrak{g}$. From [16], we have the following theorem:

Theorem 2.1. *Let $g_{ij}\omega^i\otimes\omega^j$ be a left invariant Lorentzian metric on G. Under the action (2.2) of $Aut\mathfrak{g}$, the metric g is equivalent to a left invariant Lorentzian metric of one of the following forms:*

$$\begin{aligned}
g_1 &= a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4 - \omega^5\otimes\omega^5,\ a,b\in\mathbb{R}_{>0},\\
g_2 &= -a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4 + \omega^5\otimes\omega^5,\ a,b\in\mathbb{R}_{>0},\\
g_3 &= a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 - \omega^4\otimes\omega^4 + \omega^5\otimes\omega^5,\ a,b\in\mathbb{R}_{>0},\\
g_4 &= \omega^1\otimes\omega^1 + 2\omega^2\otimes\omega^4 + a\omega^3\otimes\omega^3 + \omega^5\otimes\omega^5,\ a\in\mathbb{R}_{>0},\\
g_5 &= \omega^1\otimes\omega^1 + 2\omega^2\otimes\omega^3 + a\omega^4\otimes\omega^4 + \omega^5\otimes\omega^5,\ a\in\mathbb{R}_{>0},\\
g_6 &= a\omega^1\otimes\omega^1 + \omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + 2\omega^4\otimes\omega^5,\ a\in\mathbb{R}_{>0},\\
g_7 &= 2a\omega^1\otimes\omega^5 + \omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4,\ a\in\mathbb{R}_{>0}.
\end{aligned}$$

3. Main results and their proofs

We will now investigate the Ricci bi-conformal vector fields on G with the left invariant Lorentzian metrics.

3.1. The metrics g_1, g_2 and g_3

We can denote the families of metrics g_1, g_2 and g_3 as follows.

$$g_\mu = a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + c\omega^3\otimes\omega^3 + d\omega^4\otimes\omega^4 + f\omega^5\otimes\omega^5,$$

where $a,b,c,d,f\in\mathbb{R}$. The Levi-Civta connection ∇ of the left invariant Lorentzian metric g_μ is described by

$$\nabla_{e_i}e_j = \begin{pmatrix} 0 & 0 & -\frac{c}{2f}e_5 & 0 & \frac{1}{2}e_3\\ 0 & 0 & 0 & -\frac{1}{2}\frac{d}{f}e_5 & \frac{1}{2}e_4\\ -\frac{c}{2f}e_5 & 0 & 0 & 0 & \frac{c}{2a}e_1\\ 0 & -\frac{d}{2f}e_5 & 0 & 0 & \frac{d}{2b}e_2\\ -\frac{1}{2}e_3 & -\frac{1}{2}e_4 & \frac{c}{2a}e_1 & \frac{d}{2b}e_2 & 0 \end{pmatrix}, \tag{3.1}$$

and the Ricci tensor of g_μ is determined by

$$S = \begin{pmatrix} -\frac{c}{2f} & 0 & 0 & 0 & 0\\ 0 & -\frac{d}{2f} & 0 & 0 & 0\\ 0 & 0 & \frac{c^2}{2af} & 0 & 0\\ 0 & 0 & 0 & \frac{d^2}{2bf} & 0\\ 0 & 0 & 0 & 0 & -\frac{ad+bc}{2ab} \end{pmatrix}, \tag{3.2}$$

with respect to the basis $\{e_1,e_2,e_3,e_4,e_5\}$. For left invariant Lorentzian metric g_μ and any vector fields $X = X^ie_i$ where the X^i are smooth functions on G, we have

$$\begin{cases}
(\mathcal{L}_Xg)_{11} = 2a\partial_xX^1, & (\mathcal{L}_Xg)_{12} = b\partial_xX^2 + a\partial_yX^1,\\
(\mathcal{L}_Xg)_{13} = c\partial_xX^3 + a\partial_uX^1 + cX^5, & (\mathcal{L}_Xg)_{14} = d\partial_xX^4 + a\partial_vX^1,\\
(\mathcal{L}_Xg)_{15} = f\partial_xX^5 + ax\partial_uX^1 + ay\partial_vX^1 + a\partial_zX^1, & (\mathcal{L}_Xg)_{22} = 2b\partial_yX^2,\\
(\mathcal{L}_Xg)_{24} = d\partial_yX^4 + b\partial_vX^2 + dX^5, & (\mathcal{L}_Xg)_{23} = c\partial_yX^3 + b\partial_uX^2,\\
(\mathcal{L}_Xg)_{25} = f\partial_yX^5 + bx\partial_uX^2 + by\partial_vX^2 + b\partial_zX^2, & (\mathcal{L}_Xg)_{33} = 2c\partial_uX^3,\\
(\mathcal{L}_Xg)_{35} = f\partial_uX^5 + cx\partial_uX^3 + cy\partial_vX^3 + c\partial_zX^3 - cX^1, & (\mathcal{L}_Xg)_{34} = d\partial_uX^4 + c\partial_vX^3,\\
(\mathcal{L}_Xg)_{45} = f\partial_vX^5 + dx\partial_uX^4 + dy\partial_vX^4 + d\partial_zX^4 - dX^2, & (\mathcal{L}_Xg)_{44} = 2d\partial_vX^4,\\
(\mathcal{L}_Xg)_{55} = 2fx\partial_uX^5 + 2fy\partial_vX^5 + 2f\partial_zX^5, &
\end{cases} \tag{3.3}$$

and

$$\begin{cases}(\mathcal{L}_XS)_{11}=-\frac{c}{f}\partial_xX^1,\ \ (\mathcal{L}_XS)_{12}=-\frac{1}{2f}(d\partial_xX^2+c\partial_yX^1),\\ (\mathcal{L}_XS)_{13}=\frac{c}{2af}(c\partial_xX^3-a\partial_uX^1+cX^5),\ (\mathcal{L}_XS)_{14}=\frac{d^2}{2bf}\partial_xX^4-\frac{c}{2f}\partial_vX^1,\\ (\mathcal{L}_XS)_{15}=-\frac{ad+bc}{2ab}\partial_xX^5-\frac{c}{2f}(x\partial_uX^1+y\partial_vX^1+\partial_zX^1),\ (\mathcal{L}_XS)_{22}=-\frac{d}{f}\partial_yX^2,\\ (\mathcal{L}_XS)_{24}=\frac{d^2}{2bf}(\partial_yX^4+X^5)-\frac{d}{2f}\partial_vX^2,\ (\mathcal{L}_XS)_{23}=\frac{c^2}{2af}\partial_yX^3-\frac{d}{2f}\partial_uX^2,\\ (\mathcal{L}_XS)_{25}=-\frac{ad+bc}{2ab}\partial_yX^5-\frac{d}{2f}(x\partial_uX^2+y\partial_vX^2+\partial_zX^2),\ (\mathcal{L}_XS)_{33}=\frac{c^2}{af}\partial_uX^3,\\ (\mathcal{L}_XS)_{35}=-\frac{ad+bc}{2ab}\partial_uX^5+\frac{c^2}{2af}(x\partial_uX^3+y\partial_vX^3+\partial_zX^3-X^1),\\ (\mathcal{L}_XS)_{34}=\frac{d^2}{2bf}\partial_uX^4+\frac{c^2}{2af}\partial_vX^3,\\ (\mathcal{L}_XS)_{45}=-\frac{ad+bc}{2ab}\partial_vX^5+\frac{d^2}{2bf}(x\partial_uX^4+y\partial_vX^4+\partial_zX^4-X^2),\\ (\mathcal{L}_XS)_{44}=\frac{d^2}{bf}\partial_vX^4,\ (\mathcal{L}_XS)_{55}=-\frac{ad+bc}{ab}(x\partial_uX^5+y\partial_vX^5+\partial_zX^5),\end{cases} \tag{3.4}$$

where $(\mathcal{L}_Xg)_{ij}=\mathcal{L}_Xg(e_i,e_j)$ and $(\mathcal{L}_XS)_{ij}=\mathcal{L}_XS(e_i,e_j)$ for $1\leq i,j\leq 5$. Applying (3.1), (3.2), (3.3) and (3.4) in (1.1) and (1.2), we get

$$\begin{cases}2a_0\partial_xX^1=a\alpha-\frac{c}{2f}\beta, & b\partial_xX^2+a\partial_yX^1=0,\\ c\partial_xX^3+a\partial_uX^1+cX^5=0, & d\partial_xX^4+a\partial_vX^1=0,\\ f\partial_xX^5+ax\partial_uX^1+a_0y\partial_vX^1+a\partial_zX^1=0, & 2b\partial_yX^2=b\alpha-\frac{d}{2f}\beta,\\ c\partial_yX^3+b\partial_uX^2=0, & d\partial_yX^4+b\partial_vX^2+dX^5=0,\\ f\partial_yX^5+bx\partial_uX^2+by\partial_vX^2+b\partial_zX^2=0, & 2c\partial_uX^3=c\alpha+\frac{c^2}{2af}\beta,\\ f\partial_uX^5+cx\partial_uX^3+cy\partial_vX^3+c\partial_zX^3-cX^1=0, & d\partial_uX^4+c\partial_vX^3=0,\\ f\partial_vX^5+dx\partial_uX^4+dy\partial_vX^4+d\partial_zX^4-dX^2=0, & 2d\partial_vX^4=d\alpha+\frac{d^2}{2bf}\beta,\\ 2fx\partial_uX^5+2fy\partial_vX^5+2f\partial_zX^5=f\alpha-\frac{ad+bc}{2ab}\beta, & \end{cases}$$

and

$$\begin{cases}-\frac{c}{f}\partial_xX^1=-\frac{c}{2f}\alpha+a\beta, & -\frac{1}{2f}(d\partial_xX^2+c\partial_yX^1)=0,\\ \frac{c}{2af}(c\partial_xX^3-a\partial_uX^1+cX^5)=0, & \frac{d^2}{2bf}\partial_xX^4-\frac{c}{2f}\partial_vX^1=0,\\ -\frac{ad+bc}{2ab}\partial_xX^5-\frac{c}{2f}(x\partial_uX^1+y\partial_vX^1+\partial_zX^1)=0, & -\frac{d}{f}\partial_yX^2=-\frac{d}{2f}\alpha+b\beta,\\ \frac{d^2}{2bf}(\partial_yX^4+X^5)-\frac{d}{2f}\partial_vX^2=0, & \frac{c^2}{2af}\partial_yX^3-\frac{d}{2f}\partial_uX^2=0,\\ -\frac{ad+bc}{2ab}\partial_yX^5-\frac{d}{2f}(x\partial_uX^2+y\partial_vX^2+\partial_zX^2)=0, & \frac{c^2}{af}\partial_uX^3=\frac{c^2}{2af}\alpha+c\beta,\\ -\frac{ad+bc}{2ab}\partial_uX^5+\frac{c^2}{2af}(x\partial_uX^3+y\partial_vX^3+\partial_zX^3-X^1)=0, & \frac{d^2}{2bf}\partial_uX^4+\frac{c^2}{2af}\partial_vX^3=0,\\ -\frac{ad+bc}{2ab}\partial_vX^5+\frac{d^2}{2bf}(x\partial_uX^4+y\partial_vX^4+\partial_zX^4-X^2)=0, & \frac{d^2}{bf}\partial_vX^4=\frac{d^2}{2bf}\alpha+d\beta,\\ -\frac{ad+bc}{ab}(x\partial_uX^5+y\partial_vX^5+\partial_zX^5)=-\frac{ad+bc}{2ab}\alpha+f\beta. & \end{cases}$$

By solving the above equations, we obtain

$$X^1=a_1,\ X^2=a_2,\ X^3=a_3x+a_1z+a_4,\ X^4=a_2z+a_3y+a_5,\ X^5=-a_3$$

and $\alpha=\beta=0$ for some constants $a_1,\dots,a_5$. Therefore, we have the following theorem:

Theorem 3.1. *The left-invariant Lorentzian metric g_μ on Lie group G has a Ricci bi-conformal vector field X if and only if $X=a_1e_1+a_2e_2+(a_3x+a_1z+a_4)e_3+(a_2z+a_3y+a_5)e_4-a_3e_5$ and $\alpha=\beta=0$ for some constants $a_1,a_2,a_3,a_4,$ and a_5.*

Now, we consider the vector fields as $X=\nabla h$ for some smooth function h which are Ricci bi-conformal vector fields. On a five-dimensional Lorentzian Lie group G with metric g_μ, we have

$$\nabla h=\frac{1}{a}(\partial_xh)e_1+\frac{1}{b}(\partial_yh)e_2+\frac{1}{c}(\partial_uh)e_3+\frac{1}{d}(\partial_vh)e_4+\frac{1}{f}(x\partial_uh+y\partial_vh+\partial_zh)e_5. \tag{3.5}$$

From (3.5) and Theorem 3.1, we obtain

$$\partial_x h = a_1 a, \tag{3.6}$$
$$\partial_y h = b a_2, \tag{3.7}$$
$$\partial_u h = c(a_3 x + a_1 z + a_4)$$
$$\partial_v h = d(a_2 z + a_3 y + a_5)$$
$$x\partial_u h + y\partial_v h + \partial_z h = -a_3 f.$$

From equations (3.6) and (3.7), we deduce $0 = \partial_x\partial_u h = ca_3$. Then, $a_3 = 0$. Similarly, we infer $a_1 = a_2 = a_4 = a_5 = 0$. Therefore, we get the following corollary:

Corollary 3.2. *Any Ricci bi-conformal vector field X with respect to the left-invariant Lorentzian metric g_μ is gradient vector field as $X = \nabla h$ if and only if $h = \bar{a}_1$, where $\bar{a}_1$ is a real constant.*

3.2. The family of metrics g_4

The Levi-Civta connection ∇ of the left invariant Lorentzian metric g_4 on G is described by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\frac{1}{2}ae_5 & 0 & \frac{1}{2}e_3 \\ 0 & -e_5 & 0 & 0 & e_4 \\ -\frac{1}{2}ae_5 & 0 & 0 & 0 & \frac{1}{2}ae_1 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}e_3 & e_4 & \frac{1}{2}ae_1 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

and the Ricci tensor of g_4 is obtained by

$$S = \begin{pmatrix} -\frac{1}{2}a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a \end{pmatrix}, \tag{3.9}$$

with respect to the basis $\{e_1, e_2, e_3, e_4, e_5\}$. For left invariant Lorentzian metric g_4 and any vector fields $X = X^i e_i$, we deduce

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2\partial_x X^1, & (\mathcal{L}_X g)_{12} = \partial_x X^4 + \partial_y X^1, \\ (\mathcal{L}_X g)_{13} = a\partial_x X^3 + \partial_u X^1 + aX^5, & (\mathcal{L}_X g)_{14} = \partial_x X^2 + \partial_v X^1, \\ (\mathcal{L}_X g)_{15} = \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^4 + 2X^5, \\ (\mathcal{L}_X g)_{23} = a\partial_y X^3 + \partial_u X^4, & (\mathcal{L}_X g)_{24} = \partial_y X^2 + \partial_v X^4, \\ (\mathcal{L}_X g)_{25} = \partial_y X^5 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2, & (\mathcal{L}_X g)_{33} = 2a\partial_u X^3, \\ (\mathcal{L}_X g)_{34} = \partial_u X^2 + a\partial_v X^3, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^5 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3 - aX^1, & (\mathcal{L}_X g)_{44} = 2\partial_v X^4, \\ (\mathcal{L}_X g)_{45} = \partial_v X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5, & \end{cases} \tag{3.10}$$

and

$$\begin{cases}(\mathcal{L}_XS)_{11} = -a\partial_x X^1, & (\mathcal{L}_XS)_{12} = -\frac{1}{2}a\partial_y X^1,\\ (\mathcal{L}_XS)_{13} = \frac{a^2}{2}(X^5 + \partial_x X^3 - \frac{1}{a}\partial_u X^1), & (\mathcal{L}_XS)_{14} = -\frac{a}{2}\partial_v X^1,\\ (\mathcal{L}_XS)_{15} = -\frac{a}{2}(\partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1), & (\mathcal{L}_XS)_{22} = 0,\\ (\mathcal{L}_XS)_{23} = \frac{a^2}{2}\partial_y X^3, & (\mathcal{L}_XS)_{24} = 0,\\ (\mathcal{L}_XS)_{25} = -\frac{a}{2}\partial_y X^5, & (\mathcal{L}_XS)_{33} = a^2\partial_u X^3,\\ (\mathcal{L}_XS)_{35} = \frac{a}{2}(-\partial_u X^5 - aX^1 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3), & (\mathcal{L}_XS)_{34} = \frac{a^2}{2}\partial_v X^3,\\ (\mathcal{L}_XS)_{44} = 0, & (\mathcal{L}_XS)_{45} = -\frac{a}{2}\partial_v X^5,\\ (\mathcal{L}_XS)_{55} = -a(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5). \end{cases} \tag{3.11}$$

Applying (3.8), (3.9), (3.10) and (3.11) in (1.1) and (1.2), we infer

$$\begin{cases}2\partial_x X^1 = \alpha - \frac{1}{2}a\beta, & \partial_x X^4 + \partial_y X^1 = 0,\\ a\partial_x X^3 + \partial_u X^1 + aX^5 = 0, & \partial_x X^2 + \partial_v X^1 = 0,\\ \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1 = 0, & 2\partial_y X^4 + 2X^5 = 0,\\ a\partial_y X^3 + \partial_u X^4 = 0, & \partial_y X^2 + \partial_v X^4 = \alpha,\\ \partial_y X^5 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2 = 0, & 2a\partial_u X^3 = a\alpha + \frac{1}{2}a^2\beta,\\ \partial_u X^2 + a\partial_v X^3 = 0, & 2\partial_v X^4 = 0,\\ \partial_u X^5 + a(x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1) = 0, & \partial_v X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 = 0,\\ 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5 = \alpha - \frac{1}{2}a\beta, \end{cases} \tag{3.12}$$

and

$$\begin{cases}-a\partial_x X^1 = -\frac{1}{2}a\alpha + \beta, & -\frac{1}{2}a\partial_y X^1 = 0,\\ \frac{a^2}{2}(X^5 + \partial_x X^3 - \frac{1}{a}\partial_u X^1) = 0, & -\frac{a}{2}\partial_v X^1 = 0,\\ -\frac{a}{2}(\partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1) = 0, & \\ \frac{a^2}{2}\partial_y X^3 = 0, & 0 = \beta,\\ -\frac{a}{2}\partial_y X^5 = 0, & a^2\partial_u X^3 = \frac{1}{2}a^2\alpha + a\beta,\\ \frac{a}{2}(-\partial_u X^5 - aX^1 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3) = 0, & \frac{a^2}{2}\partial_v X^3 = 0,\\ -a(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = -\frac{1}{2}a\alpha + \beta, & -\frac{a}{2}\partial_v X^5 = 0. \end{cases} \tag{3.13}$$

By solving the equations systems (3.12) and (3.13), we have the following theorem:

Theorem 3.3. *The left-invariant Lorentzian metric g_4 on G has a Ricci bi-conformal vector field X if and only if $X = b_1e_1 + b_2e_2 + (b_1z + b_3x + b_4)e_3 + (b_2z + b_3y + b_5)e_4 - b_3e_5$ and $\alpha = \beta = 0$ for some constants $b_1, \ldots, b_5$.*

Similar to Corollary 3.2, we have the following result:

Corollary 3.4. *Any Ricci bi-conformal vector field X with respect to the left-invariant Lorentzian metric g_4 is gradient vector field with potential function $h = \bar{b}_1y + \bar{b}_2$ where $\bar{b}_1, \bar{b}_2$ are arbitrary real constants.*

3.3. The family of metrics g_5

The Levi-Civta connection ∇ of the left invariant Lorentzian metric g_5 is given by

$$\nabla_{e_i}e_j = \begin{pmatrix} 0 & -\frac{1}{2}e_5 & 0 & 0 & \frac{1}{2}e_3\\ -\frac{1}{2}e_5 & 0 & 0 & -\frac{1}{2}ae_5 & \frac{1}{2}e_1 + \frac{1}{2}e_4\\ 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{1}{2}ae_5 & 0 & 0 & \frac{1}{2}ae_3\\ -\frac{1}{2}e_3 & \frac{1}{2}e_1 - \frac{1}{2}e_4 & 0 & \frac{1}{2}ae_3 & 0 \end{pmatrix} \tag{3.14}$$

and the Ricci tensor of g_5 is represented by

$$S=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1-a}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.15}$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$. For left invariant Lorentzian metric g_5 and any vector fields $X = X^i e_i$ we obtain

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2\partial_x X^1, & \\ (\mathcal{L}_X g)_{12} = \partial_x X^3 + \partial_y X^1 + X^5, & \\ (\mathcal{L}_X g)_{13} = \partial_x X^2 + \partial_u X^1, & (\mathcal{L}_X g)_{14} = a\partial_x X^4 + \partial_v X^1,\\ (\mathcal{L}_X g)_{15} = \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^3,\\ (\mathcal{L}_X g)_{23} = \partial_y X^2 + \partial_u X^3, & \\ (\mathcal{L}_X g)_{24} = a\partial_y X^4 + \partial_v X^3 + aX^5, & \\ (\mathcal{L}_X g)_{25} = \partial_y X^5 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, & (\mathcal{L}_X g)_{33} = 2\partial_u X^2,\\ (\mathcal{L}_X g)_{34} = a\partial_u X^4 + \partial_v X^2, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & (\mathcal{L}_X g)_{44} = 2a\partial_v X^4,\\ (\mathcal{L}_X g)_{45} = \partial_v X^5 + ax\partial_u X^4 + ay\partial_v X^4 + a\partial_z X^4 - aX^2, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5, & \end{cases} \tag{3.16}$$

and

$$(\mathcal{L}_X S)_{ij} = \begin{pmatrix} 0 & \frac{1-a}{2}\partial_x X^2 & 0 & 0 & 0\\ & (1-a)\partial_x X^2 & \frac{1-a}{2}\partial_u X^2 & \frac{1-a}{2}\partial_v X^2 & \frac{1-a}{2}(x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2)\\ & & 0 & 0 & 0\\ & & & 0 & 0\\ & & & & 0 \end{pmatrix}. \tag{3.17}$$

Applying (3.14), (3.15), (3.16) and (3.17) in (1.1) and (1.2), we deduce

$$\begin{cases} 2\partial_x X^1 = \alpha, & \partial_x X^3 + \partial_y X^1 + X^5 = 0,\\ \partial_x X^2 + \partial_u X^1 = 0, & a\partial_x X^4 + \partial_v X^1 = 0,\\ \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1 = 0, & 2\partial_y X^3 = \frac{1-a}{2}\beta,\\ \partial_y X^2 + \partial_u X^3 = \alpha, & a\partial_y X^4 + \partial_v X^3 + aX^5 = 0,\\ \partial_y X^5 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1 = 0, & 2\partial_u X^2 = 0,\\ a\partial_u X^4 + \partial_v X^2 = 0, & \\ \partial_u X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 = 0, & \\ 2a\partial_v X^4 = a\alpha, & \\ \partial_v X^5 + ax\partial_u X^4 + ay\partial_v X^4 + a\partial_z X^4 - aX^2 = 0, & \\ 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5 = \alpha, & \end{cases}$$

and

$$
\begin{aligned}
&0=\beta,\\
&(1-a)\partial_x X^2=0,\\
&(1-a)\partial_y X^2=\frac{1-a}{2}\alpha,\\
&(1-a)\partial_u X^2=0,\\
&(1-a)\partial_v X^2=0,\\
&(1-a)\partial_z X^2=0.
\end{aligned}
$$

Solving the above equations, we get the following theorem:

Theorem 3.5. *The left-invariant Lorentzian metric g_5 has a Ricci bi-conformal vector field $X=X^ie_i$ if and only if*

$$
\begin{aligned}
X^1&=c_1\sin y+c_2\cos y+c_3y+c_4,\\
X^2&=c_5,\\
X^3&=c_6x+c_3v+c_4z+c_7,\\
X^4&=c_1\sin y+c_2\cos y+(c_3+c_6)y+c_5z+c_8-\frac{c_3}{a}y,\\
X^5&=-c_1\cos y+c_2\sin y-c_3-c_6,
\end{aligned}
$$

and $\alpha=\beta=0$ for some constants $c_1,\cdots,c_8$.

Therefore, we have the following result:

Corollary 3.6. *Any Ricci bi-conformal vector field X with respect to the left-invariant Lorentzian metric g_5 is gradient vector field with potential function $h=\bar{c}_1(x+yz-v)+\bar{c}_2y+\bar{c}_3$ where $\bar{c}_1,\bar{c}_2,\bar{c}_3$ are arbitrary real constants.*

3.4. The family of metrics g_6

The Levi-Civta connection ∇ of the left invariant Lorentzian metric g_6 is represented by

$$
\nabla_{e_i}e_j=\begin{pmatrix}
0 & 0 & -\frac{1}{2}e_4 & 0 & \frac{1}{2}e_3\\
0 & 0 & 0 & 0 & 0\\
-\frac{1}{2}e_4 & 0 & 0 & 0 & \frac{1}{2}e_1\\
0 & 0 & 0 & 0 & 0\\
-\frac{1}{2}e_3 & -e_4 & \frac{1}{2a}e_1 & 0 & e_2
\end{pmatrix} \tag{3.18}
$$

and the Ricci tensor of g_μ is given by

$$
S=\begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{1}{2a}
\end{pmatrix} \tag{3.19}
$$

with respect to the basis $\{e_1,e_2,e_3,e_4\}$. For left invariant Lorentzian metric g_6 and any vector fields $X=X^ie_i$ we have

$$\begin{cases}(\mathcal{L}_X g)_{11} = 2a\partial_x X^1, & (\mathcal{L}_X g)_{12} = \partial_x X^2 + a\partial_y X^1,\\ (\mathcal{L}_X g)_{13} = \partial_x X^3 + a\partial_u X^1 + X^5, & (\mathcal{L}_X g)_{14} = \partial_x X^5 + a\partial_v X^1,\\ (\mathcal{L}_X g)_{15} = \partial_x X^4 + ax\partial_u X^1 + ay\partial_v X^1 + a\partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^2,\\ (\mathcal{L}_X g)_{23} = \partial_y X^3 + \partial_u X^2, & (\mathcal{L}_X g)_{24} = \partial_y X^5 + \partial_v X^2,\\ (\mathcal{L}_X g)_{25} = \partial_y X^4 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 + X^5, & (\mathcal{L}_X g)_{33} = 2\partial_u X^3,\\ (\mathcal{L}_X g)_{34} = \partial_u X^5 + \partial_v X^3, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^4 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, & \\ (\mathcal{L}_X g)_{44} = 2\partial_v X^5, & \\ (\mathcal{L}_X g)_{45} = \partial_v X^4 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^4 + 2y\partial_v X^4 + 2\partial_z X^4 - 2X^2, & \end{cases} \tag{3.20}$$

and

$$(\mathcal{L}_X S)_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2a}\partial_x X^5 \\ & 0 & 0 & 0 & -\frac{1}{2a}\partial_y X^5 \\ & & 0 & 0 & -\frac{1}{2a}\partial_u X^5 \\ & & & 0 & -\frac{1}{2a}\partial_v X^5 \\ & & & & -\frac{1}{a}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) \end{pmatrix}. \tag{3.21}$$

Applying (3.18), (3.19), (3.20) and (3.21) in (1.1) and (1.2), we conclude

$$\begin{cases} 2a\partial_x X^1 = a\alpha, & \partial_x X^2 + a\partial_y X^1 = 0,\\ \partial_x X^3 + a\partial_u X^1 + X^5 = 0, & \partial_x X^5 + a\partial_v X^1 = 0,\\ \partial_x X^4 + ax\partial_u X^1 + ay\partial_v X^1 + a\partial_z X^1 = 0, & 2\partial_y X^2 = \alpha,\\ \partial_y X^3 + \partial_u X^2 = 0, & \partial_y X^5 + \partial_v X^2 = 0,\\ \partial_y X^4 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 + X^5 = 0, & 2\partial_u X^3 = \alpha,\\ \partial_u X^4 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1 = 0, & \partial_u X^5 + \partial_v X^3 = 0,\\ 2\partial_v X^5 = 0, & \partial_v X^4 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5 = \alpha,\\ 2x\partial_u X^4 + 2y\partial_v X^4 + 2\partial_z X^4 - 2X^2 = -\frac{1}{2a}\beta, & \end{cases} \tag{3.22}$$

and

$$\begin{aligned} &0 = \beta,\\ &\partial_x X^5 = 0,\\ &\partial_y X^5 = 0,\\ &\partial_u X^5 = 0,\\ &\partial_v X^5 = 0,\\ &-\frac{1}{a}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = -\frac{1}{2a}\alpha. \end{aligned} \tag{3.23}$$

By solving systems (3.22) and (3.23), we obtain the following theorem:

Theorem 3.7. *The left-invariant Lorentzian metric g_μ has a Ricci bi-conformal vector field $X = X^i e_i$ if and only if $\alpha = \beta = 0$ and*

$$\begin{aligned} X^1 &= \frac{1}{2}d_1 z^2 + d_2 z + d_3,\\ X^2 &= d_4 z + d_5,\\ X^3 &= -d_6 x + \frac{1}{6}d_1 z^3 - a d_1 z + \frac{1}{2}d_2 z^2 + d_3 z + d_7,\\ X^4 &= a d_1 u - (d_6 + d_4)y - (d_1 z + d_2)ax + \frac{1}{2}d_4 z^2 + d_5 z + d_8,\\ X^5 &= d_6, \end{aligned}$$

for some constants $d_1, \cdots, d_8$.

Therefore, we have the following result:

Corollary 3.8. *Any Ricci bi-conformal vector field X with respect to the left-invariant Lorentzian metric g_6 is gradient vector field with potential function $h = \frac{\bar{d}_1}{2}(2y + z^2) + \bar{d}_2 z + \bar{d}_3$ where $\bar{d}_1, \bar{d}_2, \bar{d}_3$ are arbitrary real constants.*

3.5. The family of metrics g_7

The Levi-Civta connection ∇ of the left invariant Lorentzian metric g_7 is described by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\frac{1}{2a}e_1 & 0 & \frac{1}{2}e_3\\ 0 & 0 & 0 & -\frac{1}{2a}e_1 & \frac{1}{2}e_4\\ -\frac{1}{2a}e_1 & 0 & 0 & 0 & \frac{1}{2a}e_5\\ 0 & -\frac{1}{2a}e_1 & 0 & 0 & \frac{1}{2}e_2\\ -\frac{1}{2}e_3 & frac12e_4 & \frac{1}{2a}e_5 & \frac{1}{2}e_2 & 0 \end{pmatrix} \tag{3.24}$$

and the Ricci tensor of g_7 is determined by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2a}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{2a^2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \frac{1}{2a} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{3.25}$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$. For left invariant Lorentz metric g_7 and any vector fields $X = X^i e_i$, we obtain

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2a\partial_x X^5, & (\mathcal{L}_X g)_{12} = \partial_x X^2 + a\partial_y X^5,\\ (\mathcal{L}_X g)_{13} = \partial_x X^3 + a\partial_u X^5 + X^5, & (\mathcal{L}_X g)_{14} = \partial_x X^4 + a\partial_v X^5,\\ (\mathcal{L}_X g)_{15} = a\partial_x X^1 + ax\partial_u X^5 + ay\partial_v X^5 + a\partial_z X^5, & (\mathcal{L}_X g)_{22} = 2\partial_y X^2,\\ (\mathcal{L}_X g)_{23} = \partial_y X^3 + \partial_u X^2, & (\mathcal{L}_X g)_{24} = \partial_y X^4 + \partial_v X^2 + X^5,\\ (\mathcal{L}_X g)_{25} = a\partial_y X^1 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & (\mathcal{L}_X g)_{33} = 2\partial_u X^3,\\ (\mathcal{L}_X g)_{34} = \partial_u X^4 + \partial_v X^3, & \\ (\mathcal{L}_X g)_{35} = a\partial_u X^1 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, & \\ (\mathcal{L}_X g)_{44} = 2\partial_v X^4, & \\ (\mathcal{L}_X g)_{45} = a\partial_v X^1 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2, & \\ (\mathcal{L}_X g)_{55} = 2ax\partial_u X^1 + 2ay\partial_v X^1 + 2a\partial_z X^1, & \end{cases} \tag{3.26}$$

and

$$
\begin{cases}
(\mathcal{L}_X S)_{11} = \frac{1}{a}\partial_x X^5, \\
(\mathcal{L}_X S)_{12} = \frac{1}{2a}\partial_y X^5, \\
(\mathcal{L}_X S)_{13} = -\frac{1}{2a^2}(X^5 + \partial_x X^3 - a\partial_u X^5), \\
(\mathcal{L}_X S)_{14} = \frac{1}{2a}\partial_v X^5, \\
(\mathcal{L}_X S)_{15} = \frac{1}{2a}(\partial_x X^1 - a\partial_x X^5 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5), \\
(\mathcal{L}_X S)_{22} = 0, \\
(\mathcal{L}_X S)_{23} = -\frac{1}{2a^2}\partial_y X^3, \\
(\mathcal{L}_X S)_{24} = 0, \\
(\mathcal{L}_X S)_{25} = \frac{1}{2a}(\partial_y X^1 - a\partial_y X^5), \\
(\mathcal{L}_X S)_{33} = -\frac{1}{a^2}\partial_u X^3, \\
(\mathcal{L}_X S)_{34} = -\frac{1}{2a^2}\partial_v X^3, \\
(\mathcal{L}_X S)_{35} = \frac{1}{2a^2}(a\partial_u X^1 - a^2\partial_u X^5 + X^1 - x\partial_u X^3 - y\partial_v X^3 - \partial_z X^3), \\
(\mathcal{L}_X S)_{44} = 0, \\
(\mathcal{L}_X S)_{45} = \frac{1}{2a}(\partial_v X^1 - a\partial_v X^5), \\
(\mathcal{L}_X S)_{55} = \frac{1}{a}(x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1) - (x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5).
\end{cases} \tag{3.27}
$$

Applying (3.24), (3.25), (3.26), and (3.27) in (1.1) and (1.2), we can write

$$
\begin{cases}
2a\partial_x X^5 = 0, & \partial_x X^2 + a\partial_y X^5 = 0, \\
\partial_x X^3 + a\partial_u X^5 + X^5 = 0, & \partial_x X^4 + a\partial_v X^5 = 0, \\
a\partial_x X^1 + ax\partial_u X^5 + ay\partial_v X^5 + a\partial_z X^5 = a\alpha + \frac{1}{2a}\beta, & 2\partial_y X^2 = \alpha, \\
\partial_y X^3 + \partial_u X^2 = 0, & \partial_y X^4 + \partial_v X^2 + X^5 = 0, \\
a\partial_y X^1 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 = 0, & 2\partial_u X^3 = \alpha - \frac{1}{2a^2}\beta, \\
a\partial_u X^1 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1 = 0, & \partial_u X^4 + \partial_v X^3 = 0, \\
a\partial_v X^1 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2 = 0, & 2\partial_v X^4 = \alpha, \\
2ax\partial_u X^1 + 2ay\partial_v X^1 + 2a\partial_z X^1 = -\frac{1}{2}\beta,
\end{cases}
$$

and

$$
\begin{cases}
\frac{1}{a}\partial_x X^5 = 0, & \frac{1}{2a}\partial_y X^5 = 0, \\
-\frac{1}{2a^2}(X^5 + \partial_x X^3 - a\partial_u X^5) = 0, & \frac{1}{2a}\partial_v X^5 = 0, \\
\frac{1}{2a}(\partial_x X^1 - a\partial_x X^5 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = \frac{1}{2a}\alpha + a\beta, & 0 = \beta, \\
-\frac{1}{2a^2}\partial_y X^3 = 0, & \frac{1}{2a}(\partial_y X^1 - a\partial_y X^5) = 0, \\
-\frac{1}{a^2}\partial_u X^3 = -\frac{1}{2a^2}\alpha + \beta, & -\frac{1}{2a^2}\partial_v X^3 = 0, \\
\frac{1}{2a^2}(a\partial_u X^1 - a^2\partial_u X^5 & \\
\qquad +X^1 - x\partial_u X^3 - y\partial_v X^3 - \partial_z X^3) = 0, & \frac{1}{2a}(\partial_v X^1 - a\partial_v X^5) = 0, \\
\frac{1}{a}(x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1) & \\
\qquad -(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = -\frac{1}{2}\alpha. &
\end{cases}
$$

Solving the above systems implies that in the following theorem:

Theorem 3.9. *The left-invariant Lorentzian metric g_7 has a Ricci bi-conformal vector field X if and only if $X = k_1e_1 + k_2e_2 + (k_1z - k_5x + k_3)e_3 + (k_2z - k_5y + k_4)e_4 + k_5e_5$ and $\alpha = \beta = 0$ for some constants $k_1, \cdots, k_5$.*

Also, we have the following result:

Corollary 3.10. *Any Ricci bi-conformal vector field X with respect to the left-invariant Lorentzian metric g_7 is gradient vector field with potential function $h = \bar{k}_1$ where $\bar{k}_1$ is an arbitrary real constant.*

Remark 3.11. From Theorems 3.1, 3.3, 3.5, 3.7, and 3.9, we conclude that all Ricci bi-conformal vector fields on five-dimensional Lorentzian nilpotent Lie groups (G, g_i) are Killing vector fields for $i = 1, 2, \ldots 7$.

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