



# Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups

Shahroud Azami<sup>\*1</sup> , Uday Chand De<sup>2</sup> 

<sup>1</sup>Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

<sup>2</sup>Department of Pure Mathematics, University of Calcutta 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India.

## Abstract

In this paper, we completely classify Ricci bi-conformal vector fields on simply-connected five-dimensional two-step nilpotent Lie groups which are also connected and we show which of them are the Killing vector fields and gradient vector fields.

**Mathematics Subject Classification (2020).** 53B30, 53A55

**Keywords.** Ricci bi-conformal vector fields, pseudo-Riemannian metrics, nilpotent Lie group

## 1. Introduction

Let  $(M, g)$  be an  $n$ -dimensional pseudo-Riemannian manifold. A vector field  $X$  on a Riemannian manifold  $(M, g)$  is said to be a Killing field [7] if  $\mathcal{L}_X g = 0$  where  $\mathcal{L}_X$  is the Lie derivative in the direction of  $X$ . Recently, various generalizations of Killing vector fields have been studied. For instance, conformal vector fields [10, 17] are generalized of Killing vector fields and a conformal vector field  $X$  on a Riemannian manifold  $(M, g)$  is defined by  $\mathcal{L}_X g = 2\psi g$  for some smooth function  $\psi$ . If the potential function  $\psi = 0$  then  $X$  is a Killing vector field. A vector field  $X$  on  $M$  is called a Kerr-Schild vector field if  $\mathcal{L}_X g = \alpha l \otimes l$ ,  $\mathcal{L}_X l = \beta l$ , where  $l$  is a null 1-form field and  $\alpha, \beta$  are smooth functions over  $M$ . Also, the generalized Kerr-Schild vector field is determined by

$$\mathcal{L}_X g = \alpha g + \beta l \otimes l, \quad \mathcal{L}_X l = \gamma l,$$

where  $\alpha, \beta, \gamma$  are smooth functions. Coll et al. [8] studied the generalized Kerr-Schild vector field. A symmetric tensor field  $h$  on  $M$  is said to be a square root of  $g$  if  $h_{ik}h_j^k = g_{ij}$ . Garcia-Parrado and Senovilla [11] introduced bi-conformal vector fields by using the concept of square root of  $g$ . A vector field  $X$  is called a bi-conformal vector field if it satisfies the following equations:

$$\mathcal{L}_X g = \alpha g + \beta h, \quad \mathcal{L}_X h = \alpha h + \beta g,$$

where  $h$  is a symmetric square root of  $g$  and  $\alpha, \beta$  are smooth functions. The functions  $\alpha$  and  $\beta$  are called gauges [8, 11] of the symmetry and they play a role analogous to the

\*Corresponding Author.

Email addresses: azami@sci.ikiu.ac.ir (S. Azami), uc\_de@yahoo.com (U. C. De)

Received: 10.05.2023; Accepted: 25.09.2023

factor  $\psi$  appearing in the definition of the conformal vector fields. Also, Ricci soliton is introduced by Hamilton [12] as follows

$$\mathcal{L}_X g + S = \lambda g, \quad \lambda \in \mathbb{R},$$

which is a natural generalization of Einstein metric. Wears in [16] studied Lorentzian Ricci solitons on simply-connected five-dimensional two-step nilpotent Lie groups which are also connected. For more details, see [1–6, 13–15]. Next, De et al. in [9] applying the metric tensor field  $g$  and the Ricci tensor field  $S$  introduced Ricci bi-conformal vector fields as follows:

**Definition 1.1.** A vector field  $X$  on a Riemannian manifold  $(M, g)$  is said to be Ricci bi-conformal vector field if it satisfies the following equations

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \quad (1.1)$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \quad (1.2)$$

for any vector fields  $Y, Z$  and some smooth functions  $\alpha$  and  $\beta$ , where  $S$  is the Ricci tensor of  $M$  with respect to the metric  $g$ .

Motivated by [9, 16], we study the Ricci bi-conformal vector fields on simply-connected five-dimensional two-step nilpotent Lie groups  $(G, g)$  with Lorentzian left invariant metric  $g$  which are also connected.

The paper is organized as follows. In Section 2, we recall some necessary concepts on simply-connected five-dimensional two-step nilpotent Lie groups with Lorentzian left invariant metric which are also connected and will be used throughout this paper. In Section 3, we give the main results and their proofs.

## 2. Preliminaries

Let  $\mathfrak{g}$  be a five-dimensional Lie algebra with basis vector fields  $e_1, \dots, e_4$  and  $e_5$  with the Lie algebra structure generated by the non-trivial Lie brackets  $[e_1, e_5] = e_3$  and  $[e_2, e_3] = e_5$ . The Lie algebra  $\mathfrak{g}$  is a two-step nilpotent with center  $\mathcal{Z} = \text{span}\{e_3, e_4\}$  and contains a four-dimensional maximal abelian subalgebra  $\mathfrak{h} = \{e_1, e_2, e_3, e_4\}$ . Suppose that  $G$  is the simply-connected five-dimensional two-step nilpotent Lie group with corresponding Lie algebra  $\mathfrak{g}$  which is also connected. We will identify  $G$  with  $\mathbb{R}^5$  equipped with coordinates  $(x, y, u, v, z)$ . The group operation  $\circ$  on  $G$  in coordinates is defined by

$$(x_1, y_1, u_1, v_1, z_1) \circ (x_2, y_2, u_2, v_2, z_2) = (x_1 + x_2, y_1 + y_2, u_1 + u_2 + x_1 z_2, v_1 + v_2 + y_1 z_2, z_1 + z_2).$$

We will identify the Lie algebra  $\mathfrak{g}$  of  $G$  with the left invariant vector fields on  $G$  by considering the following basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial u}, \quad e_4 = \frac{\partial}{\partial v}, \quad e_5 = x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} + \frac{\partial}{\partial z}. \quad (2.1)$$

The co-frame dual to the left invariant frame (2.1) is determined by

$$\omega^1 = dx, \quad \omega^2 = dy, \quad \omega^3 = du - x dz, \quad \omega^4 = dv - y dz, \quad \omega^5 = dz.$$

Identifying  $T_e G$  with  $\mathfrak{g}$ , the action of  $\text{Aut } \mathfrak{g}$  on the set of left invariant metrics is described by

$$(g, H) \rightarrow g.H \quad (2.2)$$

where  $H \in \text{Aut } \mathfrak{g}$ . From [16], we have the following theorem:

**Theorem 2.1.** *Let  $g_{ij}\omega^i\otimes\omega^j$  be a left invariant Lorentzian metric on  $G$ . Under the action (2.2) of  $\text{Aut}g$ , the metric  $g$  is equivalent to a left invariant Lorentzian metric of one of the following forms:*

$$\begin{aligned} g_1 &= a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4 - \omega^5\otimes\omega^5, \quad a, b \in \mathbb{R}_{>0}, \\ g_2 &= -a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4 + \omega^5\otimes\omega^5, \quad a, b \in \mathbb{R}_{>0}, \\ g_3 &= a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 - \omega^4\otimes\omega^4 + \omega^5\otimes\omega^5, \quad a, b \in \mathbb{R}_{>0}, \\ g_4 &= \omega^1\otimes\omega^1 + 2\omega^2\otimes\omega^4 + a\omega^3\otimes\omega^3 + \omega^5\otimes\omega^5, \quad a \in \mathbb{R}_{>0}, \\ g_5 &= \omega^1\otimes\omega^1 + 2\omega^2\otimes\omega^3 + a\omega^4\otimes\omega^4 + \omega^5\otimes\omega^5, \quad a \in \mathbb{R}_{>0}, \\ g_6 &= a\omega^1\otimes\omega^1 + \omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + 2\omega^4\otimes\omega^5, \quad a \in \mathbb{R}_{>0}, \\ g_7 &= 2a\omega^1\otimes\omega^5 + \omega^2\otimes\omega^2 + \omega^3\otimes\omega^3 + \omega^4\otimes\omega^4, \quad a \in \mathbb{R}_{>0}. \end{aligned}$$

### 3. Main results and their proofs

We will now investigate the Ricci bi-conformal vector fields on  $G$  with the left invariant Lorentzian metrics.

#### 3.1. The metrics $g_1, g_2$ and $g_3$

We can denote the families of metrics  $g_1, g_2$  and  $g_3$  as follows.

$$g_\mu = a\omega^1\otimes\omega^1 + b\omega^2\otimes\omega^2 + c\omega^3\otimes\omega^3 + d\omega^4\otimes\omega^4 + f\omega^5\otimes\omega^5,$$

where  $a, b, c, d, f \in \mathbb{R}$ . The Levi-Civita connection  $\nabla$  of the left invariant Lorentzian metric  $g_\mu$  is described by

$$\nabla_{e_i}e_j = \begin{pmatrix} 0 & 0 & -\frac{c}{2f}e_5 & 0 & \frac{1}{2}e_3 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{d}{f}e_5 & \frac{1}{2}e_4 \\ -\frac{c}{2f}e_5 & 0 & 0 & 0 & \frac{c}{2a}e_1 \\ 0 & -\frac{d}{2f}e_5 & 0 & 0 & \frac{d}{2b}e_2 \\ -\frac{1}{2}e_3 & -\frac{1}{2}e_4 & \frac{c}{2a}e_1 & \frac{d}{2b}e_2 & 0 \end{pmatrix}, \quad (3.1)$$

and the Ricci tensor of  $g_\mu$  is determined by

$$S = \begin{pmatrix} -\frac{c}{2f} & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{2f} & 0 & 0 & 0 \\ 0 & 0 & \frac{c^2}{2af} & 0 & 0 \\ 0 & 0 & 0 & \frac{d^2}{2bf} & 0 \\ 0 & 0 & 0 & 0 & -\frac{ad+bc}{2ab} \end{pmatrix}, \quad (3.2)$$

with respect to the basis  $\{e_1, e_2, e_3, e_4, e_5\}$ . For left invariant Lorentzian metric  $g_\mu$  and any vector fields  $X = X^i e_i$  where the  $X^i$  are smooth functions on  $G$ , we have

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2a\partial_x X^1, & (\mathcal{L}_X g)_{12} = b\partial_x X^2 + a\partial_y X^1, \\ (\mathcal{L}_X g)_{13} = c\partial_x X^3 + a\partial_u X^1 + cX^5, & (\mathcal{L}_X g)_{14} = d\partial_x X^4 + a\partial_v X^1, \\ (\mathcal{L}_X g)_{15} = f\partial_x X^5 + ax\partial_u X^1 + ay\partial_v X^1 + a\partial_z X^1, & (\mathcal{L}_X g)_{22} = 2b\partial_y X^2, \\ (\mathcal{L}_X g)_{24} = d\partial_y X^4 + b\partial_v X^2 + dX^5, & (\mathcal{L}_X g)_{23} = c\partial_y X^3 + b\partial_u X^2, \\ (\mathcal{L}_X g)_{25} = f\partial_y X^5 + bx\partial_u X^2 + by\partial_v X^2 + b\partial_z X^2, & (\mathcal{L}_X g)_{33} = 2c\partial_u X^3, \\ (\mathcal{L}_X g)_{35} = f\partial_u X^5 + cx\partial_u X^3 + cy\partial_v X^3 + c\partial_z X^3 - cX^1, & (\mathcal{L}_X g)_{34} = d\partial_u X^4 + c\partial_v X^3, \\ (\mathcal{L}_X g)_{45} = f\partial_v X^5 + dx\partial_u X^4 + dy\partial_v X^4 + d\partial_z X^4 - dX^2, & (\mathcal{L}_X g)_{44} = 2d\partial_v X^4, \\ (\mathcal{L}_X g)_{55} = 2fx\partial_u X^5 + 2fy\partial_v X^5 + 2f\partial_z X^5, & \end{cases} \quad (3.3)$$

and

$$\left\{ \begin{array}{l} (\mathcal{L}_X S)_{11} = -\frac{c}{f}\partial_x X^1, \quad (\mathcal{L}_X S)_{12} = -\frac{1}{2f}(d\partial_x X^2 + c\partial_y X^1), \\ (\mathcal{L}_X S)_{13} = \frac{c}{2af}(c\partial_x X^3 - a\partial_u X^1 + cX^5), \quad (\mathcal{L}_X S)_{14} = \frac{d^2}{2bf}\partial_x X^4 - \frac{c}{2f}\partial_v X^1, \\ (\mathcal{L}_X S)_{15} = -\frac{ad+bc}{2ab}\partial_x X^5 - \frac{c}{2f}(x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1), \quad (\mathcal{L}_X S)_{22} = -\frac{d}{f}\partial_y X^2, \\ (\mathcal{L}_X S)_{24} = \frac{d^2}{2bf}(\partial_y X^4 + X^5) - \frac{d}{2f}\partial_v X^2, \quad (\mathcal{L}_X S)_{23} = \frac{c^2}{2af}\partial_y X^3 - \frac{d}{2f}\partial_u X^2, \\ (\mathcal{L}_X S)_{25} = -\frac{ad+bc}{2ab}\partial_y X^5 - \frac{d}{2f}(x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2), \quad (\mathcal{L}_X S)_{33} = \frac{c^2}{af}\partial_u X^3, \\ (\mathcal{L}_X S)_{35} = -\frac{ad+bc}{2ab}\partial_u X^5 + \frac{c^2}{2af}(x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1), \\ (\mathcal{L}_X S)_{34} = \frac{d^2}{2bf}\partial_u X^4 + \frac{c^2}{2af}\partial_v X^3, \\ (\mathcal{L}_X S)_{45} = -\frac{ad+bc}{2ab}\partial_v X^5 + \frac{d^2}{2bf}(x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2), \\ (\mathcal{L}_X S)_{44} = \frac{d^2}{bf}\partial_v X^4, \quad (\mathcal{L}_X S)_{55} = -\frac{ad+bc}{ab}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5), \end{array} \right. \quad (3.4)$$

where  $(\mathcal{L}_X g)_{ij} = \mathcal{L}_X g(e_i, e_j)$  and  $(\mathcal{L}_X S)_{ij} = \mathcal{L}_X S(e_i, e_j)$  for  $1 \leq i, j \leq 5$ . Applying (3.1), (3.2), (3.3) and (3.4) in (1.1) and (1.2), we get

$$\left\{ \begin{array}{l} 2a_0\partial_x X^1 = a\alpha - \frac{c}{2f}\beta, \quad b\partial_x X^2 + a\partial_y X^1 = 0, \\ c\partial_x X^3 + a\partial_u X^1 + cX^5 = 0, \quad d\partial_x X^4 + a\partial_v X^1 = 0, \\ f\partial_x X^5 + ax\partial_u X^1 + a_0y\partial_v X^1 + a\partial_z X^1 = 0, \quad 2b\partial_y X^2 = b\alpha - \frac{d}{2f}\beta, \\ c\partial_y X^3 + b\partial_u X^2 = 0, \quad d\partial_y X^4 + b\partial_v X^2 + dX^5 = 0, \\ f\partial_y X^5 + bx\partial_u X^2 + by\partial_v X^2 + b\partial_z X^2 = 0, \quad 2c\partial_u X^3 = c\alpha + \frac{c^2}{2af}\beta, \\ f\partial_u X^5 + cx\partial_u X^3 + cy\partial_v X^3 + c\partial_z X^3 - cX^1 = 0, \quad d\partial_u X^4 + c\partial_v X^3 = 0, \\ f\partial_v X^5 + dx\partial_u X^4 + dy\partial_v X^4 + d\partial_z X^4 - dX^2 = 0, \quad 2d\partial_v X^4 = d\alpha + \frac{d^2}{2bf}\beta, \\ 2fx\partial_u X^5 + 2fy\partial_v X^5 + 2f\partial_z X^5 = f\alpha - \frac{ad+bc}{2ab}\beta, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} -\frac{c}{f}\partial_x X^1 = -\frac{c}{2f}\alpha + a\beta, \quad -\frac{1}{2f}(d\partial_x X^2 + c\partial_y X^1) = 0, \\ \frac{c}{2af}(c\partial_x X^3 - a\partial_u X^1 + cX^5) = 0, \quad \frac{d^2}{2bf}\partial_x X^4 - \frac{c}{2f}\partial_v X^1 = 0, \\ -\frac{ad+bc}{2ab}\partial_x X^5 - \frac{c}{2f}(x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1) = 0, \quad -\frac{d}{f}\partial_y X^2 = -\frac{d}{2f}\alpha + b\beta, \\ \frac{d^2}{2bf}(\partial_y X^4 + X^5) - \frac{d}{2f}\partial_v X^2 = 0, \quad \frac{c^2}{2af}\partial_y X^3 - \frac{d}{2f}\partial_u X^2 = 0, \\ -\frac{ad+bc}{2ab}\partial_y X^5 - \frac{d}{2f}(x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2) = 0, \quad \frac{c^2}{af}\partial_u X^3 = \frac{c^2}{2af}\alpha + c\beta, \\ -\frac{ad+bc}{2ab}\partial_u X^5 + \frac{c^2}{2af}(x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1) = 0, \quad \frac{d^2}{2bf}\partial_u X^4 + \frac{c^2}{2af}\partial_v X^3 = 0, \\ -\frac{ad+bc}{2ab}\partial_v X^5 + \frac{d^2}{2bf}(x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2) = 0, \quad \frac{d^2}{bf}\partial_v X^4 = \frac{d^2}{2bf}\alpha + d\beta, \\ -\frac{ad+bc}{ab}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = -\frac{ad+bc}{2ab}\alpha + f\beta. \end{array} \right.$$

By solving the above equations, we obtain

$$X^1 = a_1, \quad X^2 = a_2, \quad X^3 = a_3x + a_1z + a_4, \quad X^4 = a_2z + a_3y + a_5, \quad X^5 = -a_3$$

and  $\alpha = \beta = 0$  for some constants  $a_1, \dots, a_5$ . Therefore, we have the following theorem:

**Theorem 3.1.** *The left-invariant Lorentzian metric  $g_\mu$  on Lie group  $G$  has a Ricci bi-conformal vector field  $X$  if and only if  $X = a_1e_1 + a_2e_2 + (a_3x + a_1z + a_4)e_3 + (a_2z + a_3y + a_5)e_4 - a_3e_5$  and  $\alpha = \beta = 0$  for some constants  $a_1, a_2, a_3, a_4$ , and  $a_5$ .*

Now, we consider the vector fields as  $X = \nabla h$  for some smooth function  $h$  which are Ricci bi-conformal vector fields. On a five-dimensional Lorentzian Lie group  $G$  with metric  $g_\mu$ , we have

$$\nabla h = \frac{1}{a}(\partial_x h)e_1 + \frac{1}{b}(\partial_y h)e_2 + \frac{1}{c}(\partial_u h)e_3 + \frac{1}{d}(\partial_v h)e_4 + \frac{1}{f}(x\partial_u h + y\partial_v h + \partial_z h)e_5. \quad (3.5)$$

From (3.5) and Theorem 3.1, we obtain

$$\partial_x h = a_1 a, \quad (3.6)$$

$$\partial_y h = b a_2, \quad (3.7)$$

$$\partial_u h = c(a_3 x + a_1 z + a_4)$$

$$\partial_v h = d(a_2 z + a_3 y + a_5)$$

$$x\partial_u h + y\partial_v h + \partial_z h = -a_3 f.$$

From equations (3.6) and (3.7), we deduce  $0 = \partial_x \partial_u h = c a_3$ . Then,  $a_3 = 0$ . Similarly, we infer  $a_1 = a_2 = a_4 = a_5 = 0$ . Therefore, we get the following corollary:

**Corollary 3.2.** *Any Ricci bi-conformal vector field  $X$  with respect to the left-invariant Lorentzian metric  $g_\mu$  is gradient vector field as  $X = \nabla h$  if and only if  $h = \bar{a}_1$ , where  $\bar{a}_1$  is a real constant.*

### 3.2. The family of metrics $g_4$

The Levi-Civita connection  $\nabla$  of the left invariant Lorentzian metric  $g_4$  on  $G$  is described by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\frac{1}{2} a e_5 & 0 & \frac{1}{2} e_3 \\ 0 & -e_5 & 0 & 0 & e_4 \\ -\frac{1}{2} a e_5 & 0 & 0 & 0 & \frac{1}{2} a e_1 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} e_3 & e_4 & \frac{1}{2} a e_1 & 0 & 0 \end{pmatrix}, \quad (3.8)$$

and the Ricci tensor of  $g_4$  is obtained by

$$S = \begin{pmatrix} -\frac{1}{2} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} a \end{pmatrix}, \quad (3.9)$$

with respect to the basis  $\{e_1, e_2, e_3, e_4, e_5\}$ . For left invariant Lorentzian metric  $g_4$  and any vector fields  $X = X^i e_i$ , we deduce

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2\partial_x X^1, & (\mathcal{L}_X g)_{12} = \partial_x X^4 + \partial_y X^1, \\ (\mathcal{L}_X g)_{13} = a\partial_x X^3 + \partial_u X^1 + aX^5, & (\mathcal{L}_X g)_{14} = \partial_x X^2 + \partial_v X^1, \\ (\mathcal{L}_X g)_{15} = \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^4 + 2X^5, \\ (\mathcal{L}_X g)_{23} = a\partial_y X^3 + \partial_u X^4, & (\mathcal{L}_X g)_{24} = \partial_y X^2 + \partial_v X^4, \\ (\mathcal{L}_X g)_{25} = \partial_y X^5 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2, & (\mathcal{L}_X g)_{33} = 2a\partial_u X^3, \\ (\mathcal{L}_X g)_{34} = \partial_u X^2 + a\partial_v X^3, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^5 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3 - aX^1, & (\mathcal{L}_X g)_{44} = 2\partial_v X^4, \\ (\mathcal{L}_X g)_{45} = \partial_v X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5, & \end{cases} \quad (3.10)$$

and

$$\left\{ \begin{array}{ll} (\mathcal{L}_X S)_{11} = -a\partial_x X^1, & (\mathcal{L}_X S)_{12} = -\frac{1}{2}a\partial_y X^1, \\ (\mathcal{L}_X S)_{13} = \frac{a^2}{2}(X^5 + \partial_x X^3 - \frac{1}{a}\partial_u X^1), & (\mathcal{L}_X S)_{14} = -\frac{a}{2}\partial_v X^1, \\ (\mathcal{L}_X S)_{15} = -\frac{a}{2}(\partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1), & (\mathcal{L}_X S)_{22} = 0, \\ (\mathcal{L}_X S)_{23} = \frac{a^2}{2}\partial_y X^3, & (\mathcal{L}_X S)_{24} = 0, \\ (\mathcal{L}_X S)_{25} = -\frac{a}{2}\partial_y X^5, & (\mathcal{L}_X S)_{33} = a^2\partial_u X^3, \\ (\mathcal{L}_X S)_{35} = \frac{a}{2}(-\partial_u X^5 - aX^1 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3), & (\mathcal{L}_X S)_{34} = \frac{a^2}{2}\partial_v X^3, \\ (\mathcal{L}_X S)_{44} = 0, & (\mathcal{L}_X S)_{45} = -\frac{a}{2}\partial_v X^5, \\ (\mathcal{L}_X S)_{55} = -a(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5). \end{array} \right. \quad (3.11)$$

Applying (3.8), (3.9), (3.10) and (3.11) in (1.1) and (1.2), we infer

$$\left\{ \begin{array}{ll} 2\partial_x X^1 = \alpha - \frac{1}{2}a\beta, & \partial_x X^4 + \partial_y X^1 = 0, \\ a\partial_x X^3 + \partial_u X^1 + aX^5 = 0, & \partial_x X^2 + \partial_v X^1 = 0, \\ \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1 = 0, & 2\partial_y X^4 + 2X^5 = 0, \\ a\partial_y X^3 + \partial_u X^4 = 0, & \partial_y X^2 + \partial_v X^4 = \alpha, \\ \partial_y X^5 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2 = 0, & 2a\partial_u X^3 = a\alpha + \frac{1}{2}a^2\beta, \\ \partial_u X^2 + a\partial_v X^3 = 0, & 2\partial_v X^4 = 0, \\ \partial_u X^5 + a(x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1) = 0, & \partial_v X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 = 0, \\ 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5 = \alpha - \frac{1}{2}a\beta, & \end{array} \right. \quad (3.12)$$

and

$$\left\{ \begin{array}{ll} -a\partial_x X^1 = -\frac{1}{2}a\alpha + \beta, & -\frac{1}{2}a\partial_y X^1 = 0, \\ \frac{a^2}{2}(X^5 + \partial_x X^3 - \frac{1}{a}\partial_u X^1) = 0, & -\frac{a}{2}\partial_v X^1 = 0, \\ -\frac{a}{2}(\partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1) = 0, & \\ \frac{a^2}{2}\partial_y X^3 = 0, & 0 = \beta, \\ -\frac{a}{2}\partial_y X^5 = 0, & a^2\partial_u X^3 = \frac{1}{2}a^2\alpha + a\beta, \\ \frac{a}{2}(-\partial_u X^5 - aX^1 + ax\partial_u X^3 + ay\partial_v X^3 + a\partial_z X^3) = 0, & \frac{a^2}{2}\partial_v X^3 = 0, \\ -a(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) = -\frac{1}{2}a\alpha + \beta, & -\frac{a}{2}\partial_v X^5 = 0. \end{array} \right. \quad (3.13)$$

By solving the equations systems (3.12) and (3.13), we have the following theorem:

**Theorem 3.3.** *The left-invariant Lorentzian metric  $g_4$  on  $G$  has a Ricci bi-conformal vector field  $X$  if and only if  $X = b_1e_1 + b_2e_2 + (b_1z + b_3x + b_4)e_3 + (b_2z + b_3y + b_5)e_4 - b_3e_5$  and  $\alpha = \beta = 0$  for some constants  $b_1, \dots, b_5$ .*

Similar to Corollary 3.2, we have the following result:

**Corollary 3.4.** *Any Ricci bi-conformal vector field  $X$  with respect to the left-invariant Lorentzian metric  $g_4$  is gradient vector field with potential function  $h = \bar{b}_1y + \bar{b}_2$  where  $\bar{b}_1, \bar{b}_2$  are arbitrary real constants.*

### 3.3. The family of metrics $g_5$

The Levi-Civta connection  $\nabla$  of the left invariant Lorentzian metric  $g_5$  is given by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & -\frac{1}{2}e_5 & 0 & 0 & \frac{1}{2}e_3 \\ -\frac{1}{2}e_5 & 0 & 0 & -\frac{1}{2}ae_5 & \frac{1}{2}e_1 + \frac{1}{2}e_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}ae_5 & 0 & 0 & \frac{1}{2}ae_3 \\ -\frac{1}{2}e_3 & \frac{1}{2}e_1 - \frac{1}{2}e_4 & 0 & \frac{1}{2}ae_3 & 0 \end{pmatrix} \quad (3.14)$$

and the Ricci tensor of  $g_5$  is represented by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-a}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.15)$$

with respect to the basis  $\{e_1, e_2, e_3, e_4\}$ . For left invariant Lorentzian metric  $g_5$  and any vector fields  $X = X^i e_i$  we obtain

$$\left\{ \begin{array}{ll} (\mathcal{L}_X g)_{11} = 2\partial_x X^1, & \\ (\mathcal{L}_X g)_{12} = \partial_x X^3 + \partial_y X^1 + X^5, & \\ (\mathcal{L}_X g)_{13} = \partial_x X^2 + \partial_u X^1, & (\mathcal{L}_X g)_{14} = a\partial_x X^4 + \partial_v X^1, \\ (\mathcal{L}_X g)_{15} = \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^3, \\ (\mathcal{L}_X g)_{23} = \partial_y X^2 + \partial_u X^3, & \\ (\mathcal{L}_X g)_{24} = a\partial_y X^4 + \partial_v X^3 + aX^5, & \\ (\mathcal{L}_X g)_{25} = \partial_y X^5 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, & (\mathcal{L}_X g)_{33} = 2\partial_u X^2, \\ (\mathcal{L}_X g)_{34} = a\partial_u X^4 + \partial_v X^2, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & (\mathcal{L}_X g)_{44} = 2a\partial_v X^4, \\ (\mathcal{L}_X g)_{45} = \partial_v X^5 + ax\partial_u X^4 + ay\partial_v X^4 + a\partial_z X^4 - aX^2, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5, & \end{array} \right. \quad (3.16)$$

and

$$(\mathcal{L}_X S)_{ij} = \begin{pmatrix} 0 & \frac{1-a}{2}\partial_x X^2 & 0 & 0 & 0 \\ (1-a)\partial_x X^2 & \frac{1-a}{2}\partial_u X^2 & \frac{1-a}{2}\partial_v X^2 & \frac{1-a}{2}(x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.17)$$

Applying (3.14), (3.15), (3.16) and (3.17) in (1.1) and (1.2), we deduce

$$\left\{ \begin{array}{ll} 2\partial_x X^1 = \alpha, & \partial_x X^3 + \partial_y X^1 + X^5 = 0, \\ \partial_x X^2 + \partial_u X^1 = 0, & a\partial_x X^4 + \partial_v X^1 = 0, \\ \partial_x X^5 + x\partial_u X^1 + y\partial_v X^1 + \partial_z X^1 = 0, & 2\partial_y X^3 = \frac{1-a}{2}\beta, \\ \partial_y X^2 + \partial_u X^3 = \alpha, & a\partial_y X^4 + \partial_v X^3 + aX^5 = 0, \\ \partial_y X^5 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1 = 0, & 2\partial_u X^2 = 0, \\ a\partial_u X^4 + \partial_v X^2 = 0, & \\ \partial_u X^5 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 = 0, & \\ 2a\partial_v X^4 = a\alpha, & \\ \partial_v X^5 + ax\partial_u X^4 + ay\partial_v X^4 + a\partial_z X^4 - aX^2 = 0, & \\ 2x\partial_u X^5 + 2y\partial_v X^5 + 2\partial_z X^5 = \alpha, & \end{array} \right.$$

and

$$\begin{aligned}
0 &= \beta, \\
(1-a)\partial_x X^2 &= 0, \\
(1-a)\partial_y X^2 &= \frac{1-a}{2}\alpha, \\
(1-a)\partial_u X^2 &= 0, \\
(1-a)\partial_v X^2 &= 0, \\
(1-a)\partial_z X^2 &= 0.
\end{aligned}$$

Solving the above equations, we get the following theorem:

**Theorem 3.5.** *The left-invariant Lorentzian metric  $g_5$  has a Ricci bi-conformal vector field  $X = X^i e_i$  if and only if*

$$\begin{aligned}
X^1 &= c_1 \sin y + c_2 \cos y + c_3 y + c_4, \\
X^2 &= c_5, \\
X^3 &= c_6 x + c_3 v + c_4 z + c_7, \\
X^4 &= c_1 \sin y + c_2 \cos y + (c_3 + c_6)y + c_5 z + c_8 - \frac{c_3}{a}y, \\
X^5 &= -c_1 \cos y + c_2 \sin y - c_3 - c_6,
\end{aligned}$$

and  $\alpha = \beta = 0$  for some constants  $c_1, \dots, c_8$ .

Therefore, we have the following result:

**Corollary 3.6.** *Any Ricci bi-conformal vector field  $X$  with respect to the left-invariant Lorentzian metric  $g_5$  is gradient vector field with potential function  $h = \bar{c}_1(x + yz - v) + \bar{c}_2 y + \bar{c}_3$  where  $\bar{c}_1, \bar{c}_2, \bar{c}_3$  are arbitrary real constants.*

### 3.4. The family of metrics $g_6$

The Levi-Civita connection  $\nabla$  of the left invariant Lorentzian metric  $g_6$  is represented by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\frac{1}{2}e_4 & 0 & \frac{1}{2}e_3 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}e_4 & 0 & 0 & 0 & \frac{1}{2}e_1 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}e_3 & -e_4 & \frac{1}{2a}e_1 & 0 & e_2 \end{pmatrix} \quad (3.18)$$

and the Ricci tensor of  $g_\mu$  is given by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2a} \end{pmatrix} \quad (3.19)$$

with respect to the basis  $\{e_1, e_2, e_3, e_4\}$ . For left invariant Lorentzian metric  $g_6$  and any vector fields  $X = X^i e_i$  we have



$$\left\{ \begin{array}{ll} (\mathcal{L}_X g)_{11} = 2a\partial_x X^1, & (\mathcal{L}_X g)_{12} = \partial_x X^2 + a\partial_y X^1, \\ (\mathcal{L}_X g)_{13} = \partial_x X^3 + a\partial_u X^1 + X^5, & (\mathcal{L}_X g)_{14} = \partial_x X^5 + a\partial_v X^1, \\ (\mathcal{L}_X g)_{15} = \partial_x X^4 + ax\partial_u X^1 + ay\partial_v X^1 + a\partial_z X^1, & (\mathcal{L}_X g)_{22} = 2\partial_y X^2, \\ (\mathcal{L}_X g)_{23} = \partial_y X^3 + \partial_u X^2, & (\mathcal{L}_X g)_{24} = \partial_y X^5 + \partial_v X^2, \\ (\mathcal{L}_X g)_{25} = \partial_y X^4 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 + X^5, & (\mathcal{L}_X g)_{33} = 2\partial_u X^3, \\ (\mathcal{L}_X g)_{34} = \partial_u X^5 + \partial_v X^3, & \\ (\mathcal{L}_X g)_{35} = \partial_u X^4 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, & \\ (\mathcal{L}_X g)_{44} = 2\partial_v X^5, & \\ (\mathcal{L}_X g)_{45} = \partial_v X^4 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5, & \\ (\mathcal{L}_X g)_{55} = 2x\partial_u X^4 + 2y\partial_v X^4 + 2\partial_z X^4 - 2X^2, & \end{array} \right. \quad (3.20)$$

and

$$(\mathcal{L}_X S)_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2a}\partial_x X^5 \\ & 0 & 0 & 0 & -\frac{1}{2a}\partial_y X^5 \\ & & 0 & 0 & -\frac{1}{2a}\partial_u X^5 \\ & & & 0 & -\frac{1}{2a}\partial_v X^5 \\ & & & & -\frac{1}{a}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) \end{pmatrix}. \quad (3.21)$$

Applying (3.18), (3.19), (3.20) and (3.21) in (1.1) and (1.2), we conclude

$$\left\{ \begin{array}{ll} 2a\partial_x X^1 = a\alpha, & \partial_x X^2 + a\partial_y X^1 = 0, \\ \partial_x X^3 + a\partial_u X^1 + X^5 = 0, & \partial_x X^5 + a\partial_v X^1 = 0, \\ \partial_x X^4 + ax\partial_u X^1 + ay\partial_v X^1 + a\partial_z X^1 = 0, & 2\partial_y X^2 = \alpha, \\ \partial_y X^3 + \partial_u X^2 = 0, & \partial_y X^5 + \partial_v X^2 = 0, \\ \partial_y X^4 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2 + X^5 = 0, & 2\partial_u X^3 = \alpha, \\ \partial_u X^4 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1 = 0, & \partial_u X^5 + \partial_v X^3 = 0, \\ 2\partial_v X^5 = 0, & \partial_v X^4 + x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5 = \alpha, \\ 2x\partial_u X^4 + 2y\partial_v X^4 + 2\partial_z X^4 - 2X^2 = -\frac{1}{2a}\beta, & \end{array} \right. \quad (3.22)$$

and

$$\begin{aligned} 0 &= \beta, \\ \partial_x X^5 &= 0, \\ \partial_y X^5 &= 0, \\ \partial_u X^5 &= 0, \\ \partial_v X^5 &= 0, \\ -\frac{1}{a}(x\partial_u X^5 + y\partial_v X^5 + \partial_z X^5) &= -\frac{1}{2a}\alpha. \end{aligned} \quad (3.23)$$

By solving systems (3.22) and (3.23), we obtain the following theorem:

**Theorem 3.7.** *The left-invariant Lorentzian metric  $g_\mu$  has a Ricci bi-conformal vector field  $X = X^i e_i$  if and only if  $\alpha = \beta = 0$  and*

$$\begin{aligned} X^1 &= \frac{1}{2}d_1z^2 + d_2z + d_3, \\ X^2 &= d_4z + d_5, \\ X^3 &= -d_6x + \frac{1}{6}d_1z^3 - ad_1z + \frac{1}{2}d_2z^2 + d_3z + d_7, \\ X^4 &= ad_1u - (d_6 + d_4)y - (d_1z + d_2)ax + \frac{1}{2}d_4z^2 + d_5z + d_8, \\ X^5 &= d_6, \end{aligned}$$

for some constants  $d_1, \dots, d_8$ .

Therefore, we have the following result:

**Corollary 3.8.** *Any Ricci bi-conformal vector field  $X$  with respect to the left-invariant Lorentzian metric  $g_6$  is gradient vector field with potential function  $h = \frac{\bar{d}_1}{2}(2y + z^2) + \bar{d}_2z + \bar{d}_3$  where  $\bar{d}_1, \bar{d}_2, \bar{d}_3$  are arbitrary real constants.*

### 3.5. The family of metrics $g_7$

The Levi-Civita connection  $\nabla$  of the left invariant Lorentzian metric  $g_7$  is described by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\frac{1}{2a}e_1 & 0 & \frac{1}{2}e_3 \\ 0 & 0 & 0 & -\frac{1}{2a}e_1 & \frac{1}{2}e_4 \\ -\frac{1}{2a}e_1 & 0 & 0 & 0 & \frac{1}{2a}e_5 \\ 0 & -\frac{1}{2a}e_1 & 0 & 0 & \frac{1}{2}e_2 \\ -\frac{1}{2}e_3 & \frac{1}{2a}e_4 & \frac{1}{2a}e_5 & \frac{1}{2}e_2 & 0 \end{pmatrix} \quad (3.24)$$

and the Ricci tensor of  $g_7$  is determined by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2a^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2a} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad (3.25)$$

with respect to the basis  $\{e_1, e_2, e_3, e_4\}$ . For left invariant Lorentz metric  $g_7$  and any vector fields  $X = X^i e_i$ , we obtain

$$\left\{ \begin{aligned} (\mathcal{L}_X g)_{11} &= 2a\partial_x X^5, & (\mathcal{L}_X g)_{12} &= \partial_x X^2 + a\partial_y X^5, \\ (\mathcal{L}_X g)_{13} &= \partial_x X^3 + a\partial_u X^5 + X^5, & (\mathcal{L}_X g)_{14} &= \partial_x X^4 + a\partial_v X^5, \\ (\mathcal{L}_X g)_{15} &= a\partial_x X^1 + ax\partial_u X^5 + ay\partial_v X^5 + a\partial_z X^5, & (\mathcal{L}_X g)_{22} &= 2\partial_y X^2, \\ (\mathcal{L}_X g)_{23} &= \partial_y X^3 + \partial_u X^2, & (\mathcal{L}_X g)_{24} &= \partial_y X^4 + \partial_v X^2 + X^5, \\ (\mathcal{L}_X g)_{25} &= a\partial_y X^1 + x\partial_u X^2 + y\partial_v X^2 + \partial_z X^2, & (\mathcal{L}_X g)_{33} &= 2\partial_u X^3, \\ (\mathcal{L}_X g)_{34} &= \partial_u X^4 + \partial_v X^3, \\ (\mathcal{L}_X g)_{35} &= a\partial_u X^1 + x\partial_u X^3 + y\partial_v X^3 + \partial_z X^3 - X^1, \\ (\mathcal{L}_X g)_{44} &= 2\partial_v X^4, \\ (\mathcal{L}_X g)_{45} &= a\partial_v X^1 + x\partial_u X^4 + y\partial_v X^4 + \partial_z X^4 - X^2, \\ (\mathcal{L}_X g)_{55} &= 2ax\partial_u X^1 + 2ay\partial_v X^1 + 2a\partial_z X^1, \end{aligned} \right. \quad (3.26)$$

and

$$\left\{ \begin{array}{l} (\mathcal{L}_X S)_{11} = \frac{1}{a} \partial_x X^5, \\ (\mathcal{L}_X S)_{12} = \frac{1}{2a} \partial_y X^5, \\ (\mathcal{L}_X S)_{13} = -\frac{1}{2a^2} (X^5 + \partial_x X^3 - a \partial_u X^5), \\ (\mathcal{L}_X S)_{14} = \frac{1}{2a} \partial_v X^5, \\ (\mathcal{L}_X S)_{15} = \frac{1}{2a} (\partial_x X^1 - a \partial_x X^5 + x \partial_u X^5 + y \partial_v X^5 + \partial_z X^5), \\ (\mathcal{L}_X S)_{22} = 0, \\ (\mathcal{L}_X S)_{23} = -\frac{1}{2a^2} \partial_y X^3, \\ (\mathcal{L}_X S)_{24} = 0, \\ (\mathcal{L}_X S)_{25} = \frac{1}{2a} (\partial_y X^1 - a \partial_y X^5), \\ (\mathcal{L}_X S)_{33} = -\frac{1}{a^2} \partial_u X^3, \\ (\mathcal{L}_X S)_{34} = -\frac{1}{2a^2} \partial_v X^3, \\ (\mathcal{L}_X S)_{35} = \frac{1}{2a^2} (a \partial_u X^1 - a^2 \partial_u X^5 + X^1 - x \partial_u X^3 - y \partial_v X^3 - \partial_z X^3), \\ (\mathcal{L}_X S)_{44} = 0, \\ (\mathcal{L}_X S)_{45} = \frac{1}{2a} (\partial_v X^1 - a \partial_v X^5), \\ (\mathcal{L}_X S)_{55} = \frac{1}{a} (x \partial_u X^1 + y \partial_v X^1 + \partial_z X^1) - (x \partial_u X^5 + y \partial_v X^5 + \partial_z X^5). \end{array} \right. \quad (3.27)$$

Applying (3.24), (3.25), (3.26), and (3.27) in (1.1) and (1.2), we can write

$$\left\{ \begin{array}{ll} 2a \partial_x X^5 = 0, & \partial_x X^2 + a \partial_y X^5 = 0, \\ \partial_x X^3 + a \partial_u X^5 + X^5 = 0, & \partial_x X^4 + a \partial_v X^5 = 0, \\ a \partial_x X^1 + a x \partial_u X^5 + a y \partial_v X^5 + a \partial_z X^5 = a \alpha + \frac{1}{2a} \beta, & 2 \partial_y X^2 = \alpha, \\ \partial_y X^3 + \partial_u X^2 = 0, & \partial_y X^4 + \partial_v X^2 + X^5 = 0, \\ a \partial_y X^1 + x \partial_u X^2 + y \partial_v X^2 + \partial_z X^2 = 0, & 2 \partial_u X^3 = \alpha - \frac{1}{2a^2} \beta, \\ a \partial_u X^1 + x \partial_u X^3 + y \partial_v X^3 + \partial_z X^3 - X^1 = 0, & \partial_u X^4 + \partial_v X^3 = 0, \\ a \partial_v X^1 + x \partial_u X^4 + y \partial_v X^4 + \partial_z X^4 - X^2 = 0, & 2 \partial_v X^4 = \alpha, \\ 2ax \partial_u X^1 + 2ay \partial_v X^1 + 2a \partial_z X^1 = -\frac{1}{2} \beta, & \end{array} \right.$$

and

$$\left\{ \begin{array}{ll} \frac{1}{a} \partial_x X^5 = 0, & \frac{1}{2a} \partial_y X^5 = 0, \\ -\frac{1}{2a^2} (X^5 + \partial_x X^3 - a \partial_u X^5) = 0, & \frac{1}{2a} \partial_v X^5 = 0, \\ \frac{1}{2a} (\partial_x X^1 - a \partial_x X^5 + x \partial_u X^5 + y \partial_v X^5 + \partial_z X^5) = \frac{1}{2a} \alpha + a \beta, & 0 = \beta, \\ -\frac{1}{2a^2} \partial_y X^3 = 0, & \frac{1}{2a} (\partial_y X^1 - a \partial_y X^5) = 0, \\ -\frac{1}{a^2} \partial_u X^3 = -\frac{1}{2a^2} \alpha + \beta, & -\frac{1}{2a^2} \partial_v X^3 = 0, \\ \frac{1}{2a^2} (a \partial_u X^1 - a^2 \partial_u X^5 \\ \quad + X^1 - x \partial_u X^3 - y \partial_v X^3 - \partial_z X^3) = 0, & \frac{1}{2a} (\partial_v X^1 - a \partial_v X^5) = 0, \\ \frac{1}{a} (x \partial_u X^1 + y \partial_v X^1 + \partial_z X^1) \\ \quad - (x \partial_u X^5 + y \partial_v X^5 + \partial_z X^5) = -\frac{1}{2} \alpha. & \end{array} \right.$$

Solving the above systems implies that in the following theorem:

**Theorem 3.9.** *The left-invariant Lorentzian metric  $g_7$  has a Ricci bi-conformal vector field  $X$  if and only if  $X = k_1 e_1 + k_2 e_2 + (k_1 z - k_5 x + k_3) e_3 + (k_2 z - k_5 y + k_4) e_4 + k_5 e_5$  and  $\alpha = \beta = 0$  for some constants  $k_1, \dots, k_5$ .*

Also, we have the following result:

**Corollary 3.10.** *Any Ricci bi-conformal vector field  $X$  with respect to the left-invariant Lorentzian metric  $g_7$  is gradient vector field with potential function  $h = \bar{k}_1$  where  $\bar{k}_1$  is an arbitrary real constant.*

**Remark 3.11.** From Theorems 3.1, 3.3, 3.5, 3.7, and 3.9, we conclude that all Ricci bi-conformal vector fields on five-dimensional Lorentzian nilpotent Lie groups  $(G, g_i)$  are Killing vector fields for  $i = 1, 2, \dots, 7$ .

**Acknowledgment.** We would like to thank reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the manuscript.

## References

- [1] S. Azami, *Generalized Ricci solitons of three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections*, J. Nonlinear Math. Phys. **30** (1), 1-33, 2023.
- [2] P. Baird and L. Danielo, *Three-dimensional Ricci solitons which project to surfaces*, J. Reine Angew. Math. **608**, 65-91, 2007.
- [3] W. Batat and K. Onda, *Four-dimensional pseudo-Riemannian generalized symmetric spaces which are algebraic Ricci solitons*, Results Math. **64**, 253-267, 2013.
- [4] N. Bokan, T. Sukilovic and S. Vukmirovic, *Lorentz geometry of 4-dimensional nilpotent Lie groups*, Geom. dedicata **177**, 83-102, 2015.
- [5] A. Bouharis and B. Djebbar, *Ricci solitons on Lorentzian four-dimensional generalized symmetric spaces*, J. Math. Phys. Anal. Geom. **14** (2), 132-140, 2018.
- [6] G. Calvaruso, *Three-dimensional homogeneous generalized Ricci solitons*, Mediterr. J. Math. **14** (5), 1-21, 2017.
- [7] S. M. Carroll, *Spacetime and geometry: an introduction to general relativity*, Addison Wesley., 133-139, 2004.
- [8] B. Coll, S. R. Hildebrondt and J. M. M. Senovilla, *Kerr-Schild symmetries*, Gen. Relativ. Gravit. **33**, 649-670, 2001.
- [9] U. C. De, A. Sardar, and A. Sarkar, *Some conformal vector fields and conformal Ricci solitons on  $N(k)$ -contact metric manifolds*, AUT J. Math. Com. **2** (1), 61-71, 2021.
- [10] S. Deshmukh, *Geometry of conformal vector fields*, Arab. J. Math. **23** (1), 44-73, 2017.
- [11] A. Garcia-Parrado and J. M. M. Senovilla, *Bi-conformal vector fields and their applications*, Classical Quantum Gravity **21** (8), 2153-2177, 2004.
- [12] R. S. Hamilton, *The Ricci flow on surfaces in Mathematics and General Relativity*, Contemp. Math. **71**, Amer. Math. Soc. Providence, RI, 1988, 237-262.
- [13] J. Lauret, *Ricci solitons solvmanifolds*, J. Reine Angew. Math. **650**, 1-21, 2011.
- [14] L. Magnin, *Sur les algèbres de Lie nilpotents de dimension  $\leq 7$* , J. Geom. Phys. **3** (1), 119, 1986.
- [15] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J. Geom. Anal. **26**, 1280-1345, 2016.
- [16] T. H. Wears, *On Lorentzian Ricci solitons on nilpotent Lie groups*, Math. Nachr. **290** (8-9), 1381-1405, 2017.
- [17] K. Yano, *The theory of Lie derivatives and its applications*, Dover publications, 2020.