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F-Planar Curves on Para–Kähler Manifolds

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Abstract. This paper deals with classifications of F-planar curves on para-Kähler manifolds. Also, we give some examples related to them.

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1. Introduction

Let (M, g) be a n-dimensional pseudo-Riemannian manifold, where M is a differentiable manifold and g is a pseudo-Riemannian metric. A magnetic field on (M, g) is a closed 2-form F. The Lorentz force of the magnetic field F on manifold (M, g) is a $(1, 1)-$type tensor field Φ. For any vector fields $X, Y \in \chi(M)$, it is expressed as

$$g(\Phi(X), Y) = F(X, Y).$$

The magnetic curves on the pseudo-Riemannian manifold (M, g) are the trajectories of charged particles moving on M under the influence of the magnetic field F. The magnetic trajectories of F are the curves of M in the Lorentz equation. Hence, the Lorentz equation is as follows:

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'),$$

where the connection ∇ is the Levi-Civita connection of g. The generalized Lorentz equation obtained from the geodesics of M, that is, $\nabla_{\gamma'}\gamma' = 0$. Therefore, the magnetic curves generalize geodesics. The magnetic curves have been studied in different space. The magnetic curves on Kähler magnetic fields in complex space were studied by Adachi in [1] and were obtained some interesting results. Also, in [2], Corbrerizo obtained some different results when working on the Sasakian 3-manifold.

The Lorentz force is skew symmetric. Hence, we can write the following equation:

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\nabla_{\gamma'}\gamma', \gamma') = 0.$$

Therefore, the magnetic curves have a constant speed $\upsilon(t) = \left\|\gamma'\right\|_{\nu} = \upsilon_0$. When the magnetic curve $\gamma(t)$ is arc-lenght parametrized $(\upsilon_0 = 1)$, it is called a normal magnetic curve. The arc-lenght parameter of these curves is s.

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The magnetic field is always divergence-free (see [2]). Especially, Killing magnetic fields formed by Killing vector fields are the most important class of magnetic fields. A vector field V on M is a Killing vector field if and only if it satisfies the following Killing equation:

$$g(\nabla_U V, W) + g(\nabla_W V, U) = 0$$

for each vector field U and W on M, where the connection ∇ is the Levi-Civita connection of g on M (see [2,6,9,12]).

On the vector space $\mathbb{E}^{2n}_n$, the pseudo inner product can be defined in the form

$$\langle u, v\rangle_v = -\sum_{i=1}^{n} u_i v_i + \sum_{i=n+1}^{2n} u_i v_i$$

for $u = (u_1, ..., u_n, u_{n+1}, ..., u_{2n})$ and $v = (v_1, ..., v_n, v_{n+1}, ..., v_{2n})$ on $\mathbb{E}^{2n}_n$. Since the pseudo inner product is not positive, the vectors in this space are classified as follows:

- ∗: If $\langle u, u\rangle_v > 0$ or $u = 0$, then the vector u is spacelike.
- ∗: If $\langle u, u\rangle_v < 0$, then the vector u is timelike.
- ∗: If $\langle u, u\rangle_v = 0$ and $u \neq 0$, then the vector u is lightlike (or null),

where $u = (u_1, ..., u_n, u_{n+1}, ...u_{2n})$ is any vector in the space $\mathbb{E}^{2n}_n$ [10].

Almost para-Hermitian manifolds consist of a pseudo-Riemannian metric g and an almost product structure K ($K^2 = I, K \neq \pm I$), where I is the identity map such that

$$g(KU, KW) = -g(U, W)$$

for any vector fields U and W on M. An almost para-Hermitian manifold is called a para-Kähler manifold if $\nabla K = 0$. The para–Kähler manifolds firstly defined and studied by Rashevski in 1948 [16], and then many scientists from past to present have worked on the para-Kähler manifold. We refer to Rozenfeld, Ruse and Liberman in 1949 [11,17,18]. In addition, para–Kähler manifolds have recently been applied to supersymmetric field theories and studied in many different fields [3–5,7,14,15,19]

Let (M, g) be a n-dimensional pseudo-Riemannian and a $(1,1)-$type tensor field F. A curve $\gamma : A \subseteq \mathbb{R} \rightarrow M$ is called $F-$planar if the velocity of the curve γ, that is, the tangent vector γ' satisfies the following equation for each $t \in A$:

$$\nabla_{\gamma'}\gamma' = \mu(t)\gamma' + \xi(t)F\gamma',$$

where the functions μ and ξ are two differentiable functions depending on t and the connection ∇ is the Levi-Civita connection of g [8]. In [8] and [13], the $F-$planar curves studied on different manifolds and obtained different results. $F-$planar curves are a curve that represents a planar graph. If the nodes of the planar graph can be located on the curve without intersecting each other, then this curve is called $F-$planar. One of the important features of these curves is that there is at least one $F-$planar curve corresponding to every planar graph. In addition to the use of $F-$planar curves in planar graph drawings, it can also be used in other mathematical problems.

In this study, we will show that F-planar curves are generalized magnetic curves.

2. F-planar Curves on Para-Kähler Manifolds

Theorem 2.1. *Let (M, g) be the pseudo-Riemannian manifold and γ be a unit speed F-planar curve. Under the condition $\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}$, the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \mu(s)\dot{\gamma}(s) + \xi(s)F\dot{\gamma}(s)$ reduces to $\nabla_{\dot{\gamma}}\dot{\gamma} = \xi(s)F\dot{\gamma}(s)$.*

Proof. Let us that $\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}$. Then, the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \mu(s)\dot{\gamma}(s) + \xi(s)F\dot{\gamma}(s)$ transforms into

$$\ddot{\gamma} = \mu(s)\dot{\gamma}(s) + \xi(s)F\dot{\gamma}(s).$$

Since γ is a unit speed curve, we have

$$\begin{aligned} g(\ddot{\gamma}(s), \dot{\gamma}(s)) &= \mu(s)g(\dot{\gamma}(s), \dot{\gamma}(s)) + \xi(s)g(F\dot{\gamma}(s), \dot{\gamma}(s)) \\ &= \mu(s), \end{aligned}$$

such that $\mu(s) = 0$. □

The arc-length of a smooth curve γ is a trajectory of F_q if the following Lorentz equation is satisfied:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \xi(s)K\dot{\gamma}(s).$$

Let (M, K, g) be a para-Kähler manifold and be the two-form Ω_K, where

$$\Omega_K(U, W) = g(KU, W),$$

for each $U, W \in \chi(M)$.
Let $\gamma : A \to M$ be a smooth curve on M. If the curve γ satisfies the following equation, then it is the magnetic orbitals corresponding to para-Kähler magnetic field $F_q = \xi(s)\Omega_K$:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \xi(s)K\dot{\gamma}(s),$$

where $\xi(s) \neq 0$. Since K is skew-symmetric, we can write the following equation:

$$\frac{d}{ds}g(\dot{\gamma}, \dot{\gamma}) = 2g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = 2\xi(s)g(K\dot{\gamma}, \dot{\gamma}) = 0,$$

where $\dot{\gamma}$ depends on the s parameter.

3. F-Planar Curves on $\mathbb{E}_n^{2n}$

Let the coordinates $\mathbb{E}_n^{2n}$ be $(x_1, x_2, ..., x_n; y_1, y_2, ..., y_n)$. The definition of pseudo-Euclidean metric according to given coordinates is

$$g = -\sum_{k=1}^{n} dx_k^2 + \sum_{k=1}^{n} dy_k^2,$$

and the para-complex structure is

$$K\frac{\partial}{\partial y_k} = \frac{\partial}{\partial x_k}, K\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}.$$

The manifold $\mathbb{E}_n^{2n} = (\mathbb{R}^{2n}, K, g)$ is a flat para-Kähler manifold. Thus, its fundamental two-form is $g(KU, W) = \Omega_K(U, W)$. Assume that the magnetic field is $F_q = \xi(s)\Omega_K$, where $\xi(s) \neq 0$ and the curve $\gamma : A \subseteq \mathbb{R} \to \mathbb{E}_n^{2n}$ is the orbit corresponding to the magnetic field F_q. Then, the Lorentz equation is as follows:

$$\ddot{\gamma} = \xi(s)K\dot{\gamma}.$$

Therefore, we have the following result for the spacelike and timelike F-planar curves.

Theorem 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}_n^{2n}$ be a magnetic curve corresponding to the F-planar curve flat para-Kähler structure on $\mathbb{E}_n^{2n}$. In the given in the ambient space, the curve γ is ordered as follows:*

1.*i*: $\gamma(s) = \left(\int e^{\varphi(s)}, 0, ..., 0; \int e^{\varphi(s)}, 0, ..., 0\right)$;
1.*ii*: $\gamma(s) = \left(\int e^{-\varphi(s)}, 0, ..., 0; \int e^{-\varphi(s)}, 0, ..., 0\right)$;
2.*i*: $\gamma(s) = (\int \sinh(\varphi(s)), 0, \dots, 0; \int \cosh(\varphi(s)), 0, \dots, 0)$;
2.*ii*: $\gamma(s) = (\int \cosh(\varphi(s)), 0, \dots, 0; \int \sinh(\varphi(s)), 0, \dots, 0)$;

where $\varphi(s) = \int \xi(s)ds$.

Proof. Let $\gamma : I \longrightarrow \mathbb{E}_n^{2n}$ be a magnetic curve. The velocity vector of the curve γ is as follows:

$$\dot{\gamma} = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial y_k},$$

where the functions a_k and b_k are smooth functions. Moreover, they satisfy

$$-\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 = \delta,$$

where $\delta \in \{-1, 0, 1\}$.
From the Lorentz equation, we have the following differential equations:

$$\begin{cases} \dot{a}_k = \xi(s) b_k \\ \dot{b}_k = \xi(s) a_k \end{cases}, k = 1, \ldots, n. \tag{3.1}$$

The general solution of the equation (3.1) is

$$\begin{cases} a_k = \alpha_k \cosh(\int \xi(s)ds) + \beta_k \sinh(\int \xi(s)ds) \\ b_k = \beta_k \cosh(\int \xi(s)ds) + \alpha_k \sinh(\int \xi(s)ds) \end{cases} \alpha_k, \beta_k \in \mathbb{R}, \quad k = 1, \ldots, n. \tag{3.2}$$

Thus, the equation (3.2) satisfies the equation (3.3). Consequently, the velocity vector of curve γ is :

$$\dot{\gamma} = \cosh(\varphi(s))W + \sinh(\varphi(s))KW, \tag{3.3}$$

where

$$W = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} + \sum_{k=1}^{n} \beta_k \frac{\partial}{\partial y_k} \ , \ W \neq 0.$$

There are two cases according to whether the W and KW vectors are linearly dependent and linearly independent.
Case 1. Assume that the vectors W and KW are linear dependent. This means that the vector W is a constant lightlike vector of the form $W = \sum_{k=1}^{n} \alpha_k \left(\frac{\partial}{\partial x_k} + \varepsilon \frac{\partial}{\partial y_k} \right)$, where $\varepsilon = \pm 1$. Hence, the velocity vector of γ can be expressed as

$$\dot{\gamma} = (\cosh(\varphi(s)) + \varepsilon \sinh(\varphi(s))W$$

and using the velocity vector of this γ , we can write the curve γ as follows:

$$\gamma(s) = \gamma_0 + (\int \cosh(\varphi(s))ds) + \varepsilon \int \sinh(\varphi(s))ds)W.$$

Then, we have the following two cases:
1.i: For $\varepsilon = 1$:

$$\begin{cases} x(s) = x_0 + (\int e^{\varphi(s)}, 0, ..., 0) \\ y(s) = y_0 + (\int e^{\varphi(s)}, 0, ..., 0). \end{cases}$$

1.ii: For $\varepsilon = -1$:

$$\begin{cases} x(s) = x_0 + (\int e^{-\varphi(s)}, 0, ..., 0) \\ y(s) = y_0 + (\int e^{-\varphi(s)}, 0, ..., 0). \end{cases}$$

Case 2. Assume that the vectors W and KW are linear independent. Thus, these vectors are orthogonal. Hence, we have the following equation:

$$\delta = g(\dot{\gamma}, \dot{\gamma}) = \cosh^2(\varphi(s))g(W, W) + \sinh^2(\varphi(s))g(KW, KW) = g(W, W).$$

2.i: For $\delta = 1$: Without breaking the generality, we can get the vectors W and KW as follows: $W = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity vector of the curve γ in terms of the vectors W and KW given, we get

$$\dot{\gamma}(s) = \sinh(\varphi(s))e_1 + \cosh(\varphi(s))\bar{e}_1,$$

where the curve γ is a spacelike hyperbola:

$$\begin{cases} x(s) = x_0 + (\int \sinh(\varphi(s)), 0, \ldots, 0) \\ y(s) = y_0 + (\int \cosh(\varphi(s)), 0, \ldots, 0). \end{cases}$$

2.ii: For $\delta = -1$: Assume that $W = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity of the curve γ in terms of the vectors W and KW given, we get

$$\dot{\gamma}(s) = \cosh(\varphi(s))e_1 + \sinh(\varphi(s))\bar{e}_1,$$

where the curve γ is a timelike hyperbola:

$$\begin{cases} x(s) = x_0 + (\int \cosh(\varphi(s)), 0, \ldots, 0) \\ y(s) = y_0 + (\int \sinh(\varphi(s)), 0, \ldots, 0). \end{cases}$$

□

Now, let's concretize our work with a few examples below:

Example 3.2. If we take it as $\xi(s) = \frac{1}{s}$, $s \neq 0$, the curve will be as follows:

Case 1. Suppose that the vectors W and KW are linear dependent. Hence, we can express the velocity vector of the curve γ with the following equation:

$$\dot{\gamma} = \cosh(\int \frac{1}{s} ds)W + \sinh(\int \frac{1}{s} ds)KW. \tag{3.4}$$

From the equation (3.4), we have

$$\gamma(s) = \gamma_0 + (\int \cosh(\int \frac{1}{s} ds)ds) + \varepsilon \int \sinh(\int \frac{1}{s} ds)ds)W.$$

Then, the curve γ is written as follows.

1.i: For $\varepsilon = 1$:

$$\begin{cases} x(s) = x_0 + (\frac{s^2}{2}, 0, ..., 0) \\ y(s) = y_0 + (\frac{s^2}{2}, 0, ..., 0). \end{cases}$$

1.ii: For $\varepsilon = -1$:

$$\begin{cases} x(s) = x_0 + (\ln(s), 0, ..., 0) \\ y(s) = y_0 + (\ln(s), 0, ..., 0). \end{cases}$$

Case 2. Suppose that the vectors W and KW are linear independent.

2.i: For $\delta = 1$: We can get the vectors W and KW as follows. $W = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity vector of the curve in terms of the vectors W and KW given, we have

$$\dot{\gamma}(s) = \sinh(\int \frac{1}{s} ds)e_1 + \cosh(\int \frac{1}{s} ds)\bar{e}_1,$$

where the curve γ is a spacelike hyperbola:

$$\begin{cases} x(s) = x_0 + (\frac{s^2}{4} - \frac{1}{2}\ln(s), 0, ..., 0) \\ y(s) = y_0 + (\frac{s^2}{4} + \frac{1}{2}\ln(s), 0, ..., 0). \end{cases}$$

2.ii: For $\delta = -1$: Assume that $W = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity vector of the curve γ in terms of the vectors W and KW given, we have

$$\dot{\gamma}(s) = \cosh(\int \frac{1}{s} ds)e_1 + \sinh(\int \frac{1}{s} ds)\bar{e}_1,$$

where the curve γ is a timelike hyperbola:

$$\begin{cases} x(s) = x_0 + (\frac{s^2}{4} + \frac{1}{2}\ln(s), 0, ..., 0) \\ y(s) = y_0 + (\frac{s^2}{4} - \frac{1}{2}\ln(s), 0, ..., 0). \end{cases}$$

Example 3.3. If we take it as $\xi(s) = \frac{1}{2\sqrt{s}}$, $s > 0$, the curve will be as follows:

Case 1. Suppose that the vectors W and KW are linear dependent. Hence, we can express the velocity vector of the curve γ with the following equation:

$$\dot{\gamma} = \cosh(\int \frac{1}{2\sqrt{s}} ds)W + \sinh(\int \frac{1}{2\sqrt{s}} ds)KW. \tag{3.5}$$

From the equation (3.5), we have the following equation:

$$\gamma(s) = \gamma_0 + (\int \cosh(\int \frac{1}{2\sqrt{s}} ds)ds) + \varepsilon \int \sinh(\int \frac{1}{2\sqrt{s}} ds)ds)W$$

such that the curve γ is written as follows:

1.i: For $\varepsilon = 1$:

$$\begin{cases} x(s) = x_0 + (2e^{\sqrt{s}}\sqrt{s} - 2e^{\sqrt{s}}, 0, ..., 0) \\ y(s) = y_0 + (2e^{\sqrt{s}}\sqrt{s} - 2e^{\sqrt{s}}, 0, ..., 0). \end{cases}$$

1.ii: For $\varepsilon = -1$:

$$\begin{cases} x(s) = x_0 + (2e^{-\sqrt{s}}\sqrt{s} + 2e^{\sqrt{s}}, 0, ..., 0) \\ y(s) = y_0 + (2e^{-\sqrt{s}}\sqrt{s} + 2e^{\sqrt{s}}, 0, ..., 0). \end{cases}$$

Case 2. Suppose that the vectors W and KW are linear independent.
2.i: For $\delta = 1$: We can get W and KW as follows. $W = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity vector of the curve γ in terms of the vectors W and KW given, we have

$$\dot{\gamma}(s) = \sinh(\int \frac{1}{2\sqrt{s}} ds)e_1 + \cosh(\int \frac{1}{2\sqrt{s}} ds)\bar{e}_1,$$

where the curve γ is a spacelike hyperbola:

$$\begin{cases} x(s) = x_0 + (e^{\sqrt{s}}\sqrt{s} - e^{-\sqrt{s}} - \frac{\sqrt{s}+1}{e^{\sqrt{s}}}, 0, ..., 0) \\ y(s) = y_0 + (e^{\sqrt{s}}\sqrt{s} - e^{\sqrt{s}} + \frac{\sqrt{s}+1}{e^{\sqrt{s}}}, 0, ..., 0). \end{cases}$$

2.ii: For $\delta = -1$: Assume $W = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{E}^{2n}_n$ and $KW = \bar{e}_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{E}^{2n}_n$. If we write the velocity vector of the curve γ in terms of the vectors W and KW given, it is easily seen that

$$\dot{\gamma}(s) = \cosh(\int \frac{1}{2\sqrt{s}} ds)e_1 + \sinh(\int \frac{1}{2\sqrt{s}} ds)\bar{e}_1,$$

where the curve γ is a timelike hyperbola:

$$\begin{cases} x(s) = x_0 + (e^{\sqrt{s}}\sqrt{s} - e^{-\sqrt{s}} + \frac{\sqrt{s}+1}{e^{\sqrt{s}}}, 0, ..., 0) \\ y(s) = y_0 + (e^{\sqrt{s}}\sqrt{s} - e^{-\sqrt{s}} - \frac{\sqrt{s}+1}{e^{\sqrt{s}}}, 0, ..., 0). \end{cases}$$

Example 3.4. If we take $\xi(s) = q$, where $q \in \mathbb{R}^+$, then we can obtain the paper [9].

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed the published verison of the manuscript.

References

[1] Adachi, T., *Kähler magnetic fields on a complex projective space*, Proc. Japan Acad., **70**(1994), 12–13.
[2] Cabrerizo, J.L., Fernández, M., Gómez, J.S., *The contact magnetic flow in 3D Sasakian manifolds*, J. Phys. A. Math. Theor., **42**(2009), 195201.
[3] Cortés, V., Lawn, M.A., Schäfer, L., *Affine hyperspheres associated to special para-Kähler manifolds*, Int. J. Geom. Methods, **3**(2006), 995–1009.
[4] Cortés, V., *The special geometry of Euclidean supersymmetry: a survey*, Rev. Union Mat. Argent, **47**(2006), 29–34.
[5] Defever, F., Verstraelen, L., Deszcz, R., *On pseudosymmetric para-Kähler manifolds*, In Colloquium Mathematicae, **74**(1998), 253–260.
[6] Druţă-Romaniuc, S.L., Munteanu, M.I., *Magnetic curves corresponding to Killing magnetic fields in* $\mathbb{E}^3$, J. Math. Phys., **52**(2011), 113506.
[7] Gezer, A., Aktas, B., Durmaz, O., *Notes concerning Kähler and anti-Kähler structures on quasi-statistical manifolds*, arXiv preprint arXiv, (2023), 2307.15065.
[8] Hinterleitner, I., Mikeš, J., *On F-planar mappings of spaces with affine connections*, Note di Matematica, **27**(1)(2009), 111-118.
[9] Jleli, M., Munteanu, M.I., *Magnetic curves on flat para-Kähler manifolds*, Turkish Journal of Mathematics, **39**(2015), 963–969.
[10] Keçilioğlu, O., Gündoğan, H., *Pseudo matrix multiplication*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **66**(2017), 37–43.
[11] Libermann, P., *Sur le problème déquivalence de certains structures infinitésimales*, Ann. Mat. Pura App., **36**(1954), 27120.

[12] Munteanu, M.I., *Magnetic curves in the Euclidean space: one example, several approaches*, Publ. I Math-Beograd, **94**(2013), 141–150.
[13] Nistor, A.I., *New examples of F-planar curves in 3-dimensional warped product manifolds*, Kragujevac J. Math., **43**(2019), 247–257.
[14] Noyan, E.B., Gündüzalp, Y., *Proper semi-slant pseudo-Riemannian submersions in para-Kaehler geometry*, International Electronic Journal of Geometry, **15**(2022), 253–265.
[15] Noyan, E.B., Gunduzalp, Y., *Proper bi-slant pseudo-Riemannian submersions whose total manifolds are para-Kaehler manifolds*, Honam Math. J., **44**(2022), 370–383.
[16] Rashevski, P.K., *The scalar field in a stratified space*, Trudy Sem Vektor Tenzor Anal., **6**(1948), 225–248.
[17] Rozenfeld, B.A., *On unitary and stratified spaces*, Trudy Sem Vektor Tenzor Anal., **7**(1949), 260–275.
[18] Ruse, H.S., *On parallel fields of planes in a Riemannian manifold*, Quart J. Math. Oxford Ser., **20**(1949), 218–234.
[19] Vaisman, I., *Generalized para-Kähler manifolds*, Differential Geometry and Its Applications, **42**(2015), 84–103.