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On Generic Submersion in the Contact Context

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Abstract: In the present paper, we introduce a new type of Riemannian submersion in the contact framework such that the fibers of such submersion are generic submanifolds, as given in [10]. This type of submersion is a generalization of many kinds of submersion introduced before in the literature. Once the Reeb vector field ξ is tangent to the fibers, its position is given such that it should lie in the anti-invariant distribution D^0, which is given in the definition of the generic submersion. Moreover, we give an example and some results for such submersions.

Kontakt Geometride Kapsamlı Submersiyonlar Üzerine

Anahtar Kelimeler
Riemann submersiyon,
Hemen hemen kontakt metrik manifold,
Distribüsyon,
Reeb vektör alanı

Öz: Makalede, bu tür submersiyonların liflerinin kapsamlı alt manifoldlar ([10]'da verildiği gibi) olacağı şekilde, kontakt çerçevede, yeni bir Riemann submersiyon tanımlıyoruz. Bu tür submersiyonlar daha önce literatürde tanımalanan bir çok submersiyon çeşidinin genelleştirilmesidir. Reeb vektör alanı liflere teğet olduğu durumda kapsamlı submersiyon tanımında verilen D^0 distribüsyonunda yer alması gerektiği sonucu verildi. Dahası, kapsamlı submersiyon örneği ve bazı sonuçlar verildi.

1. INTRODUCTION

The theory of submanifolds has always been a trending topic in differential geometry. This fact inspired most geometers to define and study new submanifolds. After O'Neill [1] and Gray [2] introduced the concept of Riemannian submersion between two Riemann manifolds, the verity; fibers of a Riemannian submersion are a submanifold of the total manifold, put the geometers in a direction to focus on the theory of submersion. Besides this, Watson considered the Riemannian submersions in a complex context and defined and studied so-called almost Hermitian submersions [3]. Later, the theory of submersion became a popular field, and it has also been worked in the contact context [4,5]. Most recent submersion studies can be found in the books [6,7].

Ronsse defined generic submanifolds in the complex context [8]. Based on the given idea in this work, generic submersions were defined in the complex context [9]. Later on, the concept given by Ronsse was introduced in contact geometry [10]. In this paper, we construct generic submersions using the idea presented in [10] for generic submanifolds.

The current paper is organized as follows:
Section 2 includes the fundamental literature, which is used throughout the paper. Generic submersion in the contact context is defined and studied in Section 3. In this section, the position of the Reeb vector field, when it is tangent to the fibers, is given. Moreover, an original example is introduced and some results are given.

2. PRELIMINARIES

2.1. Riemannian Submersions

This section is devoted to Riemannian submersion and some related preliminaries.

Let (M,g) and (N,h) be Riemannian maifolds, here $dim(M) > dim(N)$. A surjective mapping

$$\pi: (M,g) \rightarrow (N,h)$$

is called a *Riemannian submersion* [1] if

- π has maximal rank, and

- the restriction of π_* on $ker\pi_*^\perp$ is a linear isometry.

In this case, for each point q in N, $\pi^{-1}(q)$ is a $k-$dimensional submanifold of M is called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called a *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to the fibers. A vector field X on M is called *basic* if X is horizontal and $\pi-$related to a vector field X_* on N, i.e. $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $ker\pi_*^\perp$, respectively. Here, the manifold (M,g) is called total manifold and (N,h) is called base manifold of the submersion $\pi:(M,g) \to (N,h)$.

The geometry of the Riemannian submersions is characterized by O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_U V = \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V \qquad (1)$$

$$\mathcal{A}_U V = \mathcal{V}\nabla_{\mathcal{H}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{H}U}\mathcal{V}V \qquad (2)$$

for any vector fields U and V on M,here ∇ is the Levi-Civita connection of g. One can see that $\mathcal{T}_U$ and $\mathcal{A}_U$ are skew-symmetric operators on the tangent bundle of the total manifold M reversing the vertical and horizontal distributions.

Here we give some of the properties of the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$, which will be helpful for the following sections.

Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \qquad (3)$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \qquad (4)$$

On the other hand, (1) and (2) yield us

$$\nabla_V W = \mathcal{T}_V W + \widehat{\nabla_V} W, \qquad (5)$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \qquad (6)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \qquad (7)$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \qquad (8)$$

where $\widehat{\nabla_V} W = \mathcal{V}\nabla_V W$. Moreover, if X is basic

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V.$$

Remark 1.

- In this paper, we will assume all the horizontal vector fields as basic.
- One can observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of it.

To see more details, we refer to the paper [1] and the book [7].

Lemma 1. [1] Let $\pi: M \to N$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields of M. Then

- $g(X,Y) = h(X_*, Y_*) \circ \pi$,
- the horizontal part $[X,Y]^{\mathcal{H}}$ of $[X,Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e $\pi_*([X,Y]^{\mathcal{H}}) = [X,Y]$,
- $[V,X]$ is vertical for any vector field V of $ker\pi_*$,
- $(\nabla_X Y)^{\mathcal{H}}$ is the basic vector field corresponding to $\nabla_X^N Y_*$.

2.2. Almost Contact Metric Manifolds

Let M be a $C^\infty-$differentiable manifold. An *almost contact structure* on M, denoted by (ϕ, ξ, η), consists of a (1,1) tensor field ϕ (called the structure tensor field), a vector field ξ (called Reeb vector field) and a 1-form η (the dual of ξ) such that

$$\phi^2 = -I + \eta \otimes \xi \qquad (9)$$

and

$$\eta(\xi) = 1, \qquad (10)$$

where I denotes the identity endomorphism of the fiber bundle TM. In this case, $(M, \phi, \xi, \eta,)$ is called an *almost contact manifold*. One can see that the manifold has odd dimension and it follows that

$$F\xi = 0, \quad \eta \circ F = 0. \qquad (11)$$

If a Riemannian metric g on M satisfies

$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, $\forall X, Y \in \Gamma(TM)$, 12

then g is said to be adapted to the almost contact structure (ϕ, ξ, η). In this case, (ϕ, ξ, η, g) (resp. (M, ϕ, ξ, η, g)) is called *almost contact metric structure* (resp. *almost contact metric manifold*). By using (9) and (12), the following relation can be obtained:

$$\eta(X) = g(X, \xi), \ \forall X \in \Gamma(TM). \qquad (13)$$

Let $\mathcal{D} = ImF = Ker\,\eta$ denote the contact distribution of the manifold M. Hence the tangent bundle decomposes into the orthogonal sum:

$$TM = \mathcal{D} \oplus span\,\xi. \qquad (14)$$

From (9) and (12), it follows that ϕ is skew-symmetric with respect to g, which allows one to define the 2-form α, called the fundamental 2-form of the almost contact metric structure on M, by [11]

$$\alpha(X,Y) = g(X, \phi Y), \ \forall X, Y \in \Gamma(TM). \qquad (15)$$

Therefore, (M, α) is an almost symplectic manifold.

3. GENERIC SUBMERSIONS

In this section we define and study the concept of generic submersion from an almost contact metric manifold onto a Riemannian manifold.
We would like to mention that there are other notions of generic submanifolds, [12,13,14].

First, we give some facts on generic submanifolds in the almost contact context, [10].

3.1. Generic Submanifolds of Almost Contact Metric Manifolds

In [Bejan and me], the concept generic submanifold of an almost contact metric manifold is defined and studied.

Now, we recall some fundamental knowledge.

Let (B,ϕ,ξ,η,g) be an almost contact metric manifold and let M be a Riemannian submanifold of B. For any $X\in\Gamma(TM)$, we may write

$$\phi X = PX + NX, \tag{18}$$

where $PX\in\Gamma(TM)$ and $NX\in\Gamma(TM^{\perp})$.

Proposition 1. [Bejan and me] Let M be a submanifold of an almost contact metric manifold (B,ϕ,ξ,η,g) and let P the operator defined by (18). Then

- P is skew-symmetric with respect to g on M;
- P^2 is symmetric with respect to g;
- all eigenvalues of P^2 are contained in [-1,0].

Remark 2. [Bejan and me] From Proposition 1, P^2 has at each point the associated matrix diagonalizable.

Let $-\beta^2$ be an eigenvalue of P^2 whose corresponding eigen distribution will be denoted by D^{β}. Since P^2 is diagonalizable we may take $-\beta_i^2(p)$, $i=1,\dots,n$, to be all distinct eigenvalues of P^2 at any point $p\in M$, which yields the decomposition of T_pM into the direct orthogonal sum, i.e.

$$T_pM = D_p^{\beta_1}\oplus D_p^{\beta_2}\oplus\dots\oplus D_p^{\beta_n}. \tag{19}$$

Corresponding to Ronsse's definition [8] of generic and skew CR-submanifolds in almost Hermtian context, it is introduced in almost contact framework the following:

Definition 1. [10] A submanifold M of an almost contact metric manifold (B,ϕ,ξ,η,g) is called generic if there exist some functions

$$\beta_1,\dots,\beta_n : M\to(0,1),$$

for a positive integer k, such that at each point $p\in M$:

- $-\beta_i^2(p)$, $i=1,\dots,n$ are distinct eigenvalues of P^2;
- the dimension of each $D_p^{\beta_1}, D_p^{\beta_2},\dots,D_p^{\beta_n}$ is independent of $p\in M$, where D_p^{β} denotes the eigenspace corresponding to the eiganvalue $-\beta_i^2(p)$ of P^2, for $\beta\in\{0,1,\beta_1,\beta_2,\dots,\beta_n\}$;
- the tangent space decomposes into the direct orthgonal sum
- $$T_pM = D_p^0\oplus D_p^1\oplus D_p^{\beta_1}\oplus\dots\oplus D_p^{\beta_n}.$$

When $\beta_1,\dots,\beta_n$ are constants, M is called a skew CR-submanifold.

3.2. Generic submersions in contact geometry

This section will define the generic submersions in the contact context. Since the fibers of a submersion is a submanifold of the total manifold, we will follow the idea of generic submanifold given in the Section 3.1. to construct the generic submersions.

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold, (N,h) be a Riemannian manifold and

$$\pi:(M,\phi,\xi,\eta,g)\to(N,h)$$

be a Riemannian submersion. For any $X\in\Gamma(ker\pi_*)$ and $U\in\Gamma(ker\pi_*^{\perp})$, we can set

$$\phi X = PX + QX, \tag{20}$$
$$\phi U = tU + nU, \tag{21}$$

where $PX, tU\in\Gamma(ker\pi_*)$ and $QU, nU\in\Gamma(ker\pi_*^{\perp})$.

Now, we define a generic submersion in the contact context.

Definition 2. Let (M,ϕ,ξ,η,g) be an almost contact metric manifold, (N,h) be a Riemannian manifold and
$$\pi:(M,\phi,\xi,\eta,g)\to(N,h)$$
be a Riemannian submersion. Then, π is called a *generic submersion* (resp. *skew CR-submersion*) if the fibers of the submersion π are generic submanifold (resp. Skew CR-submanifold) of M.

Remark 3. One can see that the definition has no limitation for the Reeb vector field ξ such as tangent or normal. This fact makes this work different than the others given in the literature.

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold, (N,h) be a Riemannian manifold and

$$\pi:(M,\phi,\xi,\eta,g)\to(N,h)$$

be a generic submersion (or skew CR-submersion). In this case, there are k functions (or constant functions) $\beta_1,\dots,\beta_k$ defined on the fibers with the values in the open interval (0,1) such that $ker\pi_*$ is decomposed as

$$ker\pi_* = D^0\oplus D^1\oplus D^{\beta_1}\oplus\dots\oplus D^{\beta_k}, \tag{22}$$

where D^1 is invariant, D^0 is anti-invariant, D^{β_i} is pointwise slant distribution (or slant distribution) with slant function θ_i and $-\beta_i^2$ is a distinct eigenvalue of P^2 for each $i = 1, \dots, k$.

Proposition 2. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, (N, h) be a Riemannian manifold and

$$\pi: (M, \phi, \xi, \eta, g) \to (N, h)$$

be a generic submersion. Then,

- P is skew-symmetric with respect to the metric g on the fibers;
- any distribution D_p^β is $P-$invariant, for $\beta \in \{0,1,\beta_1, \dots, \beta_k\}$;
- for any non-zero eigenvalue, the corresponding eigen distribution is even dimensional.

Proof. For any $X, Y \in \Gamma(ker\pi_*)$, by using the skew-symmetry of ϕ

$$g(PX, Y) = g(\phi X, Y) = -g(X, \phi Y) = -g(X, PY),$$

which shows the first claim.

Let consider an arbitrary $\beta \in \{0,1,\beta_1, \dots, \beta_k\}$ and let $-\beta_i^2$ be an eigenvalue of P^2 whose associated eigen distribution is D_p^β. For any $\delta \in \{0,1,\beta_1, \dots, \beta_k\}$, $\delta \neq \beta$, the skew-symmetry of P yields:

$$\begin{aligned}\beta^2 g(PX, Y) &= -\beta^2 g(X, PY) = g(P^2X, PY) \\ &= -g(PX, P^2Y) = \delta^2 g(PX, Y),\ \forall X \in \Gamma(D_p^\beta),\ \forall Y \in \Gamma(D_p^\delta).\end{aligned}$$

Since $\beta \neq \delta$, it follows

$$g(PX, Y) = 0, \forall X \in \Gamma(D_p^\beta), Y \in \Gamma(D_p^\delta),$$

which shows the second claim.

Last claim follows from the other claims.

The following identity gives a relation between certain canonical structures. The idea of the proof is same with Lemma 3.7. in [10].

Lemma 1. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, (N, h) be a Riemannian manifold and

$$\pi: (M, \phi, \xi, \eta, g) \to (N, h)$$

be a generic submersion. Then,

$$||QX||^2 = ||X||^2 - (\eta(X))^2 + g(X, P^2X),\ \forall X \in \Gamma(ker\pi_*). \quad (23)$$

Proposition 3. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, (N, h) be a Riemannian manifold and

$$\pi: (M, \phi, \xi, \eta, g) \to (N, h)$$

be a generic submersion. Then,

- $D^0 = kerP$;
- $D^1 = kerQ \cap \mathcal{D}$.

Proof. If $X \in \Gamma(D^0)$, i.e. $P^2X = 0$, then from the skew-symmetry of P, we obtain

$$||PX||^2 = -g(P^2X, X) = 0,$$

which shows the first equality.

Assume that $X \in \Gamma(D^1)$, i.e. $P^2X = -X$, which makes (23)

$$||QX||^2 = -(\eta(X))^2.$$

Then, the last equality above says that both QX and $\eta(X)$ are supposed to be identically zero, i.e.

$$D^1 \subseteq kerQ \cap \mathcal{D}.$$

On the other hand, for any $X \in \Gamma(kerQ \cap \mathcal{D})$, (23) becomes

$$0 = g(X, X) + g(X, P^2X),$$

Which shows that the only eigenvalue of P^2 is -1 and complete the proof.

■

In the view of Proposition 3., we have the following Lemma:

Lemma 2. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold, (N, h) be a Riemannian manifold and

$$\pi: (M, \phi, \xi, \eta, g) \to (N, h)$$

be a generic submersion. If the Reeb vector field ξ is tangent to $ker\pi_*$, then

$$D^0 \cap kerQ = span\{\xi\}.$$

Remark 4. In this case of ξ is tangent to the fibers, Lemma 2. gives a deomposition for the anti-invariant distribution D^0 such that

$$D^0 = \widehat{D^0} \oplus span\{\xi\}, \quad (24)$$

where $\widehat{D^0}$ is the orthogonal complementary of $span\{\xi\}$ in D^0. In other words, if the Reeb vector field ξ is tangent to the fibers, then it it always perpendicular to all other distributions $(\xi \perp D^\beta, \beta \in \{1, \beta_1, \dots, \beta_k\})$, which gives a new decomposition for the fibers.

Remark 5. As a natural consequence of Remark 4., one can see that

$$\eta(X) = 0, \quad (25)$$

where $X \in \Gamma(\widehat{D^0} \oplus D^1 \oplus D^{\beta_1} \oplus \dots \oplus D^{\beta_k})$, i.e. $\widehat{D^0}$ is comprised by the contact distribution $\mathcal{D}$.

Example 1. Consider a pair of almost complex structures $\{J_1, J_2\}$ on R^{12} as in the following:

$$J_1(\partial_1, \ldots \partial_{12}) = (-\partial_3, -\partial_4, \partial_1, \partial_2, -\partial_7, -\partial_8, \partial_5, \partial_6, -\partial_{11}, -\partial_{12}, \partial_9, \partial_{10}),$$
$$J_2(\partial_1, \ldots, \partial_{12}) = (-\partial_2, \partial_1, \partial_4, -\partial_3, -\partial_6, \partial_5, \partial_8, -\partial_7, -\partial_{10}, \partial_9, \partial_{12}, -\partial_{11})$$

with $\partial_i = \frac{\partial}{\partial x_i}$ for any $i \in \{1, \ldots, 12\}$. Thus, (ϕ, ξ, η, g) is an almost contact metric structure on the Euclidean space $R^{13} = R^{11} \times R$ with the coordinates $(x_1, \ldots, x_{13})$ such that

$$\phi\left(V + \tau\frac{\partial}{\partial x_{13}}\right) = (\cos f)J_1V + (\sin f)J_2V, \forall V \in \Gamma(R^{12}),$$
$$\xi = \frac{\partial}{\partial x_{13}}, \ \eta = dx_{13},$$

where g is the Euclidean metric on R^{13}, $\tau: R^{13} \to R$ and $f: R^{13} \to R - \{0, \frac{\pi}{2}\}$ are smooth functions on the fibers. Define a map $\pi: R^{13} \to R^{6}$ such that

$$\pi(x_1, \ldots, x_{13}) = (x_2, x_3, x_6, x_8, x_9, x_{12}).$$

$$ker\pi_* = \widehat{D^0} \oplus span\{\xi\} \oplus D^1 \oplus D^\beta,$$

where

$$\widehat{D^0} = span\{\partial x_1, \partial x_4\},$$
$$D^1 = span\{\partial x_9 - \partial x_{12}, \big((\cos f) + (\sin f)\big)\partial x_{10} + \big((\cos f) - (\sin f)\big)\partial x_{11}\}$$

and

$$D^\beta = span\{\partial x_5, \partial x_7\}$$

such that D^β is of the pointwise slant function f. Moreover,

$$(ker\pi_*)^\perp = span\{(\sin f)\,\partial x_2 + (\cos f)\,\partial x_3, (-\cos f)\,\partial x_2 + (\sin f)\,\partial x_3, \partial x_6, \partial x_8, \partial x_9 + \partial x_{12}, \big((\cos f) + (-\sin f)\big)\partial x_{10} + \big((-\cos f) + (-\sin f)\big)\partial x_{11}\}.$$

Therefore, the map π is a generic submersion.

Proposition 4. Let π be a generic submersion from an almost contact metric manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, h) with ξ tangent to the fibers and D^0 is parallel with respect to ξ. Then, any integral curve of ξ is a geodesic on the fibers if and only if $\widehat{D^0}$ is parallel with respect to ξ.

Proof. Assume that ξ is tangent to the fibers and D^0 is parallel with respect to the Reeb vector field, i.e.

$$\nabla_\xi X \in D^0, \forall X \in \Gamma(D^0),$$

Which gives the following equivalence with Remark 4, for any $Z \in \Gamma(\widehat{D^0})$,

$$\widehat{D^0} \text{ is parallel with respect to } \xi \Leftrightarrow \nabla_\xi Z \in \widehat{D^0}$$
$$\Leftrightarrow g(\nabla_\xi Z, \xi) = 0 \Leftrightarrow g(Z, \nabla_\xi \xi) = 0$$
$$\Leftrightarrow \nabla_\xi \xi \in span\{\xi\}.$$

On the other hand, since ξ is unitary,

$$g(\xi, \xi) = 1 \Leftrightarrow g(\nabla_\xi, \xi) + g(\xi, \nabla_\xi \xi) = 0$$
$$\Rightarrow g(\xi, \nabla_\xi \xi) = 0$$
$$\Rightarrow \nabla_\xi \xi = 0,$$

which completes the proof.

Theorem 1. Let π be a generic submersion from an almost contact metric manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, h). If M is of a closed fundamental 2-form α, then

- the anti-invariant distribution D^0 is integrable,
- if the Reeb vector field ξ is tangent to the fibers, then the distribution $\widehat{D^0}$ is integrable if and only if the restriction of η on $\widehat{D^0}$ is closed.

Proof. Let $X, Y \in \Gamma(D^0)$ and $V \in \Gamma(ker\pi_* - D^0)$. Thus, by Proposition 2, there exists $Z \in \Gamma(ker\pi_* - D^0)$ such that $PZ = V$. Since $PZ = PV = 0$, we have

$$\begin{aligned} g([X,Y],V) &= g([X,Y],PZ) \\ &= Zg(Y,PX) - Yg(Z,PX) \\ &\quad - Xg(Z,PY) - g([Z,Y],PX) \\ &\quad - g([X,Z],PY) + g([X,Y],PZ) \\ &= Z\alpha(Y,X) - Y\alpha(Z,X) - X\alpha(Z,Y) - \alpha([Z,Y],X) \\ &\quad - \alpha([X,Z],Y) + \alpha([X,Y],Z \\ &= d\alpha(Z,X,Y) \\ &= 0, \end{aligned}$$

which means $[X,Y] \in \Gamma(D^0)$, i.e. D^0 is integrable.
Now, let $X, Y \in \Gamma(\widehat{D^0})$. D^0 is integrable implies $[X,Y] \in \Gamma(D^0)$. From Remark 5, it follows that

$$g([X,Y],\xi) = \eta([X,Y]) = d\eta(X,Y).$$

Thus, $[X,Y] \in \Gamma(\widehat{D^0})$ if and only if $d\eta(X,Y) = 0$.

■

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