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# On Multi-G-Metric Spaces

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**Abstract**

Multisets have many applications in a variety of fields today, including computer science, medicine, banking, engineering, information storage, and information analysis. In this paper, we present a new generalized multi-G-metric space, a multi-G-metric space. We investigate some of its fundamental details, connections, and topological characteristics.

## 1. Introduction

Several branches of modern mathematics have developed that goes against a basic tenet of conventional mathematical theory. The underlying premise of traditional mathematics is that all mathematical objects are unique. As a result, there are two options between two numbers: they could be equal or dissimilar. In reality and in science, this is not the case. In the physical world, there seems to be a lot of repetition. For instance, between numerous DNA strands, many water molecules, or many hydrogen atoms. Even though they are independent objects, coins, electrons, and grains of sand with the same value and year appear to be the same.

Assuming that mathematical objects are not repeated, the classical set theory states that a particular element may only be written once in a set. That is, there is only an equal or different relationship between any two mathematical objects.

In reality and in science, this is not the case. In the physical world, it has been noted that there is too much repetition. For instance, between hydrogen atoms, water molecules, DNA strands, and so forth. Any two physical items can have one of three connections as a result: they can be distinct, distinct but the same, or overlap and be the same. They are unambiguously the same or equal if they cannot be distinguished, and they are the same and identical if they physically overlap. An organized collection of various items is referred to as a set in classical set theory. A multi-set is one that permits any object to be repeated within it (mset or bag for short). A multi-set is so distinct from a set. To describe this structure, it is appropriate to make the distinction between the sets a, b, and c and the collections a, a, a, b, c, and c. When viewed as a set, the second is identical to the first. However, with the latter, some components are purposefully used more than once. A mset is a group of components created with a specific multiplicity. $\{k_1/x_2, k_2/x_2, \ldots, k_n/x_n\}$ can be written so that the multiple set $x_i$ is found $k_i$ times. Here $k_i$ is an integer [1–7].

The metric idea, which is basic to mathematics and has a variety of diverse applications, plays a crucial role in topology and analysis. This idea has been examined in relation to a number of generalizations, including G-metric spaces [8], fuzzy metric spaces [9] and cone metric spaces [10].

Some of the key ideas and findings of cluster analysis can be extended to the arrangement of numerous clusters. Recently, research on many sets in mathematics has begun. Das and Roy [11] described one of these research in 2021. They began by defining the idea of multi-real numbers and studying their fundamental characteristics in this study. The idea of numerous metrics on different sets is introduced and its fundamental characteristics are investigated in this study at the same time. The topological characteristics of other metric spaces were then researched using the findings of this study [12]. Through this study, we also hope to contribute to this expansion. As a generalization of a multi-metric space, multi-G-metric spaces are used to explore the fundamental characteristics of multiple G-metrics. The multi-G-metric topology produced with the aid of multiple G-metrics will also be specified, and its fundamental characteristics have been investigated. After that, the ideas of multi-G-convergence and multi-G-Cauchy are introduced, and a few of their characteristics are examined.

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## 2. Preliminaries

**Definition 2.1** ( [13] ). *The function CountM, often known as $C_M$, is defined as $C_M : X \to N$, where N stands for the set of non-negative integers. A mset M selected from the set X is represented by this function. The number of times the element x appears in the M mset is represented here by $C_M(x)$. We write the mset M as $M = \{m_1/x_1, m_2/x_2, ...m_n/x_n\}$, where $m_i$ is the number of times the element $x_i$ appears in the mset M denoted by $x_i \in_i^m M, i = 1,2,3,...n$. But, elements that are omitted from the M mset have zero counts.*

**Definition 2.2** ( [13]). *Let M and N be two msets drawn from a set X. Then, the followings are defined:*

*(1) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$,*
*(2) $M \subset N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$,*
*(3) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ for all $x \in X$,*
*(4) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ for all $x \in X$.*

**Definition 2.3** ( [13]). *The power set of an mset is denoted by $P^*(M)$ and it is an ordinary set whose members are sub msets of M. An mset M is said to be an empty mset (multi-empty set) if for all $x \in X$, $C_M(x)$ = 0.*

**Definition 2.4** ( [13]). *Let the mset space $[X]^\omega$ be the set of all msets whose elements are in X such that no element in the mset occurs more than $\omega$ times. Let $M \in [X]^\omega$ and $\tau \subseteq P^*(M)$. Then $\tau$ is called a mset topology (M-topology) of M if $\tau$ satisfies the following properties,*

*(1) The mset M and multi empty set are in $\tau$,*
*(2) The mset union of the elements of any subcollection of $\tau$ is in $\tau$,*
*(3) The mset intersection of the element of any finite subcollection of $\tau$ is in $\tau$, Mathematically an mset topological space is an ordered pair $(M, \tau)$ consisting of an mset $M \in [X]^\omega$ and a mset topology $\tau \subseteq P^*(M)$ on M.*

**Definition 2.5** ( [11]). *Let M be a mset over the universal set X. The mapping $P_x^l : X \to \mathbb{N}$ such that $P_x^l(x) = l$ where $l \leq C_M(x)$ defines a multi-point of M, where x and l are the base and multiplicity of the multi-point $P_x^l$, respectively. $M_{pt}$ denotes the collection of all multi-points in a mset M.*

**Definition 2.6** ( [11]). *The mset produced by a set N of multi-points is represented by the symbol $MS(N)$, and its definition is given by the formula $C_{MS(N)}(x) = sup\{l : P_x^l \in N\}$. The collection of its multi-points can be used to create a mset. If $M_{pt}$ stands for the collection of all multi points of M, then $C_M(x) = sup\{l : P_x^l \in M_{pt}\}$ and $M = MS(M_{pt})$ are obvious conclusions.*

**Definition 2.7** ( [11]). *Let M be a mset over the universal set X.*

*(1) The elementary union between two collections of multi points C and D is denoted by $C \sqcup D$ and is defined as $C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D$ and $k = max\{l,m\}\}$.*
*(2) The elementary intersection between two collections of multi points C and D is denoted by $C \sqcap D$ and is defined as $C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D$ and $k = min\{l,m\}\}$.*
*(3) For two collection of multi points C and D, C is said to be an elementary subset of D, denoted by $C \sqsubset D$, iff $P_x^l \in C$ there exists $m \geq l$ such that $P_x^m \in D$.*

**Theorem 2.8** ( [11]). *Let M be a mset over the universal set X.*

*(1) For two collections of multi-points C and D, $C \cup D \supset C \sqcup D$.*
*(2) For a collection N of multi-points, $[MS(N)]_{pt} \supset N$.*
*(3) For two msets A and B, $A \subset B$ iff $A_{pt} \subset B_{pt}$.*
*(4) For two collections of multi-points C and D, $MS(C \sqcap D) = MS(C) \cap MS(D)$.*

**Definition 2.9** ( [11]). *$m\mathbb{R}^+$ denotes the mset over $\mathbb{R}^+$ (set of non-negative real numbers) having a multiplicity of each element equal to $\omega \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called non-negative multi-real points.*

**Definition 2.10** ( [11]). *Let $P_a^i$ and $P_b^j$ be two multi real points of $(m\mathbb{R}^+)_{pt}$.*

*(1) $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.*
*(2) $P_a^i + P_b^j = P_{a+b}^k$ where $k = Max\{i, j\}$.*
*(3)*

$$P_a^i \times P_b^j = \begin{cases} P_0^1, & if\ either\ P_a^i\ or\ P_b^j equal\ to\ P_0^1, \\ P_{ab}^k, & otherwise \quad where \quad k = Max\{i, j\}. \end{cases}$$

**Definition 2.11** ( [12]). *The subtraction of two multi real points in $m\mathbb{R}^+$ is defined as follows:*

$$P_a^i - P_b^j = \begin{cases} P_0^1, & if\ P_a^i = P_b^j, \\ P_{a-b}^k, & if\ P_a^i > P_b^j \quad where \quad k = min\{i, j\}. \end{cases}$$

**Definition 2.12.** *The division of two multi real points in $m\mathbb{R}^+$ is defined as follows:*

$$P_a^i / P_b^j = \begin{cases} P_1^1, & if\ P_a^i = P_b^j, \\ P_{a/b}^k, & if\ P_a^i \neq P_b^j \quad where \quad k = Max\{i, j\}. \end{cases}$$

**Definition 2.13.** *We define maximum of two multi-real points in $m\mathbb{R}^+$ as follows:*

$$max\{P_a^i, P_b^j\} = \begin{cases} P_a^i, & if\ P_a^i > P_b^j, \\ P_b^j, & otherwise. \end{cases}$$

**Definition 2.14** ( [11]). *Let's say that $d : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ (M being a multi set over a universal set X with multiplicity of any member at most equal to $\omega$) be a mapping that meets the following requirements:*

($md_1$) $md(P_x^l, P_y^m) > P_0^1$ *for all* $P_x^l, P_y^m \in M_{pt}$ *and* $P_x^l \neq P_y^m$,
($md_2$) $md(P_x^l, P_y^m) = P_0^1$ *iff* $P_x^l = P_y^m$,
($md_3$) $md(P_x^l, P_y^m) = md(P_y^m, P_x^l)$,
($md_4$) $md(P_x^l, P_y^m) + md(P_y^m, P_z^n) \geq md(P_x^l, P_z^n)$, *for all* $P_x^l, P_y^m, P_z^n \in M_{pt}$,
($md_5$) *For* $l \neq m$, $md(P_x^l, P_y^m) = P_0^k$ *iff* $x = y$ *and* $k = Max\{l, m\}$.

*Then, $(M, md)$ is referred to a multi-metric (or an M-metric) space and md is said to be a multi-metric on M.*

**Definition 2.15.** *Let $(M, md)$ be a multi-metric space. Let $\{P_{x_n}^{l_n}\}$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to convergence to $P_x^l \in M_{pt}$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $md(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$, $\forall n \geq n_0$ i.e. $n \geq n_0$ then the sequence $\{P_{x_n}^{l_n}\}$ is multi convergent (md-convergent) to $P_x^l$ and written as $\{P_{x_n}^{l_n}\} \to P_x^l$.*

**Definition 2.16.** *Let $(M, md)$ be a multi-metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to be multi-Cauchy (md-Cauchy) if every $P_\varepsilon^1 > P_0^1$, there exists a $n_0 \in \mathbb{N}$ such that $md(P_{x_n}^{l_n}, P_{x_m}^{l_m}) < P_\varepsilon^1$ for all $m, n \geq n_0$.*

**Definition 2.17.** *A multi-metric space $(M, md)$ is said to be md-complete if every md-Cauchy sequence in $(M, md)$ is md-convergent in $(M, md)$.*

**Definition 2.18** ( [8]). *Let U be a nonempty set, and let $G : U \times U \times U \to \mathbb{R}^+$ be a function satisfying the following conditions:*

($G_1$) $G(x, y, z) = 0$ *if* $x = y = z$,
($G_2$) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,
($G_3$) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,
($G_4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$,
($G_5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in U$,

*then the function G is called a generalized metric, or, more specifically, a G-metric on U, and the pair $(U, G)$ is a G-metric space.*

## 3. Multi G-Metric Spaces

The concept of multi-G-metric space is defined and its fundamental characteristics are determined in this section. Also, we investigate any relationships that may exist between multi-metric and multi-G-metric.

**Definition 3.1.** *Assume that X is a non-empty set and that M is a multi-set over X with multiplicity of any element approximately equal to $\omega$. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ is said to be a multi generalized metric or multi G-metric on M if mG satisfies the following conditions:*

($mG_1$) $mG(P_x^l, P_x^l, P_y^m) > P_0^1$ *for all* $P_x^l, P_y^m \in M_{pt}$ *with* $P_x^l \neq P_y^m$,
($mG_2$) $mG(P_x^l, P_y^m, P_z^n) = P_0^1$ *if* $P_x^l = P_y^m = P_z^n$
($mG_3$) $mG(P_x^l, P_y^m, P_z^n) = mG(P_x^l, P_z^n, P_y^m) = mG(P_y^m, P_z^n, P_x^l) = \ldots$,
($mG_4$) $mG(P_x^l, P_x^l, P_y^m) \leq mG(P_x^l, P_y^m, P_z^n)$ *for all* $P_x^l, P_y^m, P_z^n \in M_{pt}$ *with* $P_y^m \neq P_z^n$,
($mG_5$) $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^k, P_a^k) + mG(P_a^k, P_y^m, P_z^n)$ *for all* $P_x^l, P_y^m, P_z^n, P_a^k \in M_{pt}$,
($mG_6$) *For at least two of the l, m, n variables are different,* $mG(P_x^l, P_y^m, P_z^n) = P_0^r$ *iff* $x = y = z$ *and* $r = max\{l, m, n\}$.

*Then $(M, mG)$ is said to be a multi G-metric $(m\text{-}g\text{-}metric)$ space.*

**Example 3.2.** *Assume that X is a non-empty set and that M is a multi-set over X with multiplicity of any element approximately equal to $\omega$. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ are defined by*

$$mG(P_x^l, P_y^m, P_z^n) = \begin{cases} P_0^1, & if\ all\ of\ the\ variables\ P_x^l, P_y^m, P_z^n are\ equal, \\ P_2^k, & if\ all\ of\ the\ variables\ x, y, z\ are\ different, k = max\{l, m, n\}, \\ P_1^k, & if\ two\ of\ the\ variables\ P_x^l, P_y^m, P_z^n\ are\ equal, and\ theremaining\ one\ is\ distinct\ and\ k = max\{l, m, n\}, \\ P_0^k, & if\ x = y = z\ and\ for\ at\ least\ two\ of\ the\ l, m, n\ variables\ are\ different,\ k = max\{l, m, n\}. \end{cases}$$

*Then mG satisfies all the multi-G-metric axioms.*

**Example 3.3.** *Assume that $(M, md)$ is multi-metric space. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ is defined by $mG(P_x^l, P_y^m, P_z^n) = max\{md(P_x^l, P_y^m), md(P_y^m, P_n^z), md(P_x^l, P_z^n)\}$. Then mG satisfies all the multi-G-metric axioms.*

**Definition 3.4.** *A multi G-metric space $(M, mG)$ is said to be symmetric if $mG(P_x^l, P_y^m, P_y^m) = mG(P_x^l, P_x^l, P_y^m)$ for any $P_x^l, P_y^m \in M_{pt}$.*

**Proposition 3.5.** *Assume that X is a non-empty set and that M is a mset over X with a multiplicity of any element approximately equal to $\omega$. Let mG be a multi-G-metric. Then, the following hold for all $P_x^l, P_y^m, P_z^n, P_a^r \in M_{pt}$.*

*(1)* If $mG(P_x^l, P_y^m, P_z^n) = P_0^1$ then $P_x^l = P_y^m = P_z^n$.
*(2)* $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n)$.
*(3)* $mG(P_x^l, P_y^m, P_y^m) \leq P_2^1 mG(P_y^m, P_x^l, P_x^l)$.
*(4)* $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$.
*(5)* $mG(P_x^l, P_y^m, P_z^n) \leq P_{2/3}^1 (mG(P_x^l, P_y^m, P_a^r) + mG^( P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n))$.
*(6)* $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_a^r) + mG(P_y^m, P_a^r, P_a^r) + mG(P_z^n, P_a^r, P_a^r)$.

*Proof.* (1) By the definition of multi-G-metric, it is clear.
(2) **Case 1:** Let $P_x^l \neq P_y^m \neq P_z^n$. Then we have

$$mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n) \geq mG(P_y^m, P_x^l, P_z^n) = mG(P_x^l, P_y^m, P_z^n)$$

from $(mG_3)$ and $(mG_5)$.
**Case 2:** Let $P_x^l = P_y^m \neq P_z^n$. Then we have

$$mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n) = P_0^1 + mG(P_x^l, P_x^l, P_z^n) = P_0^1 + mG(P_x^l, P_y^m, P_z^n) \geq mG(P_x^l, P_y^m, P_z^n)$$

from $(mG_2)$, $(mG_4)$.
**Case 3:** Let $x = y = z$. The proof is clear.
Other cases' proofs are produced in a similar way.
(3) We know that $mG(P_x^l, P_y^m, P_y^m) \leq mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_y^m) = P_2^1 mG(P_x^l, P_x^l, P_y^m)$ by (2). Then we obtain

$$mG(P_x^l, P_y^m, P_y^m) \leq P_2^1 mG(P_y^m, P_x^l, P_x^l)$$

from $(mG_3)$.
(4) **Case 1:** Let $P_x^l \neq P_z^n$. Thus we have

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_a^r, P_a^r, P_x^l) + mG(P_a^r, P_y^m, P_z^n) \leq mG(P_a^r, P_x^l, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$$

from $(mG_5)$,$(mG_3)$,$(mG_4)$ respectively. So, we get

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$$

by $(mG_3)$.
**Case 2:** Let $P_x^l = P_z^n$ and $P_y^m \neq P_a^r$. Then, we have

$$mG(P_x^l, P_y^m, P_z^n) = mG(P_x^l, P_y^m, P_x^l) = mG(P_x^l, P_x^l, P_y^m) \leq mG(P_x^l, P_y^m, P_a^r)$$

from $(mG_3)$, $(mG_4)$. Therefore we obtain

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_a^r, P_y^m, P_x^l) + mG(P_x^l, P_a^r, P_x^l) = mG(P_a^r, P_y^m, P_z^n) + mG(P_x^l, P_a^r, P_z^n)$$

by $(mG_3)$.
**Case 3:** Let $x = y = z$ and $P_y^m = P_a^r$. Then it is obvious.
**Case4:** Let $x = y = z = a$,then we have

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_a^r) + mG(P_a^r, P_y^m, P_z^n) = mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n).$$

(5) By using (4) and $(mG_3)$, we get

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n + mG(P_a^r, P_y^m, P_z^n)$$

and

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_y^m) + mG(P_z^n, P_a^r, P_x^l)$$

and

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_y^m, P_a^r, P_x^l) + mG(P_z^n, P_a^r, P_x^l).$$

Thus we get

$$P_3^1 mG(P_x^l, P_y^m, P_z^n) \leq P_2^1 (mG(p_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n) + mG(P_a^r, P_x^l, P_y^m))$$

from $(mG_3)$. So, we obtain $mG(P_x^l, P_y^m, P_z^n) \leq P_{2/3}^1 (mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n) + mG(P_a^r, P_x^l, P_z^n))$.

(6) From $(mG_5)$, (2) and $(mG_3)$, we have

$$mG(P_x^l,P_y^m,P_z^n)\le mG(P_x^l,P_a^r,P_a^r)+mG(P_a^r,P_y^m,P_z^n)\le mG(P_x^l,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r)$$

and

$$mG(P_x^l,P_y^m,P_z^n)\le mG(P_y^m,P_a^r,P_a^r)+mG(P_a^r,P_x^l,P_z^n)\le mG(P_y^m,P_a^r,P_a^r)+mG(P_x^l,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r)$$

and

$$mG(P_x^l,P_y^m,P_z^n)\le mG(P_z^n,P_a^r,P_a^r)+mG(P_a^r,P_x^l,P_y^m)\le mG(P_z^n,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_x^l,P_a^r,P_a^r).$$

So we obtain

$$mG(P_x^l,P_y^m,P_z^n)\le mG(P_x^l,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r).$$

□

**Proposition 3.6.** *Assume that $X$ is a non-empty set and that $M$ is a multi-set over $X$ with multiplicity of any element approximately equal to $\omega$. Let $(M,mG)$ be a multi G-metric space; then the followings are equivalent:*

*(1) $(M,mG)$ is symmetric,*
*(2) $mG(P_x^l,P_y^m,P_y^m)\le mG(P_x^l,P_y^m,P_a^r)$ for all $P_x^l,P_y^m,P_a^r\in M_{pt}$,*
*(3) $mG(P_x^l,P_y^m,P_z^n)\le mG(P_x^l,P_y^m,P_a^r)$ + $mG(P_z^n,P_y^l,P_b^s)$ for all $P_x^l,P_y^m,P_z^n,P_a^r,P_b^s\in M_{pt}$.*

*Proof.* It is obvious from $(mG_3)$, $(mG_4)$ and Proposition 3.5. □

**Example 3.7.** *Let $X$ be a nonempty set and $M$ be a mset over $X$ having multiplicity of any element almost equal to $\omega$. $(M,mG)$ is an multi-G-metric space. Then $mG_1$ is multi G-metric on $M$ where $mG_1(P_x^l,P_y^m,P_z^n)=min\{P_k^t,mG(P_x^l,P_y^m,P_z^n)\}$ such that $P_k^t>P_1^0$.*

**Proposition 3.8.** *Let $(M,md)$ be a multi-metric space. Then $mG_s(d)$ and $mG_m(d)$ expressed as follows define multi G-metrics on $X$.*

*(1) $mG_s(d)(P_x^l,P_y^m,P_z^n)=P_{1/3}^1(md(P_x^l,P_y^m)+md(P_y^m,P_z^n)+md(P_x^l,P_z^n))$.*
*(2) $mG_m(d)(P_x^l,P_y^m,P_z^n)=\max\{md(P_x^l,P_y^m),md(P_y^m,P_z^n),md(P_x^l,P_z^n)\}$.*

*Proof.*$(mG_1)$ -$(mG_3)$ It is obvious.

$(mG_4)$ **Case 1:** $P_x^l=P_y^m$. Thus $mG_s(d)(P_x^l,P_x^l,P_y^m)$=$P_0^1\le mG_s(d)(P_x^l,P_y^m,P_z^n)$.

**Case 2:**

(a) $x=y$, $y\ne z$ and $l\ne m$. Then

$$mG_s(d)(P_x^l,P_x^l,P_x^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_x^m)+md(P_x^l,P_x^m))=P_{1/3}^1(P_0^1+P_2^1md(P_x^l,P_x^m))\le mG_s(d)(P_x^l,P_x^m,P_z^n).$$

(b) $x=y=z,m\ne n$ and $l\ne m$. Then

$$mG_s(d)(P_x^l,P_x^l,P_x^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_x^m)+md(P_x^l,P_x^m))=P_{1/3}^1(P_0^1+P_2^1md(P_x^l,P_x^m))=P_{1/3}^1P_0^k\le mG_s(d)(P_x^l,P_x^m,P_x^n)$$

where $k=max\{l,m\}$.

**Case 3:**

(a) $x\ne y$, $y\ne z$ and $x\ne z$. By using $(md_4)$, we have

$$P_2^1md(P_x^l,P_y^m)\le md(P_x^l,P_y^m)+md(P_x^l,P_z^n)+md(P_z^n,P_y^m).$$

Then

$$mG_s(d)(P_x^l,P_x^l,P_y^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_x^l,P_y^m))\le P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_z^n,P_y^m)+md(P_x^l,P_y^m)).$$

(b) $x\ne y,y\ne z$ and $P_x^l=P_z^n$. By using $(mG_4)$, we have $mG_s(d)(P_x^l,P_x^l,P_y^m)$= $mG_s(d)(P_x^l,P_y^m,P_x^l)$.

(c) $x\ne y,y\ne z,x=z$ and $l\ne n$. By using $(md_4)$, we have

$$md(P_x^l,P_y^m)\le md(P_x^l,P_x^n)+md(P_x^n,P_y^m).$$

Then

$$mG_s(d)(P_x^l,P_x^l,P_y^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_x^l,P_y^m))=P_{1/3}^1(P_0^1+P_2^1md(P_x^l,P_x^m))\le mG_s(d)(P_x^l,P_x^m,P_x^n).$$

The proofs of other cases are done in a similar way.

$(mG_5)$ From $(md_4)$ we get $md(P_x^l,P_y^m)\le md(P_x^l,P_a^r)+md(P_a^r,P_y^m)$ and $md(P_x^l,P_z^n)\le md(P_x^l,P_a^r)+md(P_a^r,P_z^n)$. Therefore

$$\begin{aligned}mG_s(d)(P_z^l,P_y^m,P_z^n)&=P_{1/3}^1(md(P_x^l,P_y^m)+md(P_x^l,P_z^n)+md(P_z^n,P_y^m))\\&\le P_{1/3}^1(md(P_x^l,P_a^r)+md(P_a^r,P_y^m)+md(P_x^l,P_a^r)+md(P_x^l,P_a^r)+md(P_a^r,P_z^n)+md(P_y^m,P_z^n))\\&=mG_s(d)(P_x^l,P_a^r,P_a^r)+mG_s(d)(P_a^r,P_y^m,P_z^n).\end{aligned}$$

$(mG_6)$ It is obvious from $md_6$.

□

**Proposition 3.9.** *Let $(M,mG)$ be a multi-G-metric space.Then $md_G$ defined a multi-metric on $M$ following holds:*

$$md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m).$$

*Proof.* The proofs of $(md_1)$, $(md_3)$ and $(md_4)$ obviously follow from $(mG_1)$, $(mG_3)$, $(mG_4)$ respectively.

$(md_2)$ Let $md_G(P_z^l,P_y^m) = P_0^1$. Assume that $P_x^l \neq P_y^m$. Since $mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m) = P_0^1$. We would have $mG(P_x^l,P_y^m,P_y^m) \leq P_0^1$ by Proposition 3.5. This contradicts to $(mG_1)$. Hence our assumption is not true. That is $P_x^l = P_y^m$. The converse is clear.

$(md_5)$ Let $md_G(P_x^l,P_y^m) = P_0^1$ for $l \neq m$. Thus we get $mG(P_x^l,P_y^m,P_z^n) + mG(P_x^l,P_x^l,P_y^m) = P_0^k$. Then, we get

$$mG(P_x^l,P_y^m,P_y^m) = mG(P_x^l,P_x^l,P_y^m) = P_0^k \quad \text{and} \quad x = y, \quad k = \max\{l,m\}.$$

Conversely, let $x = y$ and $k = \max\{l,m\}$. Thus we have

$$md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m) = mG(P_x^l,P_x^m,P_x^m) + mG(P_x^l,P_x^l,P_x^m) = P_0^k.$$

□

**Proposition 3.10.** *Let $(M,mG)$ be a multi-G-metric space.The function $md : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ defined by $md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m)$ satisfies the following properties.*

*(1) If $md_G(P_x^l,P_y^m) = P_0^1$ if and only if $P_x^l = P_y^m$.*
*(2) $md_G(P_x^l,P_y^m) \leq md_G(P_x^l,P_z^n)$+ $md_G(P_y^m,P_z^n)$ for all $P_x^l,P_y^m,P_z^n \in M_{pt}$.*
*(3) For at least two of the l, m variables are different, $md_G(P_x^l,P_y^m) = P_0^k$ iff $x = y$ and $k = max\{l,m\}$.*

*Proof.* (1) Let $md_G(P_z^l,P_y^m) = P_0^1$. By hypothesis, $mG(P_x^l,P_y^m,P_y^m)$=$P_0^1$. From Proposition 3.5, we get $P_x^l = P_y^m$. The converse is clear.
(2) From $(mG_5)$, we get $md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) \leq mG(P_x^l,P_z^n,P_z^n)$+ $mG(P_z^n,P_x^m,P_y^m)$=$md_G(P_x^l,P_z^n)$ +$md_G(P_z^n,P_y^m)$.
(3) Let $md_G(P_x^l,P_y^m) = P_0^k$. Then, we have $mG(P_x^l,P_y^m,P_y^m) = P_0^k$. By $(mG_6)$, we have $x = y$ and $k = max\{l,m\}$. Conversely, let $x = y$ and $k = \max\{l,m\}$. Thus, we have $mG(P_x^l,P_y^m,P_y^m)$ =$md_G(P_x^l,P_y^m) = P_0^k$.

□

## 4. Some topological properties of multi-G-metric spaces

In this section, we establish a few topological concepts on multi-G-metric spaces and explore a few of their associated characteristics.

**Definition 4.1.** *Let $(M,mG)$ be a multi-G- metric space. For $P_a^l \in M_{pt}$ and $P_r^1 > P_0^1$ the mG-open ball with centre $P_a^l$ and radius $P_r^1$ is defined by*

$$B_{mG}(P_a^l,P_r^1) = \{P_y^m \in M_{pt} : mG(P_x^l,P_y^m,P_y^m) < P_r^1\}.$$

*The mset $MS[B_{mG}(P_a^l,P_r^1)]$ will be called a multi-open ball with centre $P_a^l$ and radius $P_r^1 > P_0^1$.*

**Proposition 4.2.** *Let $(M,mG)$ be a multi-G- metric space. Let $P_x^l \in M_{pt}$ and $P_r^1 > P_0^1$. If $mG(P_x^l,P_y^m,P_z^n) < P_r^1$ then $P_x^l,P_y^m \in B_{mG}(P_x^l,P_r^1)$.*

*Proof.* It is obvious from $(mG_{m4})$. □

**Example 4.3.** *Consider the multi-G-metric space. $(M,mG)$ given in Example 3.2. Then we have*

$$B_{mG}(P_a^l,P_r^1) = \begin{cases} M_{pt}, & if P_r^1 > P_1^1, \\ \{P_a^n : 1 \leq n \leq \omega\}, & if P_r^1 \leq P_1^1, \end{cases}$$

*for any $P_a^l \in M_{pt}$.*

**Definition 4.4.** *Let $(M,mG)$ be a multi-G-metric space. For $P_a^l \in M_{pt}$ and $P_r^1 > P_0^1$ the mG-closed ball with $P_a^l$ and radius $P_1^r$ is defined by*

$$B_{mG}[P_a^l,P_r^1] = \{P_y^m \in M_{pt} : mG(P_x^l,P_y^m,P_y^m) < P_r^1\}.$$

*$MS[B_{mG}[P_a^l,P_r^1]]$ will be called a multi-closed ball with center $P_a^l$ and radius $P_r^1 > P_0^1$. The empty mset $\emptyset$ is separately considered as multi-G-open in $(M,mG)$.*

**Definition 4.5.** *Let $(M,mG)$ be a multi-G- metric space. Then, $O_M$, a collection of multi-points of M, is said to be mG-open if for each $P_x^l \in O_M$ there exists an mG-open ball $B_{mG}(P_a^l,P_r^1)$ with center at $P_a^l$ and radius $P_r^1 > P_0^1$ such that $B_{mG}(P_a^l,P_r^1) \subseteq O_M$. $N \subset M$ is multi-mG-open iff there exists a collection O of multi points of N such that O is mG-open and $MS(O) = N$.*

**Proposition 4.6.** *Every mG-open ball is mG-open in a multi-G-metric space.*

*Proof.* Let $P_y^m \in B_{mG}(P_x^l, P_r^1)$. Suppose $P_z^n \in B_{mG}(P_y^m, P_s^1)$. Then, we have $mG(P_z^n, P_z^n, P_y^m) < P_s^1$. By $(mG_5)$ we get

$$mG(P_z^n, P_z^n, P_x^l) \leq mG(P_x^l, P_y^m, P_y^m) + mG(P_y^m, P_z^n, P_z^n) < mG(P_x^l, P_y^m, P_y^m) + P_s^1.$$

Let $P_s^1 = P_r^1 - mG(P_x^l, P_y^m, P_y^m)$. We obtain

$$mG(P_z^n, P_z^n, P_x^l) < P_r^1.$$

Hence $P_z^n \in B_{mG}(P_x^l, P_r^1)$. □

**Theorem 4.7.** *Every multi-G-metric space generates a multi-G-topology as follows:*

$$\tau_{mG} = \{N : for\ every\ P_a^l \in N_{pt},\ there\ exists\ a P_r^1 such\ that\ B_{mG}(P_a^l, P_r^1) \subseteq O_N\ and\ MS(O_N) = N\}.$$

*This topology is said to be multi-topology produced by multi-G-metric.*

*Proof.* (1) Let $M_{pt}$ be the collection of for all multi-points in $(M, mG)$ multi-G-metric space. Then $M_{pt}$ is mG-open. Hence, $M = MS(M_{pt})$ is multi-G-open.

(2) Let $N_i$ $i = 1,2$ be two multi-G-open sets in $(M, mG)$. Then there exists $O_{N_i}$ such that $N_i = MS(O_{N_i})$ and $O_{N_i}$ is mG-open set of multi-points in $(M, mG)$. Let $P_x^l \in O_{N_1} \cap O_{N_2}$. Then, there exist $P_r^1, P_s^1 > P_0^1$ such that $B_{mG}(P_x^l, P_r^1) \subset O_{N_1}$ and $B_{mG}(P_x^l, P_s^1) \subset O_{N_2}$. Let $t = min\{r, s\}$. Then, we have $B_{mG}(P_x^l, P_1^t) \subset B_{mG}(P_x^l, P_1^r) \subset O_{N_1}$ and $B_{mG}(P_x^l, P_t^1) \subset B_{mG}(P_x^l, P_s^1) \subset O_{N_2}$. Therefore, we have $B_{mG}(P_x^l, P_t^1) \subset O_{N_1} \sqcap O_{N_2}$ and $O_{N_1} \sqcap O_{N_2}$ is mG-open. Since from Theorem 2.8, $N_1 \cap N_2 = MS(O_{N_1}) \cap MS(O_{N_2}) = MS(O_{N_1} \sqcap O_{N_2})$. Hence, $N_1 \cap N_2$ is multi-G-open.

(3) The proof can be done in a similar way (2).

□

**Definition 4.8.** *Let $(M, mG)$ be a multi-G- metric space. and $N_{mG}$ be a mset in this G-multi metric space. Then $N_{mG}$ is said to be multi-closed if its complement $N_{mG}^c$ is multi-open in this multi-G-metric space.*

**Theorem 4.9.** *Let $(M, mG)$ be a multi-G- metric space. The followings are held:*

*(1) The multi-empty set is multi-closed,*
*(2) The absolute mset M is multi closed,*
*(3) Arbitrary intersection of multi-closed sets is multi-closed,*
*(4) Finite union of multi-closed sets is multi-closed.*

*Proof.* The proofs are obvious from Theorem 4.7 and Definition 4.8. □

**Definition 4.10.** *Let $(M, mG)$ be a multi-G-metric space. Let $\{P_{x_n}^{l_n}\}$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to multi G-convergent ( mG-convergent ) to $P_x^l \in M_{pt}$, if for every $P_\varepsilon^1 > P_0^1$, there exists $n_0 \in \mathbb{N}$ such that $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$, $\forall n \geq n_0$ i.e. $n \geq n_0$ implies $\{P_{x_n}^{l_n}\} \in MS(B_{mG}(P_x^l, P_\varepsilon^1))$. We denote the sequence $\{P_{x_n}^{l_n}\}$ is multi G-convergent to $P_x^l$ and written as $\{P_{x_n}^{l_n}\} \to P_x^l$.*

**Proposition 4.11.** *In a multi-G- metric space, a sequence of multi-points multi G-converges at most one multi-point of the space.*

*Proof.* The proof is easily obtained from Definition 4.10. □

**Proposition 4.12.** *Let $(M, mG)$ be a multi-G- metric space. For the sequence $\{P_{x_n}^{l_n}\}_n \subset M_{pt}$ and a point $P_x^l \in M_{pt}$ the followings are equivalent:*

*(1) $\{P_{x_n}^{l_n}\}$ is mG-convergent to $P_x^l$,*
*(2) $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$,*
*(3) $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$,*
*(4) $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) \to P_0^1$ as as $n \to \infty$,*
*(5) $mG(P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as m, $n \to \infty$.*

*Proof.* (1) $\Rightarrow$(2) It is obvious from Proposition 3.5.

(2) $\Rightarrow$(3) Let $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$. Then, for each $P_\varepsilon^1 > P_0^1$, there exists a natural number $n_0$ such that $md(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$ whenever $n \geq n_0$. By Proposition 3.9, we have $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) = md_G(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$. Thus, we obtain $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$.

(3) $\Rightarrow$(4) It is clear since $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) \leq P_2^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l)$ by Proposition 3.5.

(4) $\Rightarrow$(5) It is follows from Proposition 3.5 since $mG((P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \leq mG(P_{x_m}^{l_m}, P_{x_m}^{l_m}, P_{x_n}^{l_n}) + mG(P_{x_m}^{l_m}, P_{x_m}^{l_m}, P_x^l)$.

(5) $\Rightarrow$(2) Let $mG(P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as m, $n \to \infty$. Since

$$md_G(P_{x_n}^{l_n}, P_x^l) = mG(P_{x_n}^{l_n}, P_x^l, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \leq P_2^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) = P_3^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l)$$

from Proposition 3.9 and Proposition 3.5, we have $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$.

□

**Definition 4.13.** *Let $(M,mG)$ be a multi-G-metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi points in M. Then the sequences $\{P_{x_n}^{l_n}\}_n$ is said to be multi G-bounded if there exists a positive multi real point $P_a^1 > P_0^1$ such that*

$$mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_{x_m}^{l_m}) \leq P_a^1$$

*for each m, $n \in \mathbb{N}$.*

**Definition 4.14.** *Let $(M,mG)$ be a multi-G- metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi-points in M. Then the sequence $\{P_{x_n}^{l_n}\}_n$ is said to be multi-G-Cauchy (mG-Cauchy) if every $P_\varepsilon^1 > P_0^1$, there exists a $n_0 \in \mathbb{N}$ such that $mG(P_{x_n}^{l_n}, P_{x_m}^{l_m}, P_{x_p}^{l_p}) < P_\varepsilon^1$ whenever $m,n,p \geq n_0$.*

**Proposition 4.15.** *Let $(M,mG)$ be a multi-G- metric space. Then the followings are equivalent:*

*(1) The sequence $\{P_{x_n}^{l_n}\}_n$ is mG-Cauchy.*
*(2) For every $P_\varepsilon^1 > P_0^1$, there exists a natural number $n_0$ such that $mG(P_{x_n}^{l_n}, P_{x_m}^{l_m}, P_{x_m}^{l_m}) < P_\varepsilon^1$ for all $n,m \geq n_0$.*
*(3) $\{P_{x_n}^{l_n}\}_n$ is a mG-Cauchy sequence in the multi-metric space $(M,md_G)$.*

*Proof.* (1) $\Rightarrow$(2) It is obvious by $(mG_4)$,
(2) $\Leftrightarrow$(3) It is clear from Proposition 3.9,
(2) $\Rightarrow$(1) It is obvious by $(mG_5)$ if it set $a = x_m$.

□

**Corollary 4.16.** *Every mG-convergent sequence in any multi-G-metric space is mG-Cauchy.*

*Proof.* It is obvious by $(mG_5)$ and Proposition 4.12. □

**Corollary 4.17.** *Every mG-Cauchy sequence is multi-G-bounded.*

*Proof.* It is obvious by Definition 4.13 and Proposition 4.12. □

**Definition 4.18.** *A mG-multi metric space $(M,mG)$ is said to be mG- complete if every mG-Cauchy sequence in $(M,mG)$ is mG-convergent in $(M,mG)$.*

**Proposition 4.19.** *A multi G-metric space $(M,mG)$ is mG- complete if and only if $(M,md_G)$ is a complete multi-metric space.*

*Proof.* It is obvious from Proposition 3.9 and Proposition 4.15 □

**Proposition 4.20.** *Let $(M,mG)$ be a multi-G-metric space. Let $md : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ be the function defined by $m\delta_G(P_x^l, P_y^m) = max\{mG(P_x^l, P_y^m, P_y^m), mG(P_y^m, P_x^l, P_x^l)\}$. Thus the followings hold:*

*(1) $(M, m\delta_G)$ is multi-metric space,*
*(2) $\{P_{x_n}^{l_n}\}_n$ is a mG-convergent to $P_x^l \in M_{pt}$ if and only if $\{P_{x_n}^{l_n}\}_n$ is a convergent to $P_x^l \in M_{pt}$ in $(M, m\delta_G)$,*
*(3) $\{P_{x_n}^{l_n}\}_n$ is a mG-Cauchy if and only if $\{P_{x_n}^{l_n}\}_n$ is Cauchy in $(M, m\delta_G)$,*
*(4) $(M,mG)$ is a mG-complete if and only if $(M, m\delta_G)$ is multi-complete.*

*Proof.* The proofs are clear from the definition of multi-G-metric and Proposition 4.12. □

## 5. Conclusion

We introduced and studied multi-G-metric spaces as generalizations of G-metric spaces and multi-metric spaces. We think this research will advance and increase future investigations into multi-topology and multi-metric systems by providing a broad framework for their practical applications.

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# References


[1] W. D. Blizard, *Mset theory*, Notre Dame J. Form. Log., **30**(1) (1989), 36-66.
[2] W. D. Blizard, *Real-valued multisets and fuzzy sets*, Fuzzy Sets Syst., **33**(1), (1989), 77-97.
[3] G. F. Clements, *On mset k-families*, Discrete Math., **69**(2) (1988), 153-164.
[4] M. Conder, S. Marshall, A. M. Slinko, *Orders on multisets and discrete cones*, Order-A Journal on The Theory of Ordered Sets and Its Applications, **24** (2007), 277-296.
[5] D. Singh, A. M. Ibrahim, T. Yohana, J. N. Singh, *Some combinatorics of multisets*, Int. J. Math. Educ. Sci. Technol., **34**(4) (2003), 489–499.
[6] D. Singh, A. M. Ibrahim, T. Yohana, J. N. Singh, *An overview of the applications of multisets*, Novi Sad J. Math., **37**(2) (2007), 73–92.
[7] D. Singh, A. M. Ibrahim, T. Yohana, J. N. Singh, *Complementation in mset theory*, Int. Math. Forum, **6**(38) (2011), 1877–1884.
[8] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**(2) (2006), 289-297.
[9] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994), 395-399.
[10] L. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468-1476.
[11] S. Das, R. Roy, *An introduction to multi metric spaces*, Adv. Dyn. Syst. Appl., **16**(2) (2021), 605-618.
[12] S. Das, R. Roy, *Some topological properties of multi metric spaces*, J. Math. Comput. Sci., **11** (2021), 7253-7268.
[13] K. P. Girish, S. J. John, *Mset topologies induced by mset relations*, Inf. Sci., **188** (2012), 298-313.