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Research Article

Certain topological properties for almost convergent Catalan-Motzkin sequence space

Muhammet Cihat Dağlı

Department of Mathematics, Akdeniz University, 07058-Antalya, Turkey

Abstract

The purpose of this paper is to introduce a new almost convergent sequence space by means of a matrix involving Catalan and Motzkin numbers. It is demonstrated that this novel sequence space and the space of all almost convergent sequences are linearly isomorphic and the $\beta-$dual of this new space is deduced. In addition, by defining Catalan-Motzkin core of a complex-valued sequence, certain inclusion relations are derived for it.

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1. Introduction and preliminaries

A sequence space is a linear subspace of the space ω of all real valued sequences. We denote the spaces of all bounded, convergent, null and absolutely $p-$summable real valued sequences by ℓ_∞, c, c_0 and ℓ_p, respectively. It is well known that these are Banach spaces according to the following norms

$$\|z\|_{\ell_\infty} = \|z\|_c = \|z\|_{c_0} = \sup_{k\in\mathbb{N}} |z_k|$$

and

$$\|z\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |z_k|^p\right)^{1/p}.$$

For real entries t_{nk}, if $T = (t_{nk})$ is an infinite matrix and T_n is the sequence in the nth row of the matrix T for each $n \in \mathbb{N}$. Then, for a given $z = (z_k) \in \omega$ the sequence $Tz = (Tz)_n = \sum_k t_{nk} z_k$ is referred as $T-$transform of z subject to the condition that the convergence of series is satisfied for each $n \in \mathbb{N}$. For brevity, in the sequel, the notation $\sum_k$ means that $\sum_{k=0}^{\infty}$. T is named as a matrix mapping from a sequence space Λ to a sequence space Ω if the sequence Tz exists and $Tz \in \Omega$ for all $z \in \Lambda$. Denote (Λ, Ω) as the collection of all infinite matrices from Λ to Ω. If Tz converges to L, then, we say that z is $T-$summable to L and so L is $T-$limit of z. Also, we say that T regularly maps Λ into Ω by writing $T \in (\Lambda, \Omega)_{reg}$ if $T \in (\Lambda, \Omega)$ and $\lim_n (Tz)_n = \lim_n z_n$ for all $z = (z_n) \in \Lambda$.

Email address: mcihatdagli@akdeniz.edu.tr (M.C. Dağlı)
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Since Fast [24] and also Steinhaus [46] initiated the almost convergence as generalization of convergence of a sequence, this concept has been widely discussed by researchers and lots of substantial papers have been assigned.

A sequence $z = (z_n)$ is called as almost convergent to L iff

$$\lim_{r\to\infty} \sum_{n=0}^{r} \frac{z_{m+n}}{r+1} = L, \quad \text{uniformly in } m.$$

and this is denoted by $h' - \lim_n z_n = L$. By h' and h'_0, we denote the spaces of all almost convergent and almost null sequences, namely,

$$h' = \left\{ z = (z_m) \in \omega : \exists L \in \mathbb{C} \ni \lim_{r\to\infty} \sum_{n=0}^{r} \frac{z_{m+n}}{r+1} = L \text{ uniformly in } m \right\}$$

and

$$h'_0 = \left\{ z = (z_m) \in \omega : \lim_{r\to\infty} \sum_{n=0}^{r} \frac{z_{m+n}}{r+1} = 0 \text{ uniformly in } m \right\},$$

respectively. Here, $\mathbb{C}$ denotes the set of all complex numbers.

The matrix domain h_T of an infinite matrix T in the sequence space h is defined by

$$h_T = \{z \in \omega : Tz \in h\},$$

which is a sequence space. For recorded papers concerning almost convergence from the point of view of matrix operators, one can suggest [4, 7, 8, 10–12, 23, 35, 38, 48], and cited references therein.

The concept of Knopp Core (or in short $\mathcal{K}-$core) of a sequence $z = (z_n) \in \omega$ is the intersection of all Q_n, the least convex closed region of complex plane containing $z_n, z_{n+1}, ...$ for $n = 1, 2, ...$ (cf. [14, p. 137]). The following relation is established in [42]

$$\mathcal{K} - core\,(z) = \cap_{u\in\mathbb{C}} M_z\,(u)\,,$$

where $M_z\,(u)$ is the set given by

$$M_z\,(u) = \left\{ w \in \mathbb{C} : |w - u| \le \limsup_n |z_n - u| \right\},$$

The notion of natural density of $E \subset \mathbb{N}$, denoted as $\delta\,(E)$ can be defined by

$$\delta\,(E) = \lim_n \frac{1}{n} |k \le n : k \in E|\,,$$

in which $|A|$ is the number of elements of A. Given any $\varepsilon > 0$ such that

$$\delta\,(\{k : |z_k - z_0| \ge \varepsilon\}) = 0,$$

then, the sequence $z = (z_k)$ is referred as statistically convergent to z_0. In this case, we write $sc - \lim z = z_0$ (see [47]). We remark that the spaces of all statistically convergent and statistically null convergent sequences will be denoted by sc and sc_0, respectively.

Fridy and Orhan [25] presented the notion of the statistically core and derived the following relation for any statistically bounded sequence z,

$$sc - core\,(z) = \cap_{u\in\mathbb{C}} D_z\,(u)\,,$$

where $D_z\,(u) = \{w \in \mathbb{C} : |w - u| \le sc - \limsup_n |z_n - u|\}$. Additional results concerning with core theorems can be found in [13, 15–17] and cited references therein. As well, investigating the monographs [5] and [40] and the papers [1, 9, 39, 41] will be useful for comprehensive background in this research area.

In recent years, construction of conservative matrix for the aim of discussion its matrix domain in certain important sequence spaces has been considered with regard to special numbers such as Catalan [27, 31, 34], Schröder [18–20], Bell [36], Fibonacci [32, 33], Tribonacci [49, 50], and some special functions [22, 26, 28–30]. Moreover, various remarkable topologic properties including almost convergence and core theorems, special duals, Hausdorff measure of noncompactness, matrix mappings, etc. have been studied.

Recall the properties of Catalan numbers, which are deeply studied in number theory and combinatorics. Historically, one of the first interpretation given to the Catalan number $(C_n)_{n\geq 0}$ was through the number of ways to triangulate a regular $n+2$-sided polygon, known as Euler problem. Another example where this sequence appears is counting the ways of constructing binary trees. In particular, C_n represents the number of ways to construct a binary tree with n nodes (see [45]).

The Catalan numbers C_n are defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. These numbers can also be written via Motzkin number M_n as

$$C_{n+1} = \sum_{k=0}^{n} \binom{n}{k} M_k. \tag{1.1}$$

Besides, they possess the recurrence relations:

$$C_{n+1} = \frac{2\,(2n+1)}{n+2} C_n, \quad C_0 = 1$$

and

$$C_n = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1. \tag{1.2}$$

By adopting the recurrence relation (1.2), the authors of [27, 34] dealt with the matrix domain of a conservative matrix in famous sequence spaces. Besides, a number of interesting and meaningful conclusions have been deduced therein.

Motivated by another representation of Catalan numbers, given by (1.1), Karakas and Dağlı [37] presented a new conservative matrix $\tilde{C} = (\tilde{c}_{nk})$ as

$$\tilde{c}_{nk} = \begin{cases} \binom{n}{k} \dfrac{M_k}{C_{n+1}}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n; \end{cases} \tag{1.3}$$

and gave the spaces $\ell_p\left(\tilde{C}\right)$ and $\ell_\infty\left(\tilde{C}\right)$ as the domains of this conservative matrix. The Schauder basis and the $\alpha-$dual, $\beta-$dual and $\gamma-$dual of these spaces were determined. Furthermore, the characterizations of some classes $\left(\ell_p\left(\tilde{C}\right), \Omega\right)$ for $1 \leq p \leq \infty$ and $\Omega \in \{c_0, c, \ell_1, \ell_\infty\}$ were discussed and certain geometric properties were investigated.

The current paper concerns a novel almost convergent sequence space by means of the matrix, given by (1.3). We provide that this new space is linearly isomorphic to almost convergent sequence spaces and we give its $\beta-$dual. Furthermore, we define the Catalan-Motzkin core of a complex-valued sequence and characterize the class of matrices for which $\tilde{C} - core\,(Tz) \subseteq \mathcal{K} - core\,(z)$ and $\tilde{C} - core\,(Tz) \subseteq sc - core\,(z)$ for all $z \in \ell_\infty$.

2. New almost convergent sequence space

We devote this section to present the space of almost convergent Catalan-Motzkin sequence as well as its some properties.

Throughout this study, the sequence y_n represents the $\tilde{C}-$transform of a sequence $z = (z_k)$ as for all $n \in \mathbb{N}$

$$y_n = \left(\tilde{C}z\right)_n = \frac{1}{C_{n+1}} \sum_{k=0}^{n} \binom{n}{k} M_k z_k,$$

while the relation

$$z_k = \left(\tilde{C}^{-1}z\right)_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{C_{j+1}}{M_k} y_j \tag{2.1}$$

holds for all $k \in \mathbb{N}$.

Define the almost convergent Catalan-Motzkin sequence space by

$$C'\left(\tilde{C}\right) = \left\{ z = (z_m) \in \omega : \exists L \in \mathbb{C} \ni \lim_{r\to\infty} \sum_{n=0}^{r} \frac{y_{m+n}}{r+1} = L \text{ uniformly in } m \right\}.$$

Theorem 2.1. *The sequence space $C'\left(\tilde{C}\right)$ is linearly isomorphic to h'.*

Proof. Let the transformation $\psi : C'\left(\tilde{C}\right) \to h'$ be given by $\psi z = \tilde{C}z$. It is obvious that ψ is linear. As well, we have $z = \theta$ while $\psi z = \theta$, thereby ψ is injective. For any sequence $y = (y_n) \in h'$, apply (2.1) to write that

$$\begin{aligned} &\frac{1}{C_{n+j+1}} \sum_{k=0}^{n+j} \binom{n+j}{k} M_k z_k \\ &= \frac{1}{C_{n+j+1}} \sum_{k=0}^{n+j} \binom{n+j}{k} M_k \sum_{i=0}^{k} \left((-1)^{k-i} \binom{k}{i} \frac{C_{i+1}}{M_k} \right) y_i \\ &= \frac{1}{C_{n+j+1}} \sum_{i=0}^{n+j} \left(\sum_{k=0}^{n+j-i} \binom{n+j}{k+i} (-1)^k \right) C_{i+1} y_i \\ &= y_{n+j}. \end{aligned}$$

Thus,

$$\frac{1}{r+1} \sum_{j=0}^{r} \frac{1}{C_{n+j+1}} \sum_{k=0}^{n+j} \binom{n+j}{k} M_k z_k = \frac{1}{r+1} \sum_{j=0}^{r} y_{n+j},$$

which implies that the following limit exists uniformly in n:

$$\lim_{r\to\infty} \frac{1}{r+1} \sum_{j=0}^{r} \left(\tilde{C}z\right)_{n+j} = \lim_{r\to\infty} \frac{1}{r+1} \sum_{j=0}^{r} y_{n+j}.$$

So that, $z \in C'\left(\tilde{C}\right)$ and $\psi z = y$. Consequently, ψ is surjective. Hereby, we have established the existence of a linear bijection between the spaces $C'\left(\tilde{C}\right)$ and h'. The proof is complete. □

The multiplier space of Λ and Ω is the set $S(\Lambda, \Omega)$ defined by

$$S(\Lambda, \Omega) = \{ u \in \omega : zu \in \Omega \text{ for all } z \in \Lambda \}.$$

Adopting this notation, the $\beta-$dual of a sequence space Λ is defined as $\Lambda^\beta = S(\Lambda, cs)$, where cs denotes the space of convergent series.

Let us mention a lemma, which we use as our favorable tool for the presentation of $\beta-$dual of the space $C'\left(\tilde{C}\right)$.

Lemma 2.2 ([43])**.** *$T = (t_{nk}) \in (h', c)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}| < \infty$$

is valid and there exists $\lambda_k \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\lim_{n\to\infty} t_{nk} = \lambda_k$$

for each $k \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sum_k t_{nk} = \lambda$$

and

$$\lim_{n\to\infty} \sum_k |\Delta(t_{nk} - \lambda_k)| = 0, \quad (\text{for each } k \in \mathbb{N})$$

where $\Delta(t_{nk} - \lambda_k) = t_{nk} - \lambda_k - (t_{n,k+1} - \lambda_{k+1})$.

Theorem 2.3. *Define the sets* $\xi_1, \xi_2, \xi_3, \xi_4$ *and* ξ_5 *as*

$$\xi_1 = \left\{ b = (b_k) \in \omega : \sup_{r\in\mathbb{N}} \sum_{m=0}^{r} \left| \sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \frac{C_{m+1}}{M_k} b_k \right| < \infty \right\},$$

$$\xi_2 = \left\{ b = (b_k) \in \omega : \lim_{r\to\infty} \sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \frac{C_{m+1}}{M_k} b_k \text{ exists} \right\},$$

$$\xi_3 = \left\{ b = (b_k) \in \omega : \lim_{r\to\infty} \sum_{m=0}^{r} \left(\sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \frac{C_{m+1}}{M_k} b_k \right) \text{ exists} \right\},$$

$$\xi_4 = \left\{ b = (b_k) \in \omega : \lim_{r\to\infty} \sum_{m=0}^{r} \left| \sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \frac{C_{m+1}}{M_k} b_k \right| = 0 \right\}$$

and

$$\xi_5 = \left\{ b = (b_k) \in \omega : \lim_{r\to\infty} \sum_{m} \left| \Delta \left(\sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \frac{C_{m+1}}{M_k} b_k - \lambda_m \right) \right| = 0 \right\}.$$

Then, we have $\left(C' \left(\tilde{C} \right) \right)^{\beta} = \xi_1 \cap \xi_2 \cap \xi_3 \cap \xi_4 \cap \xi_5$.

Proof. For any $b = (b_n) \in \omega$ and for all $r \in \mathbb{N}$, it follows from (2.1) that

$$\sum_{m=0}^{r} b_m z_m = \sum_{m=0}^{r} b_m \left(\sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \frac{C_{n+1}}{M_m} y_n \right)$$
$$= \sum_{m=0}^{r} \left(\sum_{n=m}^{r} (-1)^{n-m} \binom{n}{m} \frac{C_{m+1}}{M_n} b_n \right) y_m = (\mathcal{A}y)_r \,,$$

where $\mathcal{A} = (a_{rm})$ is the matrix, defined by

$$a_{rm} = \begin{cases} \sum_{n=m}^{r} (-1)^{n-m} \binom{n}{m} \frac{C_{m+1}}{M_n} b_n, & \text{if } 0 \le m \le r; \\ 0, & \text{otherwise}; \end{cases}$$

for all $r, m \in \mathbb{N}$. So, we disclose that $bz = (b_m z_m) \in cs$ when $z = (z_m) \in C' \left(\tilde{C} \right)$ iff $\mathcal{A}y \in c$ for all $y = (y_m) \in h'$. Therefore, $b = (b_m) \in \left(C' \left(\tilde{C} \right) \right)^{\beta}$ if and only if $\mathcal{A} \in (h', c)$. If we apply Lemma 2.2, we conclude the proof. □

3. The Catalan-Motzkin core

The aim of this final section is to offer Catalan-Motzkin core (or in short $\tilde{C} - core$) of a complex valued sequence and to compute certain characterizations of the class of matrices with $\tilde{C} - core\,(Tz) \subseteq \mathcal{K} - core\,(z)$ and $\tilde{C} - core\,(Tz) \subseteq sc - core\,(z)$ for all bounded sequences z.

The concept of Knopp Core Theorem [14, p. 138] clarifies that $\mathcal{K} - core\,(Tz) \subseteq \mathcal{K} - core\,(z)$ for a positive matrix $T \in (c, c)_{reg}$ and all real valued sequences z. The reader can refer to the monograph [5] which contains the chapter entitled Core of a Sequence, and the recent papers [2,3,6] of this research area.

The $\tilde{C} - core$ of z is defined by

$$\tilde{C} - core\,(z) = \cap_{n=1}^{\infty} H_n,$$

where H_n denotes the least closed convex hull containing $\left(\tilde{C}z\right)_n, \left(\tilde{C}z\right)_{n+1}, \ldots$. The following relation between $\tilde{C} - core$ of z and $\mathcal{K} - core$ of the sequence $\left(\tilde{C}z\right)_n$ is valid for any $p \in \mathbb{C}$

$$\tilde{C} - core\,(z) = \cap_{p\in\mathbb{C}} \mathcal{V}_z\,(p)\,, \qquad (\text{for } z \in \ell_\infty),$$

where

$$\mathcal{V}_z(p) = \left\{ w \in \mathbb{C} : |w - p| \leq \limsup_n \left| \left(\tilde{C} z \right)_n - p \right| \right\}.$$

Consider the following matrix $\tilde{T} = (\tilde{t}_{nk})$ via $T = (t_{nk})$ by

$$\tilde{t}_{nk} = \frac{1}{C_{n+1}} \sum_{k=0}^{n} \binom{n}{k} M_k t_{nk}, \quad \text{for all } n, k \in \mathbb{N}.$$

By $c_0\left(\tilde{C}\right)$ and $c\left(\tilde{C}\right)$, we denote all sequences whose $\tilde{C}-$transforms are in the spaces c_0 and c, respectively. We need some lemmas so as to reach the proofs of the conclusions.

Lemma 3.1. *$T = (t_{nk}) \in \left(\ell_\infty, c\left(\tilde{C}\right)\right)$ iff*

$$\left\| \tilde{T} \right\| = \sup_n \sum_k |\tilde{t}_{nk}| < \infty, \tag{3.1}$$

$$\lim_n \tilde{t}_{nk} = \tau_k, \text{ for each } k \in \mathbb{N} \tag{3.2}$$

and

$$\lim_n \sum_k |\tilde{t}_{nk} - \tau_k| = 0. \tag{3.3}$$

Lemma 3.2. *$T = (t_{nk}) \in \left(c, c\left(\tilde{C}\right)\right)_{reg}$ iff (3.1) is satisfied and (3.2) is also satisfied with $\tau_k = 0$ for all $k \in \mathbb{N}$ and the following limit is valid:*

$$\lim_n \sum_k \tilde{t}_{nk} = 1. \tag{3.4}$$

Lemma 3.3. *$T = (t_{nk}) \in \left(sc \cap \ell_\infty, c\left(\tilde{C}\right)\right)_{reg}$ if and only if $T \in \left(c, c\left(\tilde{C}\right)\right)_{reg}$ and*

$$\lim_n \sum_{k \in E,\ \delta(E)=0} |\tilde{t}_{nk}| = 0. \tag{3.5}$$

Proof. From the fact $c \subset sc \cap \ell_\infty$, we have $T \in \left(c, c\left(\tilde{C}\right)\right)_{reg}$. Given any $z \in \ell_\infty$ and $E \subset \mathbb{N}$ with $\delta(E) = 0$, let us take the sequence $\tilde{z} = (\tilde{z}_k)$ such that

$$\tilde{z}_k = \begin{cases} z_k, & \text{if } k \in E; \\ 0, & \text{if } k \notin E. \end{cases}$$

Observe that if $\tilde{z} \in sc_0$ then $T\tilde{z} \in c_0\left(\tilde{C}\right)$. Also, we conclude from the relation

$$\sum_k \tilde{t}_{nk} \tilde{z}_k = \sum_{k \in E} \tilde{t}_{nk} z_k,$$

that the matrix $W = (W_{nk})$ belongs to the class $\left(\ell_\infty, c\left(\tilde{C}\right)\right)$, where $W_{nk} = \tilde{t}_{nk}$ if $k \in E$ and $W_{nk} = 0$ if $k \notin E$. So, using Lemma 3.1 yields (3.5).

Conversely, if $z \in sc \cap \ell_\infty$ and $sc - \lim z = \chi$, then, we have for any $\varepsilon > 0$,

$$\delta(E) = \delta(\{k : |z_k - \chi| \geq \varepsilon\}) = 0.$$

Now, let

$$\sum_k \tilde{t}_{nk} z_k = \sum_k \tilde{t}_{nk} (z_k - \chi) + \chi \sum_k \tilde{t}_{nk}. \tag{3.6}$$

Then, in the light of the inequality

$$\sum_k \tilde{t}_{nk} (z_k - \chi) \leq \|z\| \sum_{k \in E} |\tilde{t}_{nk}| + \varepsilon \left\| \tilde{T} \right\|,$$

passing the limit $n \to \infty$ in (3.6), and using (3.4) and (3.5) give that

$$\lim_n \sum_k \tilde{t}_{nk} z_k = \chi,$$

from which we get $T\in\left(sc\cap\ell_{\infty},c\left(\tilde{C}\right)\right)_{reg}$, as desired. So, the proof is complete. □

Lemma 3.4 ([44]). *Let $T=(t_{nk})$ be a matrix, satisfying that $\sum_k|t_{nk}|<\infty$ and $\lim_n t_{nk}=0$. Then, there exists $z\in\ell_\infty$ with $\|z\|\le 1$ such that*

$$\limsup_n\sum_k t_{nk}z_k=\limsup_n\sum_k|t_{nk}|.$$

Now, let us present some inclusion theorems.

Theorem 3.5. *Assume that $T\in\left(c,c\left(\tilde{C}\right)\right)_{reg}$. For all $z\in\ell_\infty$, the relation $\tilde{C}-core\,(Tz)\subseteq\mathcal{K}-core\,(z)$ holds iff*

$$\lim_n\sum_k\left|\tilde{t}_{nk}\right|=1. \tag{3.7}$$

Proof. Since the matrix satisfy the conditions of Lemma 3.4, there exists $z\in\ell_\infty$ with $\|z\|\le1$ such that

$$\left\{l\in\mathbb{C}:|l|\le\limsup_n\sum_k\tilde{t}_{nk}z_k\right\}=\left\{l\in\mathbb{C}:|l|\le\limsup_n\sum_k\left|\tilde{t}_{nk}\right|\right\}.$$

Also, since $\mathcal{K}-core\,(z)\subset T_1(0)$, by

$$\left\{l\in\mathbb{C}:|l|\le\limsup_n\sum_k\left|\tilde{t}_{nk}\right|\right\}\subseteq T_1(0)=\{l\in\mathbb{C}:|l|\le1\},$$

which yields (3.7). On the contrary, for $l\in\mathcal{K}-core\,(Tz)$, one arrives the inequality

$$\begin{aligned}|l-p|&\le\limsup_n\left|\tilde{C}_n(Tz)-p\right|\\&=\limsup_n\left|p-\sum_k\tilde{t}_{nk}z_k\right|\\&\le\limsup_n\left|\sum_k\tilde{t}_{nk}(p-z_k)\right|+\limsup_n|p|\left|1-\sum_k\tilde{t}_{nk}\right|\\&=\limsup_n\left|\sum_k\tilde{t}_{nk}(p-z_k)\right|,\quad(\text{for }p\in\mathbb{C}).\end{aligned} \tag{3.8}$$

Now, let $\limsup_k|z_k-p|=\chi$. Then, can write for any $\varepsilon>0$ that $|z_k-p|\le\chi+\varepsilon$ for $k\ge k_0$. Hence, one finds that

$$\begin{aligned}\left|\sum_k\tilde{t}_{nk}(p-z_k)\right|&=\left|\sum_{k<k_0}\tilde{t}_{nk}(p-z_k)+\sum_{k\ge k_0}\tilde{t}_{nk}(p-z_k)\right|\\&\le\sup_k|p-z_k|\sum_{k<k_0}\left|\tilde{t}_{nk}\right|+(\chi+\varepsilon)\sum_{k\ge k_0}\left|\tilde{t}_{nk}\right|\\&\le\sup_k|p-z_k|\sum_{k<k_0}\left|\tilde{t}_{nk}\right|+(\chi+\varepsilon)\sum_k\left|\tilde{t}_{nk}\right|.\end{aligned} \tag{3.9}$$

If we gather (3.8) and (3.9), then, we have

$$|l-p|\le\limsup_n\left|\sum_k\tilde{t}_{nk}(p-z_k)\right|\le\chi+\varepsilon,$$

from which we conclude that $l\in\mathcal{K}-core\,(z)$. Thus, the proof is complete. □

Theorem 3.6. *Assume that $T\in\left(sc\cap\ell_\infty,c\left(\tilde{C}\right)\right)_{reg}$. Then, $\tilde{C}-core\,(Tz)\subseteq sc-core\,(z)$ for all $z\in\ell_\infty$ if and only if (3.7) holds.*

Proof. Since $sc - core\,(z) \subseteq \mathcal{K} - core\,(z)$, it follows from our previous theorem that (3.7) holds. Conversely, consider $l \in \tilde{C}{-}core\,(Tz)$. It is aware from [21] that $\delta\left(\{k : |z_k - p| > \chi' + \varepsilon\}\right) = 0$ if $sc - \limsup |z_k - p| = \chi'$. Thus,

$$\begin{aligned}
&\left|\sum_k \tilde{t}_{nk}\,(p - z_k)\right| \\
&= \left|\sum_{k\in E} \tilde{t}_{nk}\,(p - z_k) + \sum_{k\notin E} \tilde{t}_{nk}\,(p - z_k)\right| \\
&\leq \sup_k |p - z_k| \sum_{k\in E} |\tilde{t}_{nk}| + (\chi' + \varepsilon) \sum_{k\notin E} |\tilde{t}_{nk}| \\
&\leq \sup_k |p - z_k| \sum_{k\in E} |\tilde{t}_{nk}| + (\chi' + \varepsilon) \sum_k |\tilde{t}_{nk}|.
\end{aligned}$$

Apply (3.5) and (3.7) to reach that

$$\limsup_n \left|\sum_k \tilde{t}_{nk}\,(p - z_k)\right| \leq \chi' + \varepsilon \tag{3.10}$$

and thus gather (3.8) and (3.10) to get

$$|l - p| \leq sc - \limsup_k |z_k - p|.$$

Consequently, we have $l \in sc - core\,(z)$, as desired. □

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References

[1] B. Altay, F. Başar and E. Malkowsky, *Matrix transformations on some sequence spaces related to strong Cesàro summability and boundedness,* Appl. Math. Comput. **211,** 255–264, 2009.

[2] C. Aydın and F. Başar, *On the new sequence spaces which include the spaces c_0 and c,* Hokkaido Math. J. **33** (2), 383–398, 2004.

[3] C. Aydın and F. Başar, *Some new difference sequence spaces,* Appl. Math. Comput. **157** (3), 677–693, 2004.

[4] F. Başar, *$f-$conservative matrix sequences,* Tamkang J. Math. **22**, 205–212 1991.

[5] F. Başar, *Summability Theory and Its Applications, 2nd ed.,* CRC Press/Taylor & Francis Group, Boca Raton · London · New York, 2022.

[6] F. Başar and B. Altay, *On the space of sequences of $p-$bounded variation and related matrix mappings,* (English, Ukrainian summary) Ukrain. Mat. Zh. 55 (1) (2003), 108–118; reprinted in Ukrainian Math. J. **55** (1), 136–147, 2003.

[7] F. Başar and R. Çolak, *Almost-conservative matrix transformations,* Turkish J. Math. **13**, 91–100, 1989.

[8] F. Başar and M. Kirişçi, *Almost convergence and generalized difference matrix,* Comput. Math. App. **61** (3), 602–611, 2011.

[9] F. Başar, E. Malkowsky and B. Altay, *Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1-summable and bounded sequences,* Publ. Math. Debrecen **73** (1-2), 193–213, 2008.

[10] F. Başar and İ. Solak, *Almost-coercive matrix transformations,* Rend. Mat. Appl. **11–2**, 249–256, 1991.

[11] M. Candan, *Vector-valued FK-spaces defined by a modulus function and an infinite matrix,* Thai J. Math. **12** (1), 155–165, 2014.

[12] M. Candan, *Almost convergence and double sequential band matrix,* Acta Math. Sci. **34** (2), 354–366 2014.

[13] J. Connor, J.A. Fridy and C. Orhan, *Core equality results for sequences,* J. Math. Anal. Appl. **321**, 515–523, 2006.

[14] R.G. Cooke, *Infinite matrices and sequence spaces,* MacMillan, New York, 1950.

[15] C. Çakan and H. Coşkun, *Some new inequalities related to the invariant means and uniformly bounded function sequences,* Appl. Math. Lett. **20**, 605–609, 2007.

[16] H. Çoşkun and C. Çakan, *A class of statistical and* $\sigma-$*conservative matrices,* Czechoslovak Math. J. **55**, 791–801, 2005.

[17] H. Çoşkun, C. Çakan and M. Mursaleen, *On the statistical and* $\sigma-$*cores,* Stud. Math. **154**, 29–35, 2003.

[18] M.C. Dağlı, *A novel conservative matrix arising from Schröder numbers and its properties,* Linear Multilinear Algebra **71** (8), 1338–1351, 2023.

[19] M.C. Dağlı, *A new almost convergent sequence space defined by Schröder matrix,* Linear Multilinear Algebra, **71** (11), 1863–1874, 2023.

[20] M.C. Dağlı, *Matrix mappings and compact operators for Schröder sequence spaces,* Turkish J. Math. **46** (6), 2304–2320, 2022.

[21] K. Demirci, *A-statistical core of a sequence,* Demonstr. Math. **33**, 43–51, 2000.

[22] S. Demiriz, M. İlkhan and E.E. Kara, *Almost convergence and Euler totient matrix,* Ann. Funct. Anal. **11**, 604–616, 2020.

[23] J.P. Duran, *Infinite matrices and almost convergence,* Math. Z. **128**, 75–83, 1972.

[24] H. Fast, *Sur la convergence statistique,* Colloq. Math. **2**, 241–244, 1951.

[25] J.A. Fridy and C. Orhan, *Statistical core theorems,* J. Math. Anal. Appl. **208**, 520–527, 1997.

[26] M. İlkhan, *Matrix domain of a regular matrix derived by Euler totient function in the spaces* c_0 *and* c*,* Mediterr. J. Math. **17** (1), Article no: 27, 2020.

[27] M. İlkhan, *A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces* c *and* c_0, Linear Multilinear Algebra **68** (2), 417–434, 2020.

[28] M. İlkhan and E.E. Kara, *A new Banach space defined by Euler totient matrix operator,* Oper. Matrices **13** (2), 527–544, 2019.

[29] M.İlkhan, E.E.Kara and F. Usta, *Compact operators on the Jordan totient sequence spaces,* Math. Methods Appl. Sci. **44** (9), 7666–7675, 2021.

[30] M. İlkhan, N. Simsek and E.E. Kara, *A new regular infinite matrix defined by Jordan totient function and its matrix domain in* ℓ_p*,* Math. Methods Appl. Sci. **44** (9), 7622–7633, 2020.

[31] S. Jasrotia, U.P. Singh and K. Raj, *Some new observations on Catalan almost convergent sequence spaces and the Catalan core,* Acta Sci. Math. (Szeged) **87** (1-2), 153–163, 2021.

[32] E.E. Kara and S. Demiriz, *Some new paranormed difference sequence spaces derived by Fibonacci numbers,* Miskolc Math. Notes **16** (2), 907–923, 2015.

[33] E.E. Kara and M. İlkhan, *Some properties of generalized Fibonacci sequence spaces,* Linear Multilinear Algebra **64** (11), 2208–2223, 2016.

[34] M.İ. Kara and E.E. Kara, *Matrix transformations and compact operators on Catalan sequence spaces,* J. Math. Anal. Appl. **498** (1), Article no: 124925, 2021.

[35] A. Karaisa and F. Özger, *On almost convergence and difference sequence spaces of order* m *with core theorems,* Gen. Math. Notes **26** (1), 102–125, 2015.

[36] M. Karakas, *On the sequence spaces involving bell numbers,* Linear Multilinear Algebra **71** (14), 2298–2309, 2023.

[37] M. Karakas and M.C. Dağlı, *Some topologic and geometric properties of new Catalan sequence spaces,* Adv. Oper. Theory, **8** (1), Article number: 14, 2023.

[38] G. Kılınç and M. Candan, *Some generalized Fibonacci difference spaces defined by a sequence of modulus functions,* Facta Univ. Ser. Math. Inf. **32**, 95–116, 2017.

[39] M. Kirişçi and F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix,* Comput. Math. Appl. **60** (5), 1299–1309, 2010.

[40] M. Mursaleen and F. Başar, *Sequence Spaces: Topics in Modern Summability Theory,* CRC Press/Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton · London · New York, 2020.

[41] F. Özger and F. Başar, *Domain of the double sequential band matrix* $B(\tilde{r}, \tilde{s})$ *on some Maddox's spaces,* Acta Math. Sci. **34**, 394–408, 2014.

[42] A.A. Shcherbakov, *Kernels of sequences of complex numbers and their regular transformations,* Math. Notes **22**, 948–953, 1977.

[43] J.A. Sıddıqi, *Infinite matrices summing every almost periodic sequences,* Pacific J. Math. **39**, 235–251, 1971.

[44] S. Simons, *Banach limits, infinite matrices and sublinear functionals,* J. Math. Anal. Appl. **26**, 640–655, 1969.

[45] R.P. Stanley, *Catalan Numbers,* Cambridge University Press, New York, 2015.

[46] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique,* Colloq. Math. **2**, 73–74, 1951.

[47] H. Steinhaus, *Quality control by sampling,* Colloq. Math. **2**, 98–108, 1951.

[48] Q. Qamaruddin and S.A. Mohuiddine, *Almost convergence and some matrix transformations,* Filomat **21**, 261–266, 2007.

[49] T. Yaying and B. Hazarika, *On sequence spaces defined by the domain of a regular Tribonacci matrix,* Math. Slovaca **70** (3), 697–706, 2020.

[50] T. Yaying and M.İ. Kara, *On sequence spaces defined by the domain of tribonacci matrix in* c_0 *and* c*,* Korean J. Math. **29** (1), 25–40, 2021.