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Research Article

Numerical-analytic successive approximation method for the investigation of periodic solutions of nonlinear integro-differential systems with piecewise constant argument of generalized type

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Abstract

In this paper, we focus on investigating the existence and approximation of periodic solutions for a nonlinear integro-differential system with a piecewise alternately advanced and retarded argument of generalized type, referred to as DEPCAG. The argument is a general step function, and we obtain criteria for the existence of periodic solutions for such equations. Our approach involves converting the given DEPCAG into an equivalent integral equation and using a new approach for periodic solutions. We construct appropriate mappings and employ a numerical-analytic method to investigate periodic solutions of the ordinary differential equation given by A. M. Samoilenko [32]. Additionally, we use the contraction mapping principle to demonstrate the existence of a unique periodic solution.

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Keywords. numerical-analytic method, piecewise constant argument of generalized type, convergence of approximate solutions, periodic solutions, hybrid equations

1. Introduction

Within the realm of functional differential equations, Myshkis [26] focused on researching differential equations with piecewise constant arguments, known as DEPCA. The theory of scalar DEPCA of the type

$$\frac{dx(t)}{dt} = f(t, x(t), x(\gamma(t))), \quad \gamma(t) = [t] \quad \text{or} \quad \gamma(t) = 2\left[\frac{t+1}{2}\right], \tag{1.1}$$

where $[\cdot]$ signifies the greatest integer function, was first introduced in [33] and further developed by various authors [1, 5, 21, 25, 28, 37], including the first book on DEPCA by Wiener [38]. Applications of DEPCA have been discussed in [6, 12, 16, 29, 35, 38]. These equations are often referred to as hybrid equations due to their possession of properties from both continuous systems and discrete equations.

Over the years, significant attention has been dedicated to investigating the existence of periodic and almost periodic solutions for various types of differential equations. For

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further details and specific references, please see [1,5,6,8–11,13–15,17–20,23,25,37,38]. In their interesting paper and research monograph [21,22], the authors analyzed the existence and uniqueness of asymptotically Bloch-periodic solutions for abstract fractional nonlinear differential inclusions with piecewise constant argument in Banach spaces.

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of all integers, natural, real and complex numbers, respectively. Denote by $|\cdot|$ a norm in $\mathbb{R}^n, n \in \mathbb{N}$. Fix two real sequences $t_i, \gamma_i, i \in \mathbb{Z}$, such that $t_i < t_{i+1}$, $t_i \leq \gamma_i \leq t_{i+1}$ for all $i \in \mathbb{Z}$, $t_i \to \pm\infty$ as $i \to \pm\infty$.

Let $\gamma : \mathbb{R} \to \mathbb{R}$ represent a step function defined as $\gamma(t) = \gamma_i$ for $t \in I_i = [t_i, t_{i+1})$, and let us consider the DEPCA (1.1) under this general γ. In this scenario, the argument takes the form of a piecewise function, alternating between advanced and delayed situations. Specifically, when $\gamma(t) = [t]$, it corresponds to a sequence where $t_i = i \in \mathbb{Z}$, whereas for $\gamma(t) = 2[\frac{t+1}{2}]$, it corresponds to a sequence where $t_i = 2i-1$ and $\gamma_i = 2i$, with $i \in \mathbb{Z}$. In the case where $\gamma_i = t_i$ for all $i \in \mathbb{Z}$, it represents a purely delayed situation. Conversely, when $\gamma_i = t_{i+1}$, it signifies a purely advanced situation. Any other case indicates alternately advanced and delayed situations, where $I_i^+ = [t_i, \gamma_i]$ denotes the advanced intervals, and $I_i^- = [\gamma_i, t_{i+1})$ denotes the delayed intervals.

Applications of DEPCAG are discussed in [6,11,16], and the importance of the advanced and delayed intervals has been emphasized by M. Pinto in [28]. This decomposition will be evident in all of our results, as well as in previous works [6, 9, 11, 13–15, 28, 30]. The integration or solution of a DEPCA, as originally proposed by its pioneers [1,33], relies on the reduction of the DEPCA to discrete equations. In our study of nonlinear DEPCAGs, we will utilize an innovative approach, which involves constructing an equivalent integral equation. However, we also emphasize the profound impact of the discrete component and its associated difference equations, which will play a pivotal role in our analysis. For further details, please refer to [6, 7, 28].

In the year 2010, Chiu and Pinto [6] derived sufficient conditions for the existence and uniqueness of periodic (or harmonic) and subharmonic solutions for a quasilinear differential equation with a general piecewise constant argument, defined as:

$$y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))),$$

where $t \in \mathbb{R}$, $y \in \mathbb{C}^p$, $A(t)$ is a $p \times p$ matrix for $p \in \mathbb{N}$, and f is continuous in the first argument, $\gamma(t) = \gamma_i$, if $t_i \leq t < t_{i+1}$, $i \in \mathbb{Z}$. Within this research paper, the authors compare three inequalities of Gronwall type for DEPCAG and emphasize the introduction of a novel Gronwall's Lemma. This new lemma not only imposes less stringent conditions than the other existing Gronwall's Lemmas but also provides better estimation.

It is widely recognized that mathematical methods play a crucial role in many fields of physics and technology, particularly in the study of linear and nonlinear integro-differential equations. In recent years, the problem of determining the existence of periodic solutions and their algorithmic structure has gained increased importance.

Numerous studies and research, such as those by Butris et al. [2–4], Dorociakova et al. [18], Guerfi et al. [19], Mitropolsky et al. [24], Perestyuk [27], Ronto et al. [31], Shslapk [34], and Vakhobov [36], have focused on the treatment of both autonomous and non-autonomous periodic systems using integral and differential equations, encompassing both linear and nonlinear cases. These studies generally delve into the theory of periodic solutions and modern methodologies for addressing periodic differential equations with high precision.

A. M. Samoilenko [32] developed a numerical-analytic method for studying the algorithmic structure and periodic solutions of ordinary differential equations. This method involves sequences of periodic functions that uniformly converge, and its results have found applications in various fields, including new process industries and technology. In his work, Samoilenko investigated the existence and approximation of periodic solutions for nonlinear systems of integro-differential equations, which take the following form:

$$x'(t) = f\left(t, x(t), \int_t^{t+T} g(s, x(s))ds\right).$$

Here, $x \in D \subset \mathbb{R}^n$, where D is a closed and bounded domain. The vector functions $f(t,x,y)$ and $g(t,x)$ are continuous in t, x, and y, and periodic in t with a period of T.

R. N. Butris [3] investigated the periodic solutions of a nonlinear system of integro-differential equations that depend on the gamma distribution. In this study, a numerical-analytic method was employed, and the system is given by:

$$x'(t) = f\left(t, \vartheta(t,\alpha), x(t), \int_t^{t+T} g(s, \vartheta(s,\alpha), x(s))ds\right), \quad t \in \mathbb{R}.$$

Here, $x \in D \subset \mathbb{R}^n$, where D represents a closed and bounded domain. The vector functions $f(t, \vartheta(t,\alpha), x)$ and $g(t, \vartheta(t,\alpha), x)$ are defined within the domain: $(t, \vartheta(t,\alpha), x) \in \mathbb{R} \times [0,T] \times D \times D$.

In this current paper, we investigate the existence of periodic solutions in a nonlinear integro-differential system with a piecewise alternately advanced and retarded argument of generalized type, described by the following equation:

$$z'(t) = f\left(t, z(t), z(\gamma(t)), \int_t^{t+\omega} g(s, z(s), z(\gamma(s)))ds\right), \quad t \in \mathbb{R}, \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in their respective arguments.

In our analysis, we employ a novel approach to determine periodic solutions. We transform the nonlinear integro-differential system with the DEPCAG (1.2) into an equivalent integral equation. To investigate the periodic solutions, we utilize a numerical-analytic method, as proposed by A. M. Samoilenko in [32]. We demonstrate the existence of periodic solutions for the nonlinear integro-differential system with the DEPCAG (1.2) in Theorems 3.1 and 4.3. Additionally, in Theorem 4.6, we establish the existence of a unique periodic solution using the contraction mapping principle as our fundamental mathematical method.

In our paper, we assume that the solutions of the nonlinear integro-differential systems with DEPCAG (1.2) are continuous functions, while the deviating argument $\gamma(t)$ is discontinuous. Consequently, in general, the right-hand side of the nonlinear DEPCAG system (1.2) has discontinuities at points $t_i \in \mathbb{R}$, $i \in \mathbb{Z}$. Therefore, we consider the solutions of the DEPCAG system as functions that are continuous and continuously differentiable within intervals $[t_i, t_{i+1})$, $i \in \mathbb{Z}$. In other words, when we refer to a solution $z(t)$ of the nonlinear DEPCAG system (1.2), we mean a continuous function on $\mathbb{R}$ such that the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$, except possibly at points $t_i \in \mathbb{R}$, $i \in \mathbb{Z}$, where a one-sided derivative exists. Additionally, the nonlinear integro-differential systems with the DEPCAG (1.2) are satisfied by $z(t)$ on each interval (t_i, t_{i+1}), $i \in \mathbb{Z}$.

The paper is structured as follows. In the next section, some definitions and preliminary results are introduced. In Section 3, we establish criteria for the existence and uniqueness of periodic approximate solutions for the nonlinear DEPCAG system (1.2). In Section 4, we investigate the existence of periodic solutions for the nonlinear DEPCAG system (1.2). Additionally, we provide suitable examples in Section 5 to demonstrate the feasibility of our results.

2. Preliminaries and definition

In this section, our focus is on presenting some preliminary results that will be used to establish the existence of periodic solutions for the nonlinear integro-differential system with a piecewise alternately advanced and retarded argument (1.2).

In the subsequent analysis, we assume that for each $t \in \mathbb{R}$, there exists a unique integer $i = i(t) \in \mathbb{Z}$ such that t belongs to the interval $I_i = [t_i, t_{i+1})$.

The following assumptions will be necessary from this point onward:

Continuous condition:

(C) The vector functions $f(t, x, y, z)$ and $g(t, x, y)$ are continuous functions and defined on the domain:

$$f(t, x, y, z) \in \mathbb{R} \times D \times D \times D, \quad g(t, x, y) \in \mathbb{R} \times D \times D, \tag{2.1}$$

where D is a non-empty compact set in $\mathbb{R}^n$.

Lipschitz conditions:

(L_f) For $t \in \mathbb{R}$, $x_1, y_1, z_1, x_2, y_2, z_2 \in D \subset \mathbb{R}^n$, there exist positive constants $\mathcal{L}_i^f$, $i = 1, 2, 3$, such that

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq \mathcal{L}_1^f |x_1 - x_2| + \mathcal{L}_2^f |y_1 - y_2| + \mathcal{L}_3^f |z_1 - z_2| . \tag{2.2}$$

(L_g) For $t \in \mathbb{R}$ and $x_1, y_1, x_2, y_2 \in D \subset \mathbb{R}^n$, there exist positive constants $\mathcal{L}_i^g$, $i = 1, 2$, such that

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \mathcal{L}_1^g |x_1 - x_2| + \mathcal{L}_2^g |y_1 - y_2| . \tag{2.3}$$

Estimation condition:

(M_f) For all $t \in \mathbb{R}$ and $x, y, z \in D$, where D is a non-empty compact set in $\mathbb{R}^n$, there exists a positive constant M_f such that

$$|f(t, x, y, z)| \leq M_f. \tag{2.4}$$

Periodic conditions:

(P) There exists $\omega > 0$ such that:

1) $f(t, x_1, y_1, z_1)$ and $g(t, x_2, y_2)$ are periodic functions in t with a period ω for all $t \geq \tau$.

2) There exists $p \in \mathbb{Z}^+$, for which the sequences $\{t_i\}_{i\in\mathbb{Z}}$, $\{\gamma_i\}_{i\in\mathbb{Z}}$, satisfy the (ω, p) condition, that is

$$t_{i+p} = t_i + \omega, \quad \gamma_{i+p} = \gamma_i + \omega, \text{ for } i \in \mathbb{Z}. \tag{2.5}$$

Now, we solve the nonlinear DEPCAG system (1.2) on $I_{i(\tau)} = [t_{i(\tau)}, t_{i(\tau)+1})$:

$$z'(t) = f\left(t, z(t), z(\gamma_{i(\tau)}), \int_t^{t+\omega} g(s, z(s), z(\gamma(s)))ds\right), \quad t \in [t_{i(\tau)}, t_{i(\tau)+1}),$$

which has the solution given by:

$$\begin{aligned} z(t) = z(\tau) + \int_\tau^t \Bigg[f\Bigg(& s, z(s), z(\gamma_{i(\tau)}), \int_s^{t_{i(\tau)+1}} g(u, z(u), z(\gamma_{i(\tau)}))du \\ & + \sum\nolimits_{k=i(\tau)+1}^{i(s+\omega)-1} \int_{t_k}^{t_{k+1}} g(u, z(u), z(\gamma_k))du + \int_{t_{i(s+\omega)}}^{s+\omega} g(u, z(u), z(\gamma_{i(s+\omega)}))du \Bigg) \Bigg] ds. \end{aligned} \tag{2.6}$$

For $t \to t_{i(\tau)+1}$ in (2.6), we have

$$\begin{aligned} z(t_{i(\tau)+1}) = z(\tau) + \int_\tau^{t_{i(\tau)+1}} \Bigg[f\Bigg(& s, z(s), z(\gamma_{i(\tau)}), \int_s^{t_{i(\tau)+1}} g(u, z(u), z(\gamma_{i(\tau)}))du \\ & + \sum\nolimits_{k=i(\tau)+1}^{i(s+\omega)-1} \int_{t_k}^{t_{k+1}} g(u, z(u), z(\gamma_k))du + \int_{t_{i(s+\omega)}}^{s+\omega} g(u, z(u), z(\gamma_{i(s+\omega)}))du \Bigg) \Bigg] ds \end{aligned}$$

and in general, by induction, for any $i(t) \geq i(\tau)$:

$$
\begin{aligned}
z(t) = z(\tau) &+ \int_{\tau}^{t_{i(\tau)+1}} \Big[f\Big(s, z(s), z(\gamma_{i(\tau)}), \int_{s}^{t_{i(\tau)+1}} g(u, z(u), z(\gamma_{i(\tau)}))du \\
&\quad + \sum\nolimits_{k=i(\tau)+1}^{i(s+\omega)-1} \int_{t_k}^{t_{k+1}} g(u, z(u), z(\gamma_k))du + \int_{t_{i(s+\omega)}}^{s+\omega} g(u, z(u), z(\gamma_{i(s+\omega)}))du \Big) \Big] ds \\
&+ \sum\nolimits_{j=i(\tau)+1}^{i(t)-1} \int_{t_j}^{t_{j+1}} \Big[f\Big(s, z(s), z(\gamma_j), \int_{s}^{t_{j+1}} g(u, z(u), z(\gamma_j))du \\
&\quad + \sum\nolimits_{k=j+1}^{i(s+\omega)-1} \int_{t_k}^{t_{k+1}} g(u, z(u), z(\gamma_k))du + \int_{t_{i(s+\omega)}}^{s+\omega} g(u, z(u), z(\gamma_{i(s+\omega)}))du \Big) \Big] ds \\
&+ \int_{t_{i(t)}}^{t} \Big[f\Big(s, z(s), z(\gamma_{i(t)}) \int_{s}^{t_{i(t)+1}} g(u, z(u), z(\gamma_{i(t)}))du \\
&\quad + \sum\nolimits_{k=i(t)+1}^{i(s+\omega)-1} \int_{t_k}^{t_{k+1}} g(u, z(u), z(\gamma_k))du + \int_{t_{i(s+\omega)}}^{s+\omega} g(u, z(u), z(\gamma_{i(s+\omega)}))du \Big) \Big] ds.
\end{aligned}
$$

On the other hand, it is evident that

$$
\begin{aligned}
\int_{\tau}^{t} g(u, z(u), z(\gamma(u)))du = &\int_{\tau}^{t_{i(\tau)+1}} g(u, z(u), z(\gamma_{i(\tau)}))du \\
&+ \sum\nolimits_{j=i(\tau)+1}^{i(t)-1} \int_{t_j}^{t_{j+1}} g(u, z(u), z(\gamma_j))du + \int_{t_{i(t)}}^{t} g(u, z(u), z(\gamma_{i(t)}))du.
\end{aligned}
$$

Therefore, any solution to the nonlinear DEPCAG system (1.2) with the initial condition $z(\tau) = z_0$ can be expressed as:

$$
z(t) = z_0 + \int_{\tau}^{t} \Big[f\Big(s, z(s), z(\gamma(s)), \int_{s}^{s+\omega} g(u, z(u), z(\gamma(u)))du \Big) \Big] ds, \quad \tau \in \mathbb{R}. \tag{2.7}
$$

Among these solutions, the one that is ω-periodic is characterized by the property that $z(\tau) = z_0 = z(\tau + \omega)$. Using (2.7), we obtain:

$$
\begin{aligned}
z(t) = z_0 &+ \int_{\tau}^{t} f\Big(s, z(s), z(\gamma(s)), \int_{s}^{s+\omega} g(u, z(u), z(\gamma(u)))du \Big) ds \\
&- \frac{t-\tau}{\omega} \int_{\tau}^{\tau+\omega} f\Big(s, z(s), z(\gamma(s)), \int_{s}^{s+\omega} g(u, z(u), z(\gamma(u)))du \Big) ds, \quad \tau \in \mathbb{R}.
\end{aligned} \tag{2.8}
$$

It follows that

$$
\begin{aligned}
z(t) = z_0 &+ \Big(1 - \frac{t-\tau}{\omega} \Big) \int_{\tau}^{t} f\Big(s, z(s), z(\gamma(s)), \int_{s}^{s+\omega} g(u, z(u), z(\gamma(u)))du \Big) ds \\
&- \frac{t-\tau}{\omega} \int_{t}^{\tau+\omega} f\Big(s, z(s), z(\gamma(s)), \int_{s}^{s+\omega} g(u, z(u), z(\gamma(u)))du \Big) ds.
\end{aligned} \tag{2.9}
$$

In such a case, the nonlinear DEPCAG system (1.2) has an ω-periodic solution $z(t)$ given by the integral equation (2.9). Before we delve into studying the existence of a periodic approximate solution for integral equation (2.9) in the next section, we first present the following lemma, which will be used to establish the existence and uniqueness of a periodic approximate solution for the nonlinear integro-differential system with a piecewise alternately advanced and retarded argument (1.2).

Lemma 2.1. *Let $f(t)$ be a continuous vector function defined on the interval $[\tau, \tau + \omega]$. Then,*

$$
\Big| \int_{\tau}^{t} \Big(f(s) - \frac{1}{\omega} \int_{\tau}^{\tau+\omega} f(u)du \Big) ds \Big| \leq \alpha(t) \max_{t \in [\tau, \tau+\omega]} |f(t)|, \tag{2.10}
$$

where $\alpha(t) = 2(t - \tau)\left(1 - \frac{t-\tau}{\omega}\right)$.

Proof. It is clear that

$$\int_\tau^t \left(f(s) - \frac{1}{\omega}\int_\tau^{\tau+\omega} f(u)du\right)ds = \int_\tau^t f(s)ds - \frac{t-\tau}{\omega}\int_\tau^t f(u)du - \frac{t-\tau}{\omega}\int_t^{\tau+\omega} f(u)du.$$

Therefore,

$$\begin{aligned}&\left|\int_\tau^t \left(f(s) - \frac{1}{\omega}\int_\tau^{\tau+\omega} f(u)du\right)ds\right| \le \left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t |f(s)|ds + \frac{t-\tau}{\omega}\int_t^{\tau+\omega}|f(s)|ds\\ &\le \left(1-\frac{t-\tau}{\omega}\right)(t-\tau)\max_{t\in[\tau,\tau+\omega]}|f(t)| + \frac{t-\tau}{\omega}(\tau+\omega-t)\max_{t\in[\tau,\tau+\omega]}|f(t)|\\ &= \alpha(t)\max_{t\in[\tau,\tau+\omega]}|f(t)|,\end{aligned} \tag{2.11}$$

where $\alpha(t) = 2(t-\tau)\left(1-\frac{t-\tau}{\omega}\right)$. Moreover, it can be noted that $|\alpha(t)| \le \frac{\omega}{2}$, for $t \in [\tau, \tau+\omega]$. □

3. The periodic approximate solution for the DEPCAG system

The approach for investigating periodic solutions proposed by Samoilenko in [32] has been referred to as the numerical-analytic method or the approach of periodic successive approximations. The method is described in a form suitable for our purposes by Theorem 3.1 below, and deals with the investigation of the equation:

$$\begin{aligned}z(t,z_0) = z_0 &+ \int_\tau^t f\left(s, z(s,z_0), z(\gamma(s),z_0), \int_s^{s+\omega} g(u, z(u,z_0), z(\gamma(u),z_0))du\right)ds\\ &- \frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(s, z(s,z_0), z(\gamma(s),z_0), \int_s^{s+\omega} g(u, z(u,z_0), z(\gamma(u),z_0))du\right)ds,\end{aligned}$$

for $\tau \in \mathbb{R}$. In accordance with [31], we will now outline the original, unmodified periodic successive approximations scheme for the ω-periodic problem (1.2). Subsequently, we will modify this scheme as needed. Given the ω-periodic problem (1.2), we define the sequence of functions $z_n(\cdot, z_0)$, $n \ge 1$, according to the following rule:

$$\begin{cases} z_0(t,z_0) = z_0\\ z_n(t,z_0) = z_0 + \displaystyle\int_\tau^t f\left(\begin{matrix} s, z_{n-1}(s,z_0), z_{n-1}(\gamma(s),z_0),\\ \int_s^{s+\omega} g(u, z_{n-1}(u,z_0), z_{n-1}(\gamma(u),z_0))du\end{matrix}\right)ds\\ \qquad\qquad - \displaystyle\frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(\begin{matrix} s, z_{n-1}(s,z_0), z_{n-1}(\gamma(s),z_0),\\ \int_s^{s+\omega} g(u, z_{n-1}(u,z_0), z_{n-1}(\gamma(u),z_0))du\end{matrix}\right)ds,\end{cases} \tag{3.1}$$

for $\tau, t \in \mathbb{R}$.

We define a d-neighborhood of a point $z \in \mathbb{R}^n$ as the set of points satisfying $||z-z_0|| \le d$. For the nonlinear DEPCAG system (1.2) and the region D, we consider a subset $D_{M_f\omega/2}$ of $\mathbb{R}^n$ consisting of points in D and their $\frac{M_f\omega}{2}$-neighborhoods.

The investigation of periodic approximate solutions for the nonlinear DEPCAG system (1.2) is presented in the following theorem.

Theorem 3.1. *Let (C), (M_f), (P), and the functions f and g satisfy the Lipschitz conditions (L_f) and (L_g) with a constant β, such that the inequality:*

$$\beta := \frac{\omega}{2}\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f(\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) < 1 \tag{3.2}$$

holds, and furthermore:

$$(t, z_0) \in \mathbb{R} \times D_{M_f\omega/2}. \tag{3.3}$$

Then,

(i) *the sequence* (3.1) *is uniformly convergent:*

$$z(t,z_0)=\lim_{n\to\infty} z_n(t,z_0). \tag{3.4}$$

Moreover,

$$|z_n(t,z_0)-z_0|\le M_f\alpha(t), \tag{3.5}$$

and

$$|z(t,z_0)-z_n(t,z_0)|\le \frac{\beta^n}{1-\beta}M_f\alpha(t). \tag{3.6}$$

(ii) *the function* $z(t,z_0)$ *is the unique* ω*-periodic solution of the integral equation*

$$\begin{aligned} z(t,z_0)=z_0+\int_\tau^t f\left(\begin{matrix} s,z(s,z_0),z(\gamma(s),z_0),\\ \int_s^{s+\omega} g(u,z(u,z_0),z(\gamma(u),z_0))du\end{matrix}\right)ds\\ -\frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(\begin{matrix} s,z(s,z_0),z(\gamma(s),z_0),\\ \int_s^{s+\omega} g(u,z(u,z_0),z(\gamma(u),z_0))du\end{matrix}\right)ds, \end{aligned} \tag{3.7}$$

on the domain (3.3).

Proof. Consider the sequence of functions $z_1(t,z_0)$, $z_2(t,z_0)$, …, $z_n(t,z_0)$, …, defined by the recurrence relation (3.1). Each function in the sequence is continuous and ω-periodic on the domain (2.1).

Now, by applying Lemma 2.1 and using (3.1) with $n=1$, we obtain:

$$\begin{aligned} &|z_1(t,z_0)-z_0|\\ &\le\left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t\left|f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)\right|ds\\ &\quad+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\left|f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)\right|ds\\ &\le\left(1-\frac{t-\tau}{\omega}\right)M_f\cdot(t-\tau)+\frac{t-\tau}{\omega}M_f\cdot(\tau+\omega-t)\\ &=2(t-\tau)\left(1-\frac{t-\tau}{\omega}\right)M_f=M_f\alpha(t). \end{aligned}$$

Then,

$$|z_1(t,z_0)-z_0|\le M_f\alpha(t)\le\frac{M_f\omega}{2}, \tag{3.8}$$

i.e. $z_1(t,z_0)\in D$, for all $t\in\mathbb{R}$, $z_0\in D_{M_f\omega/2}$.

Also from (3.8), we have:

$$\begin{aligned} &\left|\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du-\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right|\\ &\le\int_s^{s+\omega}|g(u,z_1(u,z_0),z_1(\gamma(u),z_0))-g(u,z_0(u,z_0),z_0(\gamma(u),z_0))|\,du\\ &\le\int_s^{s+\omega}[\mathcal{L}_1^g|z_1(u,z_0)-z_0(u,z_0)|+\mathcal{L}_2^g|z_1(\gamma(u),z_0)-z_0(\gamma(u),z_0)|]du\\ &\le\int_s^{s+\omega}[\mathcal{L}_1^g\alpha(u)M_f+\mathcal{L}_2^g\alpha(\gamma(u))M_f]du\\ &\le\frac{1}{2}\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)M_f\omega^2. \end{aligned}$$

Hence,

$$\begin{aligned} &\left|\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du-\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right|\\ &\le\frac{1}{2}\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)M_f\omega^2. \end{aligned} \tag{3.9}$$

Hence, by mathematical induction, we find that:

$$|z_n(t,z_0)-z_0| \le M_f\alpha(t), \quad n\in\mathbb{N}. \tag{3.10}$$

For all $t\in\mathbb{R}$ and $z_0\in D_{M_f\omega/2}$. Furthermore, from (3.10), we have:

$$\begin{aligned}&\left|\int_s^{s+\omega} g(u,z_n(u,z_0),z_1(\gamma(u),z_0))du-\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right|\\&\le\frac{1}{2}\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)M_f\omega^2,\end{aligned} \tag{3.11}$$

for all $t\in\mathbb{R}$ and $z_0\in D_{M_f\omega/2}$. We assert that the sequence of functions (3.1) is uniformly convergent on the domain (3.3). By applying Lemma 2.1 and setting $n=1$ in (3.1), we obtain:

$$\begin{aligned}&|z_2(t,z_0)-z_1(t,z_0)|\\
&=\Bigg|\Bigg[z_0+\int_\tau^t f\left(s,z_1(s,z_0),z_1(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du\right)ds\\
&\qquad-\frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(s,z_1(s,z_0),z_1(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du\right)ds\Bigg]\\
&\quad-\Bigg[z_0+\int_\tau^t f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)ds\\
&\qquad-\frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)ds\Bigg]\Bigg|\\
&\le\left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t\Bigg|f\left(s,z_1(s,z_0),z_1(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du\right)\\
&\qquad-f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)\Bigg|ds\\
&\quad+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Bigg|f\left(s,z_1(s,z_0),z_1(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du\right)\\
&\qquad-f\left(s,z_0(s,z_0),z_0(\gamma(s),z_0),\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right)\Bigg|ds\\
&\le\left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t\Bigg(\mathcal{L}_1^f|z_1(s,z_0)-z_0|+\mathcal{L}_2^f|z_1(\gamma(s),z_0)-z_0|\\
&\quad+\mathcal{L}_3^f\left|\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du-\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right|\Bigg)ds\\
&\quad+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Bigg(\mathcal{L}_1^f|z_1(s,z_0)-z_0|+\mathcal{L}_2^f|z_1(\gamma(s),z_0)-z_0|\\
&\quad+\mathcal{L}_3^f\left|\int_s^{s+\omega} g(u,z_1(u,z_0),z_1(\gamma(u),z_0))du-\int_s^{s+\omega} g(u,z_0(u,z_0),z_0(\gamma(u),z_0))du\right|\Bigg)ds\end{aligned}$$

$$\leq \left(1-\frac{t-\tau}{\omega}\right)\int_{\tau}^{t}\left(\mathcal{L}_1^f\alpha(s)M_f+\mathcal{L}_2^f\alpha(\gamma(s))M_f+\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g\alpha(u)M_f+\mathcal{L}_2^g\alpha(\gamma(u))M_f]du\right)ds$$

$$+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\left(\mathcal{L}_1^f\alpha(s)M_f+\mathcal{L}_2^f\alpha(\gamma(s))M_f+\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g\alpha(u)M_f+\mathcal{L}_2^g\alpha(\gamma(u))M_f]du\right)ds$$

$$\leq \left(1-\frac{t-\tau}{\omega}\right)\int_{\tau}^{t}\left[\frac{\mathcal{L}_1^f+\mathcal{L}_2^f}{2}\omega M_f+\frac{\mathcal{L}_3^f}{2}(\mathcal{L}_1^g+\mathcal{L}_2^g)\omega^2M_f\right]ds$$

$$+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\left[\frac{\mathcal{L}_1^f+\mathcal{L}_2^f}{2}\omega M_f+\frac{\mathcal{L}_3^f}{2}(\mathcal{L}_1^g+\mathcal{L}_2^g)\omega^2M_f\right]ds$$

$$=\frac{\omega}{2}\left(\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f(\mathcal{L}_1^g+\mathcal{L}_2^g)\omega\right)M_f\alpha(t).$$

Let $\beta:=\frac{\omega}{2}\left(\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f(\mathcal{L}_1^g+\mathcal{L}_2^g)\omega\right)$. Then, the inequality

$$|z_2(t,z_0)-z_1(t,z_0)|\leq \beta M_f\alpha(t) \tag{3.12}$$

is true. We assume the following inequality holds:

$$|z_{\mathrm{n}}(t,z_0)-z_{n-1}(t,z_0)|\leq \beta^{n-1}M_f\alpha(t), \tag{3.13}$$

for all $n\geq 1$. Now, we shall prove the following:

$$|z_{n+1}(t,z_0)-z_n(t,z_0)|$$

$$\leq \left(1-\frac{t-\tau}{\omega}\right)\int_{\tau}^{t}\left[\mathcal{L}_1^f|z_n(s,z_0)-z_{n-1}(s,z_0)|+\mathcal{L}_2^f|z_n(\gamma(s),z_0)-z_{n-1}(\gamma(s),z_0)|\right.$$

$$\left.+\mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g|z_n(u,z_0)-z_{n-1}(s,z_0)|+\mathcal{L}_2^g|z_n(\gamma(u),z_0)-z_{n-1}(\gamma(s),z_0)|du\right]ds$$

$$+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\left[\mathcal{L}_1^f|z_n(s,z_0)-z_{n-1}(s,z_0)|+\mathcal{L}_2^f|z_n(\gamma(s),z_0)-z_{n-1}(\gamma(s),z_0)|\right.$$

$$\left.+\mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g|z_n(u,z_0)-z_{n-1}(s,z_0)|+\mathcal{L}_2^g|z_n(\gamma(u),z_0)-z_{n-1}(\gamma(s),z_0)|du\right]ds$$

$$\leq \left(1-\frac{t-\tau}{\omega}\right)\int_{\tau}^{t}\left[\mathcal{L}_1^f\beta^{n-1}M_f\alpha(s)+\mathcal{L}_2^f\beta^{n-1}M_f\alpha(\gamma(s))\right.$$

$$\left.+\mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g\beta^{n-1}M_f\alpha(u)+\mathcal{L}_2^g\beta^{n-1}M_f\alpha(\gamma(u))du\right]ds$$

$$+\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\left[\mathcal{L}_1^f\beta^{n-1}M_f\alpha(s)+\mathcal{L}_2^f\beta^{n-1}M_f\alpha(\gamma(s))\right.$$

$$\left.+\mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g\beta^{n-1}M_f\alpha(u)+\mathcal{L}_2^g\beta^{n-1}M_f\alpha(\gamma(u))du\right]ds$$

$$\leq \left(1-\frac{t-\tau}{\omega}\right)\left[\frac{\omega}{2}\left(\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)\omega\right)\beta^{n-1}M_f\right]$$

$$+\frac{t-\tau}{\omega}\left[\frac{\omega}{2}\left(\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)\omega\right)\beta^{n-1}M_f\right]$$

$$\leq \frac{\omega}{2}\left(\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)\omega\right)\beta^{n-1}M_f\alpha(t)=\beta^nM_f\alpha(t).$$

And hence

$$|z_{n+1}(t,z_0)-z_n(t,z_0)|\leq \beta^nM_f\alpha(t), \tag{3.14}$$

for all $n\geq 0$. Based on (3.14), it can be concluded that for $k\geq 1$, the following holds:

$$|z_{n+k}(t,z_0)-z_n(t,z_0)|\leq \sum_{i=0}^{k-1}\beta^{n+i}M_f\alpha(t), \tag{3.15}$$

such that

$$
\begin{aligned}
|z_{n+k}(t, z_0) - z_n(t, z_0)| &\leq \sum_{i=0}^{\infty} |z_{n+1+i}(t, z_0) - z_{n+i}(t, z_0)| \\
&\leq \sum_{i=0}^{\infty} \beta^{n+i} M_f \alpha(t) \leq \frac{\beta^n}{1-\beta} M_f \alpha(t),
\end{aligned}
\tag{3.16}
$$

whence, taking the limit as $k \to \infty$, we obtain

$$
|z(t, z_0) - z_n(t, z_0)| \leq \frac{\beta^n}{1-\beta} M_f \alpha(t). \tag{3.17}
$$

By the hypothesis (3.2) and (3.17), we can establish the uniform convergence of the sequence of functions (3.1) on the domain (3.3). Let

$$
\lim_{n \to \infty} z_n(t, z_0) = z(t, z_0). \tag{3.18}
$$

Since the sequence of functions (3.1) is ω-periodic, the limit function $z(t, z_0)$ is also ω-periodic.
In addition, by the Lemma 2.1 and inequality (3.17), inequalities (3.5) and (3.6) hold.
Finally, we must demonstrate that $z(t, z_0)$ is the unique solution to the nonlinear DEPCAG system (1.2). Assume that $y(t, z_0)$ is another solution of the nonlinear DEPCAG system

$$
\begin{aligned}
y(t, z_0) = z_0 + \int_{\tau}^{t} f &\left(\begin{matrix} s, y(s, z_0), y(\gamma(s), z_0), \\ \int_s^{s+\omega} g(u, y(u, z_0), y(\gamma(u), z_0)) du \end{matrix} \right) ds \\
&- \frac{t-\tau}{\omega} \int_{\tau}^{\tau+\omega} f \left(\begin{matrix} s, y(s, z_0), y(\gamma(s), z_0), \\ \int_s^{s+\omega} g(u, y(u, z_0), y(\gamma(u), z_0)) du \end{matrix} \right) ds,
\end{aligned}
\tag{3.19}
$$

Now, we will prove that $z(t, z_0) = y(t, z_0)$ for all $z_0 \in D_{M_f \omega/2}$. To achieve this, we need to establish the following inequality:

$$
|y(t, z_0) - z_n(t, z_0)| \leq \frac{\beta^n}{1-\beta} M_f^* \alpha(t), \tag{3.20}
$$

where $M_f^* = \max\limits_{n \in \mathbb{N}} \Big\{ f\Big(t, z_n(t, z_0), z_n(\gamma(t), z_0), \int_t^{t+\omega} g(s, z_n(s, z_0), z_n(\gamma(s), z_0)) ds\Big),$

$$
f\Big(t, y(t, z_0), y(\gamma(t), z_0), \int_t^{t+\omega} g(s, y(s, z_0), y(\gamma(s), z_0)) ds\Big) \Big\}.
$$

Suppose that (3.20) is true for $n = k$, i.e.,

$$
|y(t, z_0) - z_k(t, z_0)| \leq \frac{\beta^k}{1-\beta} M_f^* \alpha(t). \tag{3.21}
$$

Then,

$$
\begin{aligned}
&|y(t,z_0)-z_{k+1}(t,z_0)| \\
&\le \left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t \Big[\mathcal{L}_1^f|y(s,z_0)-z_k(s,z_0)|+\mathcal{L}_2^f|y(\gamma(s),z_0)-z_k(\gamma(s),z_0)| \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}|g(u,z(u,z_0),z(\gamma(u),z_0))-g(u,y(u,z_0),y(\gamma(u),z_0))|du\Big]ds \\
&\quad +\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Big[\mathcal{L}_1^f|y(s,z_0)-z_k(s,z_0)|+\mathcal{L}_2^f|y(\gamma(s),z_0)-z_k(\gamma(s),z_0)| \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}|g(u,z(u,z_0),z(\gamma(u),z_0))-g(u,y(u,z_0),y(\gamma(u),z_0))|du\Big]ds \\
&\le \left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t \Big[\mathcal{L}_1^f|y(s,z_0)-z_k(s,z_0)|+\mathcal{L}_2^f|y(\gamma(s),z_0)-z_k(\gamma(s),z_0)| \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g|y(u,z_0)-z_k(u,z_0)|+\mathcal{L}_2^g|y(\gamma(u),z_0)-z_k(\gamma(u),z_0)|]du\Big]ds \\
&\quad +\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Big[\mathcal{L}_1^f|y(s,z_0)-z_k(s,z_0)|+\mathcal{L}_2^f|y(\gamma(s),z_0)-z_k(\gamma(s),z_0)| \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g|y(u,z_0)-z_k(u,z_0)|+\mathcal{L}_2^g|y(\gamma(u),z_0)-z_k(\gamma(u),z_0)|du\Big]ds \\
&\le \left(1-\frac{t-\tau}{\omega}\right)\int_\tau^t \Big[\mathcal{L}_1^f\frac{\beta^k}{1-\beta}M_f^*\alpha(s)+\mathcal{L}_2^f\frac{\beta^k}{1-\beta}M_f^*\alpha(\gamma(s)) \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g\frac{\beta^k}{1-\beta}M_f^*\alpha(u)+\mathcal{L}_2^g\frac{\beta^k}{1-\beta}M_f^*\alpha(\gamma(u))]du\Big]ds \\
&\quad +\frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Big[\mathcal{L}_1^f\frac{\beta^k}{1-\beta}M_f^*\alpha(s)+\mathcal{L}_2^f\frac{\beta^k}{1-\beta}M_f^*\alpha(\gamma(s)) \\
&\qquad +\mathcal{L}_3^f\int_s^{s+\omega}[\mathcal{L}_1^g\frac{\beta^k}{1-\beta}M_f^*\alpha(u)+\mathcal{L}_2^g\frac{\beta^k}{1-\beta}M_f^*\alpha(\gamma(u))]du\Big]ds \\
&\le \left(1-\frac{t-\tau}{\omega}\right)\frac{\omega}{2}\left[\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)\omega\right]\frac{\beta^k}{1-\beta}M_f^*\cdot(t-\tau) \\
&\qquad +\frac{t-\tau}{\omega}\frac{\omega}{2}\left[\mathcal{L}_1^f+\mathcal{L}_2^f+\mathcal{L}_3^f\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)\omega\right]\frac{\beta^k}{1-\beta}M_f^*\cdot(\tau+\omega-t) \\
&\le \frac{\beta^{k+1}}{1-\beta}M_f^*\alpha(t).
\end{aligned}
$$

By induction, inequality (3.20) holds for $k=0,1,2,\ldots$. Therefore, using (3.18) and (3.20), we obtain:

$$\lim_{n\to\infty}|y(t,z_0)-z_n(t,z_0)|=0.$$

As a result,

$$\lim_{n\to\infty}z_n(t,z_0)=y(t,z_0).$$

Utilizing the relation (3.18), we conclude that $z(t,z_0)=y(t,z_0)$, which implies that $z(t,z_0)$ is a unique solution of the nonlinear DEPCAG system (1.2) on the domain (3.3). □

4. Existence of Periodic Solutions

The problem of existence of an ω-periodic solution of the nonlinear DEPCAG system (1.2) is closely related to the existence of zeros of a function given by the form:

$$\Delta(z_0) = \frac{1}{\omega}\int_{\tau}^{\tau+\omega} f\left(s, z(s,z_0), z(\gamma(s),z_0), \int_s^{s+\omega} g(u, z(u,z_0), z(\gamma(u),z_0))du\right) ds, \quad (4.1)$$

where $z(t, z_0)$ is the limiting function of the sequence of functions (3.1).

The function (4.1) can only be approximated, for example, by computing the following functions:

$$\Delta_n(z_0) = \frac{1}{\omega}\int_{\tau}^{\tau+\omega} f\left(s, z_n(s,z_0), z_n(\gamma(s),z_0), \int_s^{s+\omega} g(u, z_n(u,z_0), z_n(\gamma(u),z_0))du\right) ds \quad (4.2)$$

and $n = 0, 1, 2, ...$ Now, we prove the following theorem.

Theorem 4.1. *Under the assumptions of Theorem 3.1, the following inequality will hold for all $n \geq 0$ and $z_0 \in D_{M_f\omega/2}$:*

$$|\Delta(z_0) - \Delta_n(z_0)| \leq \frac{\beta^{n+1}}{1-\beta} M_f. \quad (4.3)$$

Proof. From equations (3.6), (4.1), and (4.2), we can derive the following estimate:

$$\begin{aligned}
&|\Delta(z_0) - \Delta_n(z_0)| \\
&\leq \frac{1}{\omega}\int_{\tau}^{\tau+\omega} \Bigg| f\left(s, z(s,z_0), z(\gamma(s),z_0), \int_s^{s+\omega} g(u, z(u,z_0), z(\gamma(u),z_0))du\right) \\
&\qquad - f\left(s, z_n(s,z_0), z_n(\gamma(s),z_0), \int_s^{s+\omega} g(u, z_n(u,z_0), z_n(\gamma(u),z_0))du\right)\Bigg| ds \\
&\leq \frac{1}{\omega}\int_{\tau}^{\tau+\omega} \Bigg[\mathcal{L}_1^f |z(s,z_0) - z_n(s,z_0)| + \mathcal{L}_2^f |z(\gamma(s),z_0) - z_n(\gamma(s),z_0)| \\
&\qquad + \mathcal{L}_3^f \int_s^{s+\omega} [\mathcal{L}_1^g |z(u,z_0) - z_n(u,z_0)| + \mathcal{L}_2^g |z(\gamma(u),z_0) - z_n(\gamma(u),z_0)|]du\Bigg] ds \\
&\leq \frac{1}{\omega}\int_{\tau}^{\tau+\omega} \Bigg[\mathcal{L}_1^f \frac{\beta^n}{1-\beta} M_f \alpha(s) + \mathcal{L}_2^f \frac{\beta^n}{1-\beta} M_f \alpha(\gamma(s)) \\
&\qquad + \mathcal{L}_3^f \int_s^{s+\omega} \left[\mathcal{L}_1^g \frac{\beta^n}{1-\beta} M_f \alpha(u) + \mathcal{L}_2^g \frac{\beta^n}{1-\beta} M_f \alpha(\gamma(u))\right] du\Bigg] ds \\
&\leq \frac{\omega}{2}\left[\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f (\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right] \frac{\beta^n}{1-\beta} M_f = \frac{\beta^{n+1}}{1-\beta} M_f.
\end{aligned}$$

Thus, the inequality (4.2) holds for all $n \geq 0$. □

Remark 4.2. When $\mathbb{R}^n = \mathbb{R}$, which means that z_0 is a scalar, the requirement for the singular point to be isolated can be relaxed in order to strengthen the existence of solutions, as shown in [32]. Therefore, we have the following result.

Theorem 4.3. *Let the nonlinear DEPCAG (1.2) be defined on the interval $[a, b]$. Suppose that for $n \geq 0$, the function $\Delta n(z_0)$ defined according to (4.2) satisfies the following inequalities:*

$$\begin{cases} \min\limits_{a+h \leq z_0 \leq b-h} \Delta_n(z_0) \leq -\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f(\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) \frac{\beta^n}{1-\beta} M_f, \\ \max\limits_{a+h \leq z_0 \leq b-h} \Delta_n(z_0) \geq \left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f(\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) \frac{\beta^n}{1-\beta} M_f. \end{cases} \quad (4.4)$$

Then there exists an ω-periodic solution $z = z(t, z_0)$ of the nonlinear DEPCAG (1.2) for which $z_0 \in [a + h, b - h]$, where $h = \frac{\omega}{2} M_f$.

Proof. Let z_1 and z_2 be any two points in the interval $[a, b]$ such that:

$$\begin{cases} \Delta_n(z_1) = \min\limits_{a+h \le z_0 \le b-h} \Delta_n(z_0), \\ \Delta_n(z_2) = \max\limits_{a+h \le z_0 \le b-h} \Delta_n(z_0), \end{cases} \tag{4.5}$$

where $h = \frac{\omega}{2} M_f$.
By using the inequalities (4.3) and (4.4), we obtain:

$$\begin{cases} \Delta(z_1) = \Delta_n(z_1) + (\Delta(z_1) - \Delta_n(z_1)) < 0, \\ \Delta(z_2) = \Delta_n(z_2) + (\Delta(z_2) - \Delta_n(z_2)) > 0. \end{cases} \tag{4.6}$$

It follows from the inequalities (4.6) and the continuity of the function $\Delta(z_0)$ that there exists an isolated singular point z_0 in the interval $[z_1, z_2]$ for which $\Delta(z_0) \equiv 0$. This implies that the nonlinear DEPCAG (1.2) has an ω-periodic solution $z = z(t, z_0)$ with z_0 belonging to the interval $[a + h, b - h]$. □

Theorem 4.4. *If the function $\Delta(z_0)$ is defined as $\Delta : D_{M_f \omega/2} \to \mathbb{R}^n$,*

$$\Delta(z_0) = \frac{1}{\omega} \int_\tau^{\tau+\omega} f\left(s, z(s, z_0), z(\gamma(s), z_0), \int_s^{s+\omega} g(u, z(u, z_0), z(\gamma(u), z_0))du\right) ds, \tag{4.7}$$

where $z(t, z_0)$ represents the limiting function of the sequence of functions (3.1)*. The following inequalities then hold:*

$$|\Delta(z_0)| \le M_f, \tag{4.8}$$

and

$$|\Delta(z_0^1) - \Delta(z_0^2)| \le \frac{2}{\omega} \frac{\beta}{1 - \beta} |z_0^1 - z_0^2|, \tag{4.9}$$

for z_0, z_0^1, $z_0^2 \in D_{M_f \omega/2}$.

Proof. From the properties of the function $z(t, z_0)$ established in Theorem 3.1, it follows that the function $\Delta(z_0)$ is continuous and bounded by M_f on the domain $D_{M_f \omega/2}$. Using (4.7), we can derive:

$$\begin{aligned} &|\Delta(z_0^1) - \Delta(z_0^2)| \\ &= \left| \frac{1}{\omega} \int_\tau^{\tau+\omega} f\left(s, z(s, z_0^1), z(\gamma(s), z_0^1), \int_s^{s+\omega} g(u, z(u, z_0^1), z(\gamma(u), z_0^1))du\right) ds \right. \\ &\quad \left. - \frac{1}{\omega} \int_\tau^{\tau+\omega} f\left(s, z(s, z_0^2), z(\gamma(s), z_0^2), \int_s^{s+\omega} g(u, z(u, z_0^2), z(\gamma(u), z_0^2))du\right) ds \right| \\ &\le \frac{1}{\omega} \int_\tau^{\tau+\omega} \left| f\left(s, z(s, z_0^1), z(\gamma(s), z_0^1), \int_s^{s+\omega} g(u, z(u, z_0^1), z(\gamma(u), z_0^1))du\right) \right. \\ &\quad \left. - f\left(s, z(s, z_0^2), z(\gamma(s), z_0^2), \int_s^{s+\omega} g(u, z(u, z_0^2), z(\gamma(u), z_0^2))du\right) \right| ds \\ &\le \frac{1}{\omega} \int_\tau^{\tau+\omega} \left[\mathcal{L}_1^f |z(s, z_0^1) - z(s, z_0^2)| + \mathcal{L}_2^f |z(\gamma(s), z_0^1) - z(\gamma(s), z_0^2)| \right. \\ &\quad \left. + \mathcal{L}_3^f \int_s^{s+\omega} \left[\mathcal{L}_1^g \left| z_1(u, z_0^1) - z_0(u, z_0^2) \right| + \mathcal{L}_2^g \left| z_1(\gamma(u), z_0^1) - z_0(\gamma(u), z_0^2) \right| \right] du \right] ds \\ &\le \left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f (\mathcal{L}_1^g + \mathcal{L}_2^g) \omega \right) ||z(\cdot, z_0^1) - z(\cdot, z_0^2)|| = \frac{2}{\omega} \beta ||z(\cdot, z_0^1) - z(\cdot, z_0^2)||, \end{aligned} \tag{4.10}$$

where $z(t, z_0^1)$ and $z(t, z_0^2)$ are the solutions of the integral equations:

$$\begin{aligned} z(t, z_0^k) = z_0^k + \int_\tau^t f\left(s, z(s, z_0), z(\gamma(s), z_0^k), \int_s^{s+\omega} g(u, z(u, z_0^k), z(\gamma(u), z_0^k))du\right) ds \\ - \frac{t-\tau}{\omega} \int_\tau^{\tau+\omega} f\left(s, z(s, z_0^k), z(\gamma(s), z_0^k), \int_s^{s+\omega} g(u, z(u, z_0^k), z(\gamma(u), z_0^k))du\right) ds, \end{aligned} \tag{4.11}$$

with $z(\tau, z_0^k) = z_0^k$, $k = 1, 2$.
The equation (4.11) yields:

$$\begin{aligned} &|z(t, z_0^1) - z(t, z_0^2)| \\ &\leq |z_0^1 - z_0^2| + \left(1 - \frac{t-\tau}{\omega}\right) \int_\tau^t \Bigg(\mathcal{L}_1^f |z(s, z_0^1) - z(s, z_0^2)| + \mathcal{L}_2^f |z(\gamma(s), z_0^1) - z(\gamma(s), z_0^2)| \\ &\qquad + \mathcal{L}_3^f \int_s^{s+\omega} \mathcal{L}_1^g |z(u, z_0^1) - z(u, z_0^2)| + \mathcal{L}_2^g |z(\gamma(u), z_0^1) - z(\gamma(u), z_0^2)| du\Bigg) ds \\ &\quad + \frac{t-\tau}{\omega} \int_t^{\tau+\omega} \Bigg(\mathcal{L}_1^f |z(s, z_0^1) - z(s, z_0^2)| + \mathcal{L}_2^f |z(\gamma(s), z_0^1) - z(\gamma(s), z_0^2)| \\ &\qquad + \mathcal{L}_3^f \int_s^{s+\omega} \mathcal{L}_1^g |z(u, z_0^1) - z(u, z_0^2)| + \mathcal{L}_2^g |z(\gamma(u), z_0^1) - z(\gamma(u), z_0^2)| du\Bigg) ds. \end{aligned}$$

Then,

$$\begin{aligned} &||z(\cdot, z_0^1) - z(\cdot, z_0^2)|| \\ &\leq |z_0^1 - z_0^2| + \left(1 - \frac{t-\tau}{\omega}\right)(t-\tau)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f (\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) ||z(\cdot, z_0^1) - z(\cdot, z_0^2)|| \\ &\qquad + \frac{t-\tau}{\omega}(\tau + \omega - t)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f (\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) ||z(\cdot, z_0^1) - z(\cdot, z_0^2)|| \\ &\leq |z_0^1 - z_0^2| + \alpha(t)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f (\mathcal{L}_1^g + \mathcal{L}_2^g)\omega\right) ||z(\cdot, z_0^1) - z(\cdot, z_0^2)|| \\ &\leq |z_0^1 - z_0^2| + \beta ||z(\cdot, z_0^1) - z(\cdot, z_0^2)||. \end{aligned}$$

Thus,

$$|z(t, z_0^1) - z(t, z_0^2)| \leq \frac{1}{1-\beta} |z_0^1 - z_0^2|. \tag{4.12}$$

Substituting the inequality (4.12) into (4.10), we obtain the desired result (4.9). □

Remark 4.5. Theorem 4.4 confirms the stability of the solutions of the nonlinear DEPCAGs system (1.2), specifically, it shows that a small change in the point z_0 results in a correspondingly small change in the function $\Delta(z_0)$. This property is also noted in [24].

The following theorem introduces the Banach method for investigating the existence of ω-periodic solutions.

Theorem 4.6. *Under the conditions (C), (L_f), (L_g), and* (3.2)*, the nonlinear DEPCAG system (1.2) has a unique ω-periodic solution.*

Proof. Define the operator $\mathcal{T} : D \to D$ by

$$\begin{aligned} \mathcal{T}(z(t), z_0) = z_0 + \int_\tau^t f\left(s, z(s), z(\gamma(s)), \int_s^{s+\omega} g(u, z(u), z(\gamma(u)))du\right) ds \\ - \frac{t-\tau}{\omega} \int_\tau^{\tau+\omega} f\left(s, z(s), z(\gamma(s)), \int_s^{s+\omega} g(u, z(u), z(\gamma(u)))du\right) ds, \end{aligned} \tag{4.13}$$

for $\tau \in \mathbb{R}$. The theorem follows if we prove that $\mathcal{T}$ has a fixed point. Let ψ, φ be two functions in D. By using the conditions (C), (L_f), and (L_g), we can deduce the following:

$$\begin{aligned}
&|\mathcal{T}(\psi(t,z_0)) - \mathcal{T}(\varphi(t,z_0))| \\
&= \Bigg| \Bigg[z_0 + \int_\tau^t f\left(s, \psi(s,z_0), \psi(\gamma(s),z_0), \int_s^{s+\omega} g(u,\psi(u,z_0),\psi(\gamma(u),z_0))du\right)ds \\
&\quad - \frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(s, \psi(s,z_0), \psi(\gamma(s),z_0), \int_s^{s+\omega} g(u,\psi(u,z_0),\psi(\gamma(u),z_0))du\right)ds\Bigg] \\
&\quad - \Bigg[z_0 + \int_\tau^t f\left(s, \varphi(s,z_0), \varphi(\gamma(s),z_0), \int_s^{s+\omega} g(u,\varphi(u,z_0),\varphi(\gamma(u),z_0))du\right)ds \\
&\quad - \frac{t-\tau}{\omega}\int_\tau^{\tau+\omega} f\left(s, \varphi(s,z_0), \varphi(\gamma(s),z_0), \int_s^{s+\omega} g(u,\varphi(u,z_0),\varphi(\gamma(u),z_0))du\right)ds\Bigg]\Bigg| \\
&\le \left(1 - \frac{t-\tau}{\omega}\right)\int_\tau^t \Bigg(\mathcal{L}_1^f|\psi(s,z_0)-\varphi(s,z_0)| + \mathcal{L}_2^f|\psi(\gamma(s),z_0)-\varphi(s,z_0)| \\
&\qquad + \mathcal{L}_3^f\left|\int_s^{s+\omega} g(u,\psi(u,z_0),\psi(\gamma(u),z_0))du - \int_s^{s+\omega} g(u,\varphi(u,z_0),\varphi(\gamma(u),z_0))du\right|\Bigg)ds \\
&\quad + \frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Bigg(\mathcal{L}_1^f|\psi(s,z_0)-\varphi(s,z_0)| + \mathcal{L}_2^f|\psi(\gamma(s),z_0)-\varphi(s,z_0)| \\
&\qquad + \mathcal{L}_3^f\left|\int_s^{s+\omega} g(u,\psi(u,z_0),\psi(\gamma(u),z_0))du - \int_s^{s+\omega} g(u,\varphi(u,z_0),\varphi(\gamma(u),z_0))du\right|\Bigg)ds \\
&\le \left(1 - \frac{t-\tau}{\omega}\right)\int_\tau^t \Bigg(\mathcal{L}_1^f|\psi(s,\psi_0)-\varphi(s,z_0)| + \mathcal{L}_2^f|\psi(\gamma(s),\psi_0)-\varphi(\gamma(s),z_0)| \\
&\qquad + \mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g|\psi(u,\psi_0)-\varphi(u,z_0)| + \mathcal{L}_2^g|\psi(\gamma(u),z_0)-\varphi(\gamma(u),z_0)|du\Bigg)ds \\
&\quad + \frac{t-\tau}{\omega}\int_t^{\tau+\omega}\Bigg(\mathcal{L}_1^f|\psi(s,z_0)-\varphi(s,z_0)| + \mathcal{L}_2^f|\psi(\gamma(s),z_0)-\varphi(\gamma(s),z_0)| \\
&\qquad + \mathcal{L}_3^f\int_s^{s+\omega}\mathcal{L}_1^g|\psi(u,z_0)-\varphi(u,z_0)| + \mathcal{L}_2^g|\psi(\gamma(u),z_0)-\varphi(\gamma(u),z_0)|du\Bigg)ds.
\end{aligned}$$

Then,

$$\begin{aligned}
&|\mathcal{T}(\psi(t,z_0)) - \mathcal{T}(\varphi(t,z_0))| \\
&\le \left(1 - \frac{t-\tau}{\omega}\right)(t-\tau)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)\omega\right)||\psi(\cdot,z_0)-\varphi(\cdot,z_0)|| \\
&\qquad + \frac{t-\tau}{\omega}(\tau+\omega-t)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)\omega\right)||\psi(\cdot,z_0)-\varphi(\cdot,z_0)|| \\
&\le \alpha(t)\left(\mathcal{L}_1^f + \mathcal{L}_2^f + \mathcal{L}_3^f\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)\omega\right)||\psi(\cdot,z_0)-\varphi(\cdot,z_0)|| \\
&\le \beta\,||\psi(\cdot,z_0)-\varphi(\cdot,z_0)||.
\end{aligned}$$

Thus,

$$|\mathcal{T}(\psi(t,z_0)) - \mathcal{T}(\varphi(t,z_0))| \le \beta\,||\psi(\cdot,z_0)-\varphi(\cdot,z_0)||. \tag{4.14}$$

Based on (3.2) and (4.14), the mapping $\mathcal{T}$ is a contraction. Therefore, the mapping $\mathcal{T}$ has a unique fixed point $\varphi^* \in D$, such that $\mathcal{T}\varphi^* = \varphi^*$. The proof of Theorem 4.6 is now complete. □

5. Examples

We will introduce appropriate examples in this section. These examples will show the feasibility of our theory.

Example 5.1. Consider the following integro-differential equation with piecewise alternately advanced and retarded argument of generalized type:

$$z'(t) = az(t) + b(\sin 4t)z(\gamma(t)) + (1 + \cos^2 4t) + \lambda \int_t^{t+\frac{\pi}{2}} z(\gamma(s))ds, \tag{5.1}$$

where, $a, b, \lambda \in \mathbb{R}$ and the sequences $\{t_i\}_{i\in\mathbb{Z}}$ and $\{\gamma_i\}_{i\in\mathbb{Z}}$, satisfy the $\left(\frac{\pi}{2}, 1\right)$ condition.

The conditions of Theorem 3.1 are fulfilled. Indeed,

i) $f(t, x, y, z)$ satisfies (L_f) with $\mathcal{L}_1^f = |a|$, $\mathcal{L}_2^f = |b|$, and $\mathcal{L}_3^f = 1$.
ii) $g(t, x, y) = \lambda y$ satisfies (L_g) with $\mathcal{L}_2^g = |\lambda|$.
iii) For every $R > 0$, $t \in \mathbb{R}$, $|x|$, $|y|$, $|z| \leq R$, the functions $f(t, x, y, z)$ and $g(t, x, y)$ are continuous functions.
iv) For all $t \in \mathbb{R}$ and $|x|$, $|y|$, $|z| \leq R$, there exists a positive constant M_f such that

$$|f(t, x, y, z)| \leq M_f := |a|R + |b|R + \frac{|\lambda|\pi R}{2} + 2.$$

Furthermore, there exists β such that $\beta := \frac{\pi}{4}\left(|a| + |b| + \frac{|\lambda|\pi}{2}\right) < 1$.

Then, by Theorem 3.1, the DEPCAG system (5.1) has a $\frac{\pi}{2}$-periodic solution.
Here, in particular, we choose the parameters $a = 0.13$, $b = -0.25$, and $\lambda = 0.5$, such that $\beta \approx 0.9153 < 1$. In this case, the integro-differential equation

$$z'(t) = 0.13z(t) - 0.25(\sin 4t)z(\gamma(t)) + (1 + \cos^2 4t) + 0.5 \int_t^{t+\frac{\pi}{2}} z(\gamma(s))ds,$$

has a $\frac{\pi}{2}$-periodic solution.

Example 5.2. Let $\Lambda : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\mu : \mathbb{R} \to \mathbb{R}^n$ be two functions satisfying

$$\sup_{t\in\mathbb{R}} \int_t^{t+\omega} |\Lambda(s)|ds = \hat{\Lambda} < \infty, \quad \sup_{t\in\mathbb{R}} \int_t^{t+\omega} |\mu(t-s)|ds = \hat{\mu} < \infty.$$

Now, consider the integro-differential system with piecewise alternately advanced and retarded argument of generalized type:

$$z'(t) = h(z(t), z(\gamma(t))) + \int_t^{t+\omega} \left[\Lambda(s)\kappa(z(\gamma(s))) + \mu(t-s)\right] ds, \tag{5.2}$$

where the sequences $\{t_i\}_{i\in\mathbb{Z}}$ and $\{\gamma_i\}_{i\in\mathbb{Z}}$, satisfy the (ω, p) condition, h, Λ, κ and μ are ω-periodic continuous functions and

i) $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and there exist positive constants $\mathcal{L}_1^h$, $\mathcal{L}_2^h$ such that

$$|h(x_1, y_1) - h(x_2, y_2)| \leq \mathcal{L}_1^h |x_1 - y_1| + \mathcal{L}_2^h |x_2 - y_2|.$$

ii) κ is a continuous function and there exists positive constant $\mathcal{L}_2^\kappa$ such that

$$|\kappa(x) - \kappa(y)| \leq \mathcal{L}_2^\kappa |x - y|.$$

The hypotheses of Theorem 4.6 are fulfilled. Therefore, if there exists a β such that

$$\beta := \frac{\omega}{2}\left(\mathcal{L}_1^h + \mathcal{L}_2^h + \hat{\Lambda}\mathcal{L}_2^\kappa\right) < 1,$$

Theorem 4.6 implies the existence of a unique ω-periodic solution of the DEPCAG system (5.2).

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