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Cesàro-Type Operator on the Dirichlet Space of the Upper Half Plane

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Abstract
We construct a Cesàro-type operator acting on Dirichlet space of the upper half plane using the approach of strongly continuous semigroups of composition operators on Banach spaces. We then determine the spectral and norm properties of the obtained Cesàro-type operator.

Keywords: Cesàro-type operator, Generator, One-parameter semigroup, Resolvent, Spectrum
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1. Introduction

Let $\mathbb{C}$ be the complex plane. We define the (open) unit disk of the complex plane as the set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, while the set $\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half plane of the complex plane, where $\Im(\omega)$ denotes the imaginary part of the complex number ω. Also $dA(\omega)$ shall denotes the (normalized) area Lebesgue measure on $\mathbb{U}$. The Cayley transform ψ is the function $\psi(z) = \frac{i(1+z)}{1-z}$ and maps the unit disc $\mathbb{D}$ conformally onto the upper half plane $\mathbb{U}$ with inverse given by $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. For more details see [1].

Let $\mathscr{H}(\Omega)$ denotes the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω for an open subset Ω of $\mathbb{C}$. For $1 \le p < \infty$, the Bergman spaces of the upper half plane, $L^p_a(\mathbb{U})$, are defined by

$$L^p_a(\mathbb{U}) := \left\{ f \in \mathscr{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U})} := \left(\int_{\mathbb{U}} |f(\omega)|^p dA(\omega) \right)^{\frac{1}{p}} < \infty \right\},$$

while the Hardy space of the upper half plane $H^p(\mathbb{U})$ are defined by

$$H^p(\mathbb{U}) = \left\{ f \in \mathscr{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

The Dirichlet space of the upper half plane, $\mathscr{D}(\mathbb{U})$, is defined by

$$\mathscr{D}(\mathbb{U}) := \left\{ f \in \mathscr{H}(\mathbb{U}) : \|f\|^2_{\mathscr{D}_1(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty \right\}, \tag{1.1}$$

and the norm is given by $\|f\|^2_{\mathscr{D}(\mathbb{U})} = |f(i)|^2 + \|f\|^2_{\mathscr{D}_1(\mathbb{U})}$ where $\|.\|_{\mathscr{D}_1(\mathbb{U})}$ is a seminorm on $\mathscr{D}(\mathbb{U})$. We refer to [1, 2, 3] for comprehensive details on spaces of analytic functions. The theory of analytic spaces of the unit disc $\mathbb{D}$ is well established in literature as opposed to their counterparts of the upper half plane $\mathbb{U}$. For instance, the reproducing kernel for $\mathscr{D}(\mathbb{U})$ has recently been computed in [4] and is given by

$$K_w(z) = 1 + \log\left(\frac{i(z+i)(\bar{\omega}-i)}{2(z-\bar{\omega})} \right), \tag{1.2}$$

where $z, \omega \in \mathbb{U}$. Consequently, every $f \in \mathscr{D}(\mathbb{U})$ satisfies the growth condition

$$|f(\omega)| \leq \|f\|_{\mathscr{D}(\mathbb{U})} \sqrt{1 + \log\left(\frac{|\omega+i|^2}{4\Im(\omega)} \right)}. \tag{1.3}$$

Let X be an arbitrary Banach space over $\mathbb{C}$ and T be closed operator on X. The resolvent set of T, $\rho(T)$ is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$. The spectrum of T, $\sigma(T) = \mathbb{C} \backslash \rho(T)$. Moreover, $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ defines the spectral radius of T and the point spectrum $\sigma_p(T)$ of T is given by $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } 0 \neq x \in \operatorname{dom}(T)\}$. A semigroup, $(T_t)_{t\geq 0}$, of bounded linear operators on X is said to be strongly continuous if $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for all $x \in X$. The infinitesimal generator Γ of $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x = \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\bigg|_{t=0} \tag{1.4}$$

for each $x \in \operatorname{dom}(\Gamma)$, where the domain of Γ is given by

$$\operatorname{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\}.$$

For details, see [5, 6, 7].
The composition operator C_φ induced by φ and is acting on $\mathscr{H}(\Omega)$ is defined by

$$C_\varphi f = f \circ \varphi, \text{ for all } f \in \mathscr{H}(\Omega).$$

On the other hand, the composition semigroup C_{φ_t} induced by the semigroup $(\varphi_t)_{t\geq 0}$ on $\mathscr{H}(\Omega)$ is defined by

$$C_{\varphi_t}(f) = f \circ \varphi_t \text{ for all } f \in \mathscr{H}(\Omega).$$

We refer to [7, 8, 9] for a comprehensive account of the theory of composition semigroups on analytic spaces of $\mathbb{D}$.
For the case of the upper half plane, the study of composition semigroups yielding Cesáro-type operators was initiated by [10] on the Hardy space, and later extended by [11] on the Hardy and weighted Bergman spaces. For the Dirichlet space of the upper half plane, even though the composition semigroups have been partially considered by [12], the Cesáro-type operators have not been studied. In this work we construct such an operator on a subspace of the Dirichlet space of $\mathbb{U}$ and determine its properties.

2. Composition Semigroup on the Dirichlet space of $\mathbb{U}$

We note that the automorphisms of the upper half plane were classified into three groups [11], that is the scaling, the translation and the rotation groups depending on the location of their fixed points. In this paper, we consider the group of composition operators on $\mathscr{D}(\mathbb{U})$ associated with the scaling group of the form $\varphi_t(\omega) = e^{-t}\omega$ for $\omega \in \mathbb{U}$ where φ_t is a self analytic map. We determine if the group of composition operator is an isometry on $\mathscr{D}(\mathbb{U})$. We then investigate both the semigroup and spectral properties of the composition semigroup. For $\varphi_t(\omega) = e^{-t}\omega$, $\omega \in \mathbb{U}$ the composition semigroup induced by the scaling group and acting on $\mathscr{D}(\mathbb{U})$ is defined as

$$\begin{aligned} C_{\varphi_t} f(\omega) &= f \circ \varphi_t(\omega) \\ &= f(e^{-t}\omega), \end{aligned} \tag{2.1}$$

for all $f \in \mathscr{D}(\mathbb{U})$. It can be easily shown that the functions $(C_{\varphi_t})_{t\in\mathbb{R}}$ form a group on $\mathscr{D}(\mathbb{U})$ under composition.

Theorem 2.1. *The operator $C_{\varphi t}$ fails to be an isometry on $\mathscr{D}(\mathbb{U})$.*

Proof. By norm definition,

$$\|C_{\varphi_t} f\|^2_{\mathscr{D}(\mathbb{U})} = \|f \circ \varphi_t\|^2_{\mathscr{D}(\mathbb{U})} = |f \circ \varphi_t(i)|^2 + \int_{\mathbb{U}} |(f \circ \varphi_t)'(\omega)|^2 dA(\omega). \tag{2.2}$$

But $(f \circ \varphi_t)(\omega) = f(e^{-t}\omega)$. Thus $(f \circ \varphi_t)'(\omega) = e^{-t} f'(e^{-t}\omega)$ implying that $|(f \circ \varphi_t)'(\omega)|^2 = e^{-2t}|f'(e^{-t}\omega)|^2$ and $|f \circ \varphi_t(i)|^2 = |f(e^{-t}i)|^2$.
By change of variables, we let $z = e^{-t}\omega$, then $\omega = e^t z$ and $dA(z) = e^{-2t} dA(\omega)$ implying that $dA(\omega) = e^{2t} dA(z)$.
Substituting in (2.2), we get

$$\begin{aligned} \|C_{\varphi_t} f\|^2_{\mathscr{D}(\mathbb{U})} &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} |(f \circ \varphi_t)'(\omega)|^2 dA(\omega). \\ &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} e^{-2t}|f'(e^{-t}\omega)|^2 dA(\omega). \\ &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} |f'(z)|^2 dA(z). \end{aligned}$$

But $|f(e^{-t}i)|^2 + \int_{\mathbb{U}} |f'(z)|^2 dA(z) \neq \|f\|^2_{\mathscr{D}(\mathbb{U})}$. Thus C_{φ_t} fails to be an isometry on $\mathscr{D}(\mathbb{U})$. □

Remark 2.2. Because of Theorem 2.1, the spectral analysis of the group $(C_{\varphi_t})_{t\in\mathbb{R}}$ gets complicated since the spectrum of $(C_{\varphi_t})_{t\in\mathbb{R}}$ cannot be exactly identified rendering the spectral mapping theorems for semigroups inapplicable. We therefore consider $\mathscr{D}_\circ(\mathbb{U})$, the subspace of $\mathscr{D}(\mathbb{U})$ consisting of functions vanishing at $i, f(i) = 0$, defined as $\mathscr{D}_\circ(\mathbb{U}) = \{f \in \mathscr{D}(\mathbb{U}) : f(i) = 0\}$ with the norm given as $\|f\|^2_{\mathscr{D}_\circ(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega)$.
However, $C_{\varphi_t} f(i) = f(e^{-t}i) \neq 0$, and so C_{φ_t} does not map $\mathscr{D}_\circ(\mathbb{U})$ into $\mathscr{D}_\circ(\mathbb{U})$ as expected for our semigroups. Therefore, we apply a correction factor and redefine C_{φ_t} as

$$\hat{C}_{\varphi_t} f(z) = f(e^{-t}z) - f(e^{-t}i). \tag{2.3}$$

Now $\hat{C}_{\varphi_t} f(i) = f(e^{-t}i) - f(e^{-t}i) = 0$ as desired so that indeed $\hat{C}_{\varphi_t} : \mathscr{D}_\circ(\mathbb{U}) \to \mathscr{D}_\circ(\mathbb{U})$.

Clearly, the functions $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ form a group on $\mathscr{D}_\circ(\mathbb{U})$.

Proposition 2.3. *The operator $\hat{C}_{\varphi_t}$ is an isometry on $\mathscr{D}_\circ(\mathbb{U})$.*

Proof. By norm definition,

$$\begin{aligned} \|\hat{C}_{\varphi_t} f\|^2_{\mathscr{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |(\hat{C}_{\varphi_t} f)'(\omega)|^2 dA(\omega) \\ &= \int_{\mathbb{U}} \left| \left(f(e^{-t}\omega) - f(e^{-t}i) \right)' \right|^2 dA(\omega) \\ &= \int_{\mathbb{U}} \left| e^{-t} f'(e^{-t}\omega) \right|^2 dA(\omega). \end{aligned} \tag{2.4}$$

By change of variables, we let $z = e^{-t}\omega$, then, $\omega = e^t z$ and applying the Jacobian, $dA(z) = e^{-2t} dA(\omega)$, implying that $dA(\omega) = e^{2t} dA(z)$.
Substituting in (2.4),

$$\begin{aligned} \|\hat{C}_{\varphi_t} f\|^2_{\mathscr{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} e^{-2t}|f'(e^{-t}\omega)|^2 dA(\omega) \\ &= \int_{\mathbb{U}} e^{-2t}|f'(z)|^2 e^{2t} dA(z) \\ &= \int_{\mathbb{U}} |f'(z)|^2 dA(z) \\ &= \|f\|^2_{\mathscr{D}_\circ(\mathbb{U})}. \end{aligned}$$

This completes our proof. □

Next, we prove that the operator $\hat{C}_{\varphi_t}$ is strongly continuous on the Dirichlet space of the upper half plane $\mathscr{D}_\circ(\mathbb{U})$.

Proposition 2.4. *The operator $\hat{C}_{\varphi_t}$ is strongly continuous on $\mathscr{D}_\circ(\mathbb{U})$.*

Proof. It is known that $\|\hat{C}_{\varphi_t} f\|^2_{\mathscr{D}_\circ(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega)$. To prove strong continuity of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, it suffices to show that $\lim_{t\to 0^+} \|\hat{C}_{\varphi_t} f - f\|_{\mathscr{D}_\circ(\mathbb{U})} = 0$ for all $f \in \mathscr{D}_\circ(\mathbb{U})$. That is to say that,
$\int_{\mathbb{U}} |(\hat{C}_{\varphi_t} f - f)'(\omega)|^2\, dA(\omega) \to 0$ as $t \to 0^+$ which is equivalent to showing that $\lim_{t\to 0^+} \int_{\mathbb{U}} |(\hat{C}_{\varphi_t} f - f)'(\omega)|^2 dA(\omega) = 0$.
Let $f \in \mathscr{D}_\circ(\mathbb{U})$ and suppose that $t_n \to 0$ in $\mathbb{R}$. Let $f_n = \hat{C}_{\varphi_{t_n}} f$, then $f_n(z) \to f(z)$ on compact subsets of $\mathbb{U}$ and $f_n' \to f'$ for each n.
Let $g_n(z) := 2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2$, then $g_n \geq 0$ and $g_n(z) \to 2^2 |f'(z)|^2$ on $\mathscr{D}_\circ(\mathbb{U})$ as $n \to \infty$.
By Fatou's lemma, we have

$$\begin{aligned}
\int_{\mathbb{U}} 2^2 \mid f'(\omega) \mid^2 dA(\omega) &= \int_{\mathbb{U}} \liminf g_n dA(\omega) \\
&\leq \liminf \int_{\mathbb{U}} g_n dA(\omega) \\
&= \liminf \int_{\mathbb{U}} 2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2) dA(\omega) \\
&= 2\int_{\mathbb{U}} |f'|^2 dA(\omega) + 2\int_{\mathbb{U}} |f'|^2 dA(\omega) - \limsup_n \int_{\mathbb{U}} |f' - f_n'| dA(\omega) \\
&= 2^2 \int_{\mathbb{U}} |f'|^2 dA(\omega) - \limsup_n \int_{\mathbb{U}} |f' - f_n'| dA(\omega)
\end{aligned}$$

Thus $0 \leq -\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 dA \leq 0$, implying that $\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 dA(\omega) = 0$. Hence $\lim_n \int_{\mathbb{U}} |f' - f_n'|^2 dA(\omega) = 0$, that is $\lim_n \int_{\mathbb{U}} \|\hat{C}_{\varphi tn} f - f\|^2 dA(\omega) = 0$.
Therefore, $\|\hat{C}_{\varphi t} f - f\|_{\mathscr{D}_\circ(\mathbb{U})} \to 0$ as $t \to 0$, implying that $\hat{C}\varphi_t$ for $t \in \mathbb{R}$ is strongly continuous as desired. □

We now obtain the infinitesimal generator Γ of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ and investigate some of its properties.

Proposition 2.5. *The infinitesimal generator Γ of $(\hat{C}_{\varphi_t})_{t\geq 0}$ is given by*

$$\Gamma f(\omega) = -\omega f'(\omega) + i f'(i),$$

with its domain dom$(\Gamma) = \{f \in \mathscr{D}_\circ(\mathbb{U}) : \omega f'(\omega) \in \mathscr{D}_\circ(\mathbb{U})\}$.

Proof. If $f \in \mathrm{dom}(\Gamma)$ in $\mathscr{D}_\circ(\mathbb{U})$, then growth condition in (1.3) implies that for all $\omega \in \mathbb{U}$ and $f \in \mathscr{D}_\circ(\mathbb{U})$,

$$\begin{aligned}
\Gamma f(\omega) &= \lim_{t\to 0^+} \frac{(f(e^{-t}\omega) - f(e^{-t}i)) - f(\omega)}{t} \\
&= \frac{\partial}{\partial t}(f(e^{-t}\omega) - f(e^{-t}i))\Big|_{t=0} \\
&= -e^{-t}\omega f'(e^{-t}\omega) + ie^{-t} f'(e^{-t}i)\big|_{t=0} \\
&= -\omega f'(\omega) + i f'(i).
\end{aligned}$$

This shows that $\mathrm{dom}(\Gamma) \subseteq \{f \in \mathscr{D}_\circ(\mathbb{U}) : \omega f'(\omega) \in \mathscr{D}_\circ(\mathbb{U})\}$. Conversely, let $f \in \mathscr{D}_\circ(\mathbb{U})$ such that $\omega f'(\omega) \in \mathscr{D}_\circ(\mathbb{U})$. Then for $\omega \in \mathbb{U}$, and by fundamental theorem of calculus, we have,

$$\begin{aligned}
\hat{C}_{\varphi_t} f(\omega) - f(\omega) &= \int_0^t \frac{\partial}{\partial s}(f(e^{-s}\omega) - f(e^{-s}i)) ds \\
&= \int_0^t -e^{-s}\omega f'(e^{-s}\omega) + e^{-s} i f'(e^{-s}i)) ds \\
&= \int_0^t e^{-s}(-\omega f'(\omega) + i f'(i)) ds \\
&= \int_0^t \hat{C}_{\varphi_s} F(\omega) ds,
\end{aligned}$$

where $F(\omega) = -\omega f'(\omega) + i f'(i)$ is a function in $\mathscr{D}_\circ(\mathbb{U})$. Thus $\lim_{t\to 0} \frac{\hat{C}_{\varphi_t} f - f}{t} = \lim_{t\to 0} \frac{1}{t} \int_0^t C_{\varphi_s}(F) ds$ and strong continuity of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ implies that $\frac{1}{t} \int_0^t \|\hat{C}_{\varphi_t} F - F\| ds \to 0$ as $t \to 0$. Thus
$\mathrm{dom}(\Gamma) \supseteq \{f \in \mathscr{D}_\circ(\mathbb{U}) : \omega f'(\omega) \in \mathscr{D}_\circ(\mathbb{U})\}$ completing the proof. □

Lemma 2.6. *A function $f \in \mathscr{D}_\circ(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$.*

Proof. By definition of $\mathscr{D}_\circ(\mathbb{U})$, $f \in \mathscr{D}_\circ(\mathbb{U})$ if and only if

$$\|f\|^2_{\mathscr{D}_1(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty,$$

Also, by definition of $L^2_a(\mathbb{U})$, (see equation (**??**)) $f' \in L^2_a(\mathbb{U})$ if and only if

$$\|f'\|^2_{L^2_a(\mathbb{U})} \quad = \quad \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty.$$

This implies that $f \in \mathscr{D}_\circ(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$, as desired. □

Lemma 2.7 ([11]). *Let X denote the space $L^p_a(\mathbb{U}), 1 \leq p < \infty$. If $c \in \mathbb{R}$ and $\lambda, v \in \mathbb{C}$, then $f(\omega) = (\omega - c)^\lambda (w+i)^v \in X$ if and only if $\Re(\lambda + v) < -1 < \Re(\lambda)$. In particular, $(\omega - c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$, and $(\omega + i)^v \in X$ if and only if $\Re(v) < -1$.*

Proposition 2.8. *Let Γ be the infinitesimal generator of the group $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, then $\sigma_p(\Gamma) = \emptyset$ and $\sigma(\Gamma) \subseteq i\mathbb{R}$.*

Proof. Let λ be an eigenvalue of Γ and let f be a corresponding eigenvector. The eigenvalue equation $\Gamma(f) = \lambda f$ is equivalent to a first order differential equation

$$-zf'(z) + if'(i) = \lambda f(z).$$

Solving this differential equation using the integrating factor technique yields

$$f(z) = \frac{B}{\lambda} + Cz^{-\lambda},$$

where $B = if'(i)$ and C is an arbitrary constant.
It remains to find for which $\lambda's$ is $f \in \mathscr{D}_\circ(\mathbb{U})$ given that $f(z) = \frac{B}{\lambda} + Cz^{-\lambda}$.
But $f \in \mathscr{D}_\circ(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$. By differentiation, $f'(z) = -\lambda C z^{-(\lambda+1)}$. It follows clearly from Lemma 2.7, that $f' \in L^2_a(\mathbb{U})$ if and only if $\Re(\lambda) < -1 < \Re(\lambda)$. No such λ exists and so $\sigma_p(\Gamma) = \emptyset$. Since $(\hat{C}_{\varphi_t})$ is an invertible isometry, $\sigma(\hat{C}_{\varphi_t}) \subseteq \partial\mathbb{D}$ and by spectral mapping theorem for semigroups, $e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{C}_{\varphi t})$. Thus

$$e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{C}_{\varphi t}) \subseteq \partial\mathbb{D}.$$

Let $\lambda \in \sigma(\Gamma)$, then

$$|e^{\lambda t}| = 1.$$

This shows that

$$\begin{aligned} e^{t\Re(\lambda)} = 1 \quad &\Rightarrow \quad t\Re(\lambda) = 0 \\ &\Rightarrow \quad \Re(\lambda) = 0. \end{aligned}$$

So $\lambda \in i\mathbb{R}$ implying that $\sigma(\Gamma) \subseteq i\mathbb{R}$. □

3. Cesáro-Type Operator on the Dirichlet Space of $\mathbb{U}$

We determine the resolvent of the infinitesimal generator as an integral operator of the Cesáro-type and then determine the point spectrum, spectrum and spectral radius as well as the norm of the operator on the Dirichlet space. Since $\sigma(\Gamma) \subseteq i\mathbb{R}$, we can consider the point $\lambda = 1$ in the resolvent set, $\rho(\Gamma)$, and then obtain the corresponding resolvent operator given by the Laplace transform.

Theorem 3.1. *Let Γ be the infinitesimal generator of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, then the following hold:*

(a) The resolvent operator $\mathscr{C} = R(1,\Gamma)$ on $\mathscr{D}_\circ(\mathbb{U})$ is given by

$$\mathscr{C}h(z) = R(1,\Gamma)h(z) = \frac{1}{z}\int_0^z \left(h(\omega) - h(\frac{\omega}{z}i) \right) d\omega. \tag{3.1}$$

The operator $\mathscr{C}$ is a Cesàro-type operator.

(b) $\sigma(\mathscr{C}) \subseteq \{\omega : |\omega - \frac{1}{2}| = \frac{1}{2}\}$.

(c) $\|\mathscr{C}\| \leq 1$.

(d) $r(\mathscr{C}) \leq 1$.

Proof. To prove (a), we consider a point $\lambda \notin i\mathbb{R}$. Then $\lambda \in \rho(\Gamma)$ since $\sigma(\Gamma) \subseteq i\mathbb{R}$. The resolvent operator, $R(\lambda, \Gamma)$, is therefore given by the Laplace transform,
$R(\lambda, \Gamma)h = \int_0^\infty e^{-\lambda t}\hat{C}_{\varphi_t} h dt$ with convergence in norm.
Now

$$R(\lambda, \Gamma)h(z) = \int_0^\infty e^{-\lambda t}(h(e^{-t}z) - h(e^{-t}i))dt.$$

By change of variables, we let $\omega = e^{-t}z$. Then $e^{-t} = \frac{\omega}{z}, dw = -e^{-t}zdt$, $t = 0 \Rightarrow \omega = z; t = \infty \Rightarrow \omega = 0$.
Therefore

$$\begin{aligned} R(\lambda, \Gamma)h(z) &= \int_z^0 (\frac{\omega}{z})^\lambda (h(\omega) - h(\frac{\omega}{z}i))\frac{-1}{\omega}d\omega \\ &= \int_0^z (\frac{\omega}{z})^\lambda \left(h(\omega) - h(\frac{\omega}{z}i)\right)\frac{1}{\omega}d\omega. \end{aligned}$$

Taking $\lambda = 1$, we obtain

$$R(1, \Gamma)h(z) = \frac{1}{z}\int_0^z \left(h(\omega) - h(\frac{\omega}{z}i)\right)d\omega,$$

which is a Cesàro-type operator of difference of two Cesàro operators.
To prove (b), we apply the spectral mapping theorem for the resolvents which asserts that

$$\sigma(R(\lambda, \Gamma))\backslash\{0\} = (\lambda - \sigma(\Gamma))^{-1} = \left\{\frac{1}{\lambda - \mu} : \mu \in \sigma(\Gamma)\right\}. \tag{3.2}$$

Thus,

$$\sigma(R(1, \Gamma))\backslash\{0\} \subseteq \left\{\frac{1}{1 - ir} : r \in \mathbb{R}\right\}. \tag{3.3}$$

Rationalizing the denominator and simplifying we get,

$$\frac{1}{1 - ir} = \frac{1 + ir}{1 + r^2}.$$

Letting $\omega = \frac{1+ir}{1+r^2}$ and subtracting $\frac{1}{2}$, we get

$$w - \frac{1}{2} = \frac{-r + i}{2(r + i)}.$$

Getting the magnitude on both sides of the equation and simplifying, we get $\left|\omega - \frac{1}{2}\right| = \frac{1}{2}$ and therefore

$$\sigma(\mathscr{C}) \subseteq \left\{\omega : \left|\omega - \frac{1}{2}\right| = \frac{1}{2}\right\}.$$

For (c), we apply the Hille Yosida theorem,

$$\|R(1, \Gamma)\| \leq 1,$$

implying that

$$\|\mathscr{C}\| \leq 1. \tag{3.4}$$

For (d), we use (3.4) and the fact that $r(\mathscr{C}) \leq \|\mathscr{C}\| \leq 1$. Clearly,

$$r(\mathscr{C}) \leq 1.$$

This completes our proof. ∎

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