Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 26(1), 180-187, (2024)**

DOI:10.25092/baunfbed. 1299271 **J. BAUN Inst. Sci. Technol., 26(1), 180-187, (2024)**

Some new properties on N semigroups

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Geliş Tarihi (Received Date): 18.05.2023
Kabul Tarihi (Accepted Date): 25.10.2023

Abstract

In this study we first show that N satisfies two important homological properties, namely Rees short exact sequence an d short five lemma. In addition, by defining inversive semigroup varieties of N we prove that strictly inverse semigroup N is isomorphic to the spined product of (C)-inversive semigroup and the idempotent semigroup of N. Moreover, we give some consequences of the results to make a detailed classification over N. It has been recently defined a new semigroup N based on Rees matrix and completely 0-simple semigroups. Further, it has been also proved finiteness conditions and the existence of some fundamental properties over N.

Keywords: *Rees matrix semigroup, spined product, completely 0-simple Semigroup.*

N yarıgrubu üzerinde bazı yeni özellikler

Öz

Bu çalışmada ilk olarak N' nin iki önemli homolojik özelliği, yani Rees kısa tam dizisi ve kısa beşli lemmayı sağladığı gösterilmiştir. Ek olarak, N'nin ters yarı grup çeşitlerini tanımlayarak, kesin olarak ters yarıgrup N 'nın (C)-ters yarıgrup ve N'nın idempotent yarıgrubunun (spined) döndürülmüş çarpımına izomorfik olduğu kanıtlanmıştır. Ayrıca, N üzerinden ayrıntılı bir sınıflandırma yapmak için bazı sonuçlar verilmiştir. Son zamanlarda Rees matrisine ve tam 0-basit yarı gruplara dayalı yeni bir yarı grup olan N tanımlanmıştır. Dahası N üzerinde bazı temel özelliklerin ve sonluluk koşullarının varlığı da kanıtlanmıştır.

Anahtar kelimeler: *Rees matris yarıgrup, döndürülmüş çarpımlar, tam basit sıfır yarıgrup.*

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1. Introduction and Preliminaries

Since their introduction into the mathematical literature in the early 1900s, semigroups have become one of the most-studied classes of algebra with many mathematician devoted to their understanding. Such important objects of study have naturally given rise to a number of different methods for their investigation. So, very important semigroup classes is emerged.

In [11], we created a new semigroup class and denoted it by N. When a new semigroup appears, it is necessary to investigate the properties of this semigroup. For example, in [13] and [14] the authors considered a new product and gave some properties on a special product of semigroups (and monoids) [1]. In this paper we more deeply investigate the place N in the literature as a continuation of the work [11].

It is a well known fact that, in group theory, a group extension is a general means of describing a group in terms of a particular normal subgroup and quotient group. If B and A are two groups, then G is an extension of B by A if there is a short exact sequence
$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1$.

Group extensions arise in the context of the extension problem, where the groups B and A are known and the properties of G are to be determined. If there is a short exact sequence for the semigroup N, this sequence gives an idea about the, just like in the G group. Therefore, it is a question of wondering whether a short exact sequence can be created for the semigroup N. In order to set the scene for what follows, it is necessary to give a definition short exact sequence of semigroup or monoid. Chen and Shum [3] introduced Rees short exact sequence of acts over monoids. In [6], the authors investigated conditions under which flatness properties of right acts A and C in the Rees short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ can be transferred to B. It is clear that short exact sequence which is called Rees short exact sequence over monoid reveals many features. So it makes sense to try to define the Rees short exact sequence for N and that is what we will do in Section 2.1. (see Theorem 2.1.) in below.

Spined products of semigroups were first defined and studied by Kimura [7]. After that, spined products have been considered many time as predominantly those a band and a semilattice of semigroups with respect to their common semilattice homomorphic image [2]. In this paper we show that N is isomorphic to spined product of Rees matrix semigroup and completely 0-simple semigroup (see Theorem 2.2.)

In [6] the authors presented short five lemma over monoid and also defined some detailed information on it. In this paper, we will create a commutative diagram by combining properties of short five lemma and spined product of N (see Corollory 2.3.)

The classification of types of semigroup has normally arisen only as an incidental problem since the difficulty of the classification problem in general. So any classification of semigroups naturally important. In [11] it has been proved that the semigroup N is completely inverse and regular. As a continuation of this classification in [11] we will continue to make some new classifications in here. We will prove that N is strictly inversive semigroup (in Theorem 2.4 and also it is a (C)-inversive semigroup (in Theorem 2.5). Furthermore we will state some corollaries as a result of these classifications (see Corollaries 2.6., 2.7. and 2.8.)

Throughout this paper all notations and terminologies are taken from [11]. For other undefined terminologies and definitions, the reader is referred to [4-5].

Now, we will recall some fundamental concepts and definitions which will be needed in this paper. First, let us recall the definition of the semigroup N given recently in [11]. We consider the following mapping

$$\gamma: (M_R \times M_C) \star (M_R \times M_C) \rightarrow (M_R \times M_C)$$

that has the binary operation $\star$ as

$$[a,b,c),(d,e,f)] \star [(k,l,m),(x,y,z)] = \begin{cases} \left((a, bp_{\{ck\}}l, m), 0\right) & if \quad p_{\{ck\}} \neq 0 \; and \; p'_{\{fx\}} = 0 \\ \left(0, (d, ep'_{\{fx\}}y, z)\right) & if \quad p_{\{ck\}} = 0 \; and \; p'_{\{fx\}} \neq 0 \\ \left((a, bp_{\{ck\}}l, m), \left(d, ep^{\{'}_{\{fx\}}\right\}y, z\right) & if \quad p_{\{ck\}} \neq 0 \; and \; p'_{\{fx\}} \neq 0 \\ (0_R, 0_C) & if \; p_{\{ck\}} = 0 \; and \; p'_{\{fx\}} = 0, \end{cases} \tag{1}$$

where $(a,b,c),(k,l,m)$ in element M_R, $(d,e,f),(x,y,z)$ in element M_C. In here completely *0*-simple semigroup $\mathrm{M}^0\left[\mathrm{G}^{\{0\}};\ \mathrm{I},\mathrm{J};\ \mathrm{P}'\right]$ defined by the set $\mathrm{I} \times \mathrm{G}^{\{0\}} \times \mathrm{J}$ and Rees matrix semigroup $\mathrm{M}^0\left[\mathrm{S}^{\{0\}};\ \mathrm{I},\mathrm{J};\ \mathrm{P}\right]$ defined by the set $\mathrm{I} \times \mathrm{S}^{\{0\}} \times \mathrm{J}$ are denoted by the notations M_R and M_C, respectively.

Then the set $M_R \times M_C$ defines the semigroup $M^0[S^{\{0\}}, G^0;\ M_R, M_C;\ P, P']$ using the operation given in (1). For simplicity, we denote this new semigroup by N. More detailed information can be found in [11]. It is obvious that a detailed chechking implies this binary operation does not turns out to be componentwise multiplication.

In (1) again for simplicity, we will use the shortcuts $(r,0)$ for the first line, $(0,c)$ for the second line, (r,c) for the third line and (0,0) for the last line, respectively. Therefore each of r,c and 0 contains a triple in itself. For example (0,0) means $[(a, 0_S, b), (c, 0_G, d)]$.

Definition 1.1. [10] A semigroup is called an externally commutative semigroup if it satisfies the identity $axb = bxa$.

Lemma 1.2. $(N,\star)$ is commutative semigroup if the following conditions are satisfied:

(i) The index sets I and J contains an unique element.

(ii) S is externally commutative semigroup.

(iii) G is commutative group.

Proof: Suppose that N is commute, i.e. for every $x', y' \in N x' \star y' = y' \star x'$. According to [11, Remark 2.1.] it is well known that third line of operarion $\star$ defines the general situation. For this reason, we consider the form (r,c) which is the most common form among the others and clearly including other cases in (1). Therefore we have

$$[(a,b,c),(d,e,f)] \star [(x,y,z),(k,l,m)] = [(x,y,z),(k,l,m)] \star [(a,b,c),(d,e,f)]$$
$$\left[(a, bp_{\{cx\}}y, z), \left(d, ep'_{\{fk\}}l, m\right)\right] = [(x, yp_{\{za\}}b, c), (k, lp'_{\{md\}}e, f)]$$

If the index sets I and J contains an unique element, i.e $I = \{i\}$ and $J = \{j\}$, then we have $a = d = x = k = i$ and $z = m = c = f = j$. In case the equality $[(a, bp_{\{cx\}}y, z)$, $(d, ep'_{\{fk\}}l, m)] = [(x, yp_{\{za\}}b, c), (k, lp'_{\{md\}}e, f)]$ implies that $[(i, bp_{\{ji\}}y, j), (i, ep'_{\{ji\}}l, j)] = [(i, yp_{\{ji\}}b, j), (i, lp'_{\{ji\}}e, j)]$. It remains to show that $bp_{\{ji\}}y = yp_{\{ji\}}b$ and $ep'_{\{ji\}}\, l = lp'_{\{ji\}}e$. If S is externally commutative semigroup, then we have $bp_{\{ji\}}y = yp_{\{ji\}}b$. Similarly, if G is commutative group, $ep'_{\{ji\}}l = lp'_{\{ji\}}e$ is satisfies. Hence the result.

The following definitions will be used in the classification of *N*.

Definition 1.3. [12] A semigroup S is called strictly inversive if the set of idempotents of S is a subband of S.

Definition 1.4. [4,5] A semigroup S is called inversive if it is satisfies one of the following conditions.

- **i.** S has an idempotent and the set of idempotents of S is a subband of S.
- **ii.** For any element x of S, there exist an element $x^{\{*\}}$ such that \$ $xx^{\{*\}} = x^{\{*\}}x$ and $xx^{\{*\}}x = x$.

Definition 1.5. [12] Let S be an inversive semigroup. If it satisfies $xy = yx$, then S is said to be (C) −inversive semigroup.

For a sequence of semigroup homomorphisms $\cdots \to L_S \to^f M_S \to^g N_S \to \cdots$, we said that it is M_S exact if M_S provided $kerg = \kappa_{Imf}$, where $\kappa_{Imf} = (Imf \times Imf) \cup \Delta_{\{M_S\}}$ such that $\Delta_{\{M_S\}}$is the identity congruence on M_S. If L_S, M_S and N_S are exact in the sequence $0 \to L_S \to M_S \to N_S \to 0$, then this sequence is called Rees short exact sequence.

Lemma 1.6. [6]

Let the following diagram be commutative with Rees short exact rows.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & N \arrow[r, "\psi"] \arrow[d, "\beta"'] & K \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varphi'"] & N' \arrow[r, "\psi'"] & K' \arrow[r] & 0
\end{tikzcd}$$

Figure 1: Commutative diagram

Then

- **i.** If α and γ are monomorhism, then β is monomorphism.
- **ii.** If α and γ are epimorphism, then β is epimorphism.
- **iii.** If α and γ are isomorphism, then β is isomorphism.

If P and Q are two semigroups having a common homomorphic image Y, then the spined product of P and Q with respect to Y is $S = \{(a,b) \in P \times Q \mid \varphi(a) = \psi(b)\}$, where $\varphi: P \to Y$ and $\psi: Q \to Y$ are homomorphisms onto Y [1]. Let us notate the spined product by $\times_S$.

The structure of this paper is designed to obtain new properties of the semigroup structure N. It is divided into several sections, each of which serves a specific purpose and contributes to the overall understanding of our findings. Firstly we will prove that N is a Rees short exact sequence in Section 2.1 (Theorem 2.1.) After we will show that semigroup N is the spined product of $\mathrm{M_R}$ and M_C in Section 2.2 (Theorem 2.2.). Combining the results obtained in Section 2.1 and 2.2, we will found an important result (in Corollary 2.3.). In Section 2.3, we will prove that $N^C \times_S E(N) \cong N'$ (in Corollary 2.6.) In fact, to obtain Corollary 2.6., we proved Lemma 1.2. and Theorem 2.5., we stated and proved necessary and sufficient conditions N is strictly inversive semigroup (in Theorem 2.4.). Combining some results in this paper and [11, Theorem 2.7.] we state some further results in Corollaries 2.7.and 2.8.

2. Results

In this section we give our main results with different subsections. N is commutative semigroup, unless otherwise stated.

2.1. N satisfies short exact sequence $\mathbf{0} \to \boldsymbol{M_R} \to \boldsymbol{N} \to \boldsymbol{M_C} \to \mathbf{0}$

Theorem 2.1. $0 \to^{\alpha} M_R \to^{f} N \to^{g} M_C \to^{\beta} 0$ is a Rees short exact sequence.

Proof: Firstly, it is must be shown that M_R, N and M_C are exact. Let us consider the mappings

$$\begin{array}{rcl} \alpha : & 0 & \to M_R \\ [(i,0_S,j),(i,0_G,j)] & \mapsto & n \star [(i,0_S,j),(i,0_G,j)], \end{array}$$

$$\begin{array}{rcl} f : & M_R & \to N \\ [(i,s,j),(i,0_G,j)] & \mapsto & n \star [(i,s,j),(i,0_G,j)], \end{array}$$

where $n \in N$. It is clear that α is well defined. It is easily seen that $n \star [(i,s,j),(i,0_G,j)] \in N$. If $[(i,s,j),(i,0_G,j)] = [i,s',j),(i,0_G,j)]$ then $n \star [(i,s,j),(i,0_G,j)] = n \star [(i,s',j),(i,0_G,j)]$. In this case f is also well defined. Clearly these mappings are all homomorphisms. The task is now to find kernels and images of these homomorphism. In fact

$$\begin{aligned} ker f &= \{[(i,s,j),(i,0_G,j)],[(i,s',j),(i,0_G,j)]\}\ and\ also \\ \kappa_{Im\alpha} &= \{[(i,0_S,j),(i,0_G,j)],[(i,0_S,j),(i,0_G,j)]\} \cup \Delta_{\{M_R\}} \\ &= \{[(i,0_S,j),(i,0_G,j)],[(i,0_S,j),(i,0_G,j)]\} \\ &\quad \cup \{[(i,s,j),(i,0_G,j)],[(i,s',j),(i,0_G,j)]\} = ker f \end{aligned}$$

which implies that M_R is exact since $kerf = \kappa_{Im\alpha}$. On the other hand, for the mappings

$$g : N \to M_C$$
$$[(i,s,j),(i,g,j)] \mapsto n \star [(i,0_S,j),(i,g,j)] ,$$

$$\beta : M_C \to 0$$
$$[(i,0_S,j),(i,g,j)] \mapsto n \star [(i,0_S,j),(i,0_G,j)] ,$$

where $n \in N$. In the same manner we can see that g and β are well defined. We can consider a similar process as in the following.

$\kappa_{Im\alpha} = (Imf \times Imf) \cup \Delta_N$

$= [(i,xp_{\{ji\}}s,j),(i,0_G,j)],[(i,xp_{\{ji\}}s,j),(i,0_G,j)]\} \cup \Delta_N$

(here $xp_{\{ji\}}s$ is an element in the semigroup S , i.e, $p_{\{ji\}}s = b \in S$)

$= \{[(i,b,j),(i,0_G,j)],[(i,b,j),(i,0_G,j)]\} \cup \{[(i,s,j),(i,g,j)],[(i,s',j),(i,g',j)]\}$

$= \{[(i,S,j),(i,G,j)],[(i,S,j),(i,G,j)]\} = ker\, f$

and so N is exact since $kerg = \kappa_{Imf}$. Similary, we have $ker\,\beta = \kappa_{Img}$ so M_C is exact. Therefore $0 \to M_R \to N \to M_C \to 0$ is a Rees short exact sequence.

2.2 N is spined product M_R and M_C

Firstly we show that N is spined product of M_R and M_C in this subsection.

Theorem 2.2. Let S be a rectangular band. Then N is the spined product of M_R and M_C.

Proof: Let us consider the mappings

$$\varphi : M_R \to N \qquad\qquad \psi : M_C \to N$$
$$(r,0) \mapsto (r,c) \star (r_{1,0}) \qquad\qquad (0,c) \mapsto (r,c) \star (0,c\_1)$$

First of all we have to prove that these mappings are well defined. There is no doubt that $(r,c) \star (r_1,0)$ and $(r,c) \star (0,c_1)$ in elements of N. If $(r,0) = (r',0)$, then $\varphi\,(r,0) = (r,c) \star (r_1,0) = (r',c) \star (r_1,0) = \varphi\,(r',0)$. One can prove, in a similar way to the proof of well defined of φ, that if $(0,c) = (0,c')$, then $\psi\,(0,c) = (r,c) \star (0,c_1) = (r,c') \star (0,c_1) = \psi\,(0,c')$. So φ and ψ are well defined.

Since S is a rectangular band,

$$(r,0) \star (r_{1,0}) = [(i,b,j),(i,0_G,j)] \star [(i,b',j),(i,0_G,j)]$$
$$= [(i,bp_{\{ji\}}b',j),(i,0_G,j)]$$
$$\varphi[(r,0) \star (r_1,0)] = [(i,s,j),(i,g,j)] \star [(i,bp_{\{ji\}}b',j),(i,0_G,j)]$$
$$= [(i,sp_{\{ji\}}b'p_{\{ji\}}b',j),(i,0_G,j)] \qquad (2)$$
$$\varphi(r,0) \star \varphi(r_1,0) = [(i,sp_{\{ji\}}b,j),(i,0_G,j)] \star [(i,sp_{\{ji\}}b',j),(i,0_G,j)]$$
$$= [(i,sp_{\{ji\}}bp_{\{ji\}}sp_{\{ji\}}b',j),(i,0_G,j)] \qquad (3)$$

The expressions (2) and (3) are equal and so φ is a homomorphism. Similarly ψ is also a homomorphism.

Therefore we obtain $M_R \times_S M_C = N$ since $\varphi(r,0) = \psi(0,c) = N$.

Corollary 2.3. The following diagram is commute.

$$\begin{tikzcd}
0 \arrow[r] & M_R \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & M_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_R \arrow[r] & M_R \times_{\mathcal{S}} M_C \arrow[r] & M_C \arrow[r] & 0
\end{tikzcd}$$

Figure 2. Commutative diagram for N

Proof: We proved that N is the spined product of M_R and M_C in Theorem 2.2. Conditions for creating a commutative diagram are given in Lemma 1.6. By using Theorem 2.2. and Lemma 1.6. we deduce immediately the truthfulness of the corollary.

2.3 $N^C \times_S E_N \cong N'$

In this part, the properties (C) $-$inversive semigroup and strictly inversive semigroup over N will be denoted by N^C and N' respectively.
It is well-known that inversive semigroups are very important classes in semigroup algebra. In fact, different varieties of inversive semigroups have been used frequently in studies. For example, every regular semigroup is an E $-$inversive semigroup which is a variant of the inversive semigroup and this is very crucial for the classification of semigroups [9]. We may also refer, for example, [8,12].

In this subsection, we focus on some varieties of inversive semigroups. In addition, by applying the spined product obtained in Section 2.2. We present some important semigroup classifications from inversive varieties of N.

Theorem 2.4. If S is a rectangular band, then N is strictly inversive semigroup.

Proof: Let us assume that S is rectangular band. In [11, Lemma 2.3.], the elements form of E(N) which is $[(i,b,j),(i,d,j)]$ such that $p'_{\{ji\}} = d^{\{-1\}}$ and S is a rectangular band. By Definition 1.4., it is known that the set of idempotent element should be a subband of N. Since every element of E(N)is idempotent and S is a rectangular band, N is strictly inversive semigroup as stated in Definition 1.4.

Theorem 2.5. If N is commutative, then N is (C) $-$inversive semigroup.

Proof: As it mentioned in the proof of Theorem 2.4., the form of $E(N)$ is depicted in [11, Lemma 2.3]. We said that for the semigroup N to be commutative, the index sets must have one element Lemma 1.2. According to this fact, the elements of the idempotent set $E(N)$ will be the formed as $[(i,b,j),(i,d,j)]$ such that S is a rectangular band and $p'_{\{ji\}} = d^{\{-1\}}$. It is clear that $E(N)$is a subband of N. Thus, by considering also Definition 1.4. we get that N is (C) $-$inversive semigroup while N is commutative.

In [12], an important theorem (*structure theorem)* states that a semigroup is isomorphic to the spined product of a (C) $-$inversive semigroup and a band if and only if it is strictly inversive. By using this result and Theorems 2.4., 2.5. we have the following corollary which is easily verified.

Corollary 2.6. Let S be a commutative regular semigroup and G be a commutative group. Then N is isomorphic to the spined product of N^C and $E(N)$.

Furthermore, by using [11, Theorem 2.7.] Lemma 2.2. and Theorem 2.5., we have the following consequence as a result.

Corollary 2.7. If N is completely inverse semigroup, then N is $(C)-$inversive semigroup.

According to [11, Theorem 2.7.] we also have the following important corollary.

Corollary 2.8. N strictly inverse if and only if it is completely inverse semigroup.

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