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TRANSLATION SURFACES GENERATING WITH SOME PARTNER CURVE

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Abstract. In this article, generating curves of translation surfaces are paired with some special curve pairs. With the results obtained from these pairings, the developable and minimal translation surfaces are characterized. In addition, the surface curvatures of the translation surface are obtained. For a better understanding of the results, examples are given and their drawings are made with the help of Mathematica.

1. Introduction

The main purpose of differential geometry is to understand and characterize the mathematical properties of any geometric object defined in space. The most important of these objects are curves and surfaces. Researchers working on this subject often have to characterize the curve and the surface in a certain way in order to understand it. One of the most important ways to characterize the curve is to use Frenet vectors. For example, Bertrand pairs of curves were characterized by J. Bertrand in 1850 as curves whose reciprocal normal vectors are linearly dependent [1]. Similarly, the Mannheim curve pairs were characterized by the normal vector of one of the curves and the binormal vector of the other as linearly dependent by A. Mannheim in 1878 [2]. In addition, the involute-evolute curve pairs are characterized as curve pairs whose mutual tangent vectors are perpendicular [3].

The study of surfaces is one of the most captivating subjects in the field of differential geometry. Consequently, researchers have extensively investigated various types of surfaces [4–6]. Much like curves, researchers endeavor to characterize

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surfaces. Moreover, another significant aspect that piques researchers' interest is whether a surface is developable or minimal [7,8]. One of the interesting surfaces in Euclidean space is the translation surface produced by the two curves. The general form of translation surface is the surface that can be generated from two arbitrary space curves by translating either of them parallel to itself. In such a way that each of its points describes a curve that is a translation of the other curve. A generalized type of translation surface parameterized by

$$\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v) \tag{1}$$

where $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\mathbf{y} : J \subset \mathbb{R} \to \mathbb{E}^3$ are arbitrary generating curves of χ according to the parameters u and v (may be the arc-length parameters), respectively. Let $\{\mathbf{t_x}, \mathbf{n_x}, \mathbf{b_x}\}$ be the Frenet frame field of $\mathbf{x}$ with curvature $\kappa_{\mathbf{x}}$ and torsion $\tau_{\mathbf{x}}$. Also, let $\{\mathbf{t_y}, \mathbf{n_y}, \mathbf{b_y}\}$ be the Frenet frame field of $\mathbf{y}$ with curvature $\kappa_{\mathbf{y}}$ and torsion $\tau_{\mathbf{y}}$. A translation surface has the property that the translations of a parametric curve $u = c$ by $\mathbf{y}(v)$ remain in χ (similarly for the parametric curves $v = c$) [9–11]. Translation surfaces are the basic modeling surfaces commonly used in computer aided geometric design and geometric modeling [12]. Also, translation surfaces are common in descriptive geometry and architecture because they can be easily modeled [13,14]. Many studies are carried out on translation surfaces so far: L. Verstraelen et al. have studied minimal translation surfaces in n-dimensional Euclidean spaces [15]. H. Liu has studied Gaussian curvature and mean curvature of translation surfaces in 3-dimensional space [16]. D. W. Yoon has studied the differential geometric properties of translation surfaces by applying the Laplace operator to the Gauss transform [17]. Additionally, numerous studies have been conducted on translation surfaces [18–22].

In this study, generating curves of translation surfaces are associated with some special curve pairs. The article investigates the conditions necessary for these translation surfaces to be both developable and minimal surface, while also characterizing the conditions that make this possible.

2. Preliminaries

In this section, for parametrized curves and surface elements some basics definitions and theorems are given.

A regular naturally parametrization of class C^k, with $k \geq 1$ of a curve in $\mathbb{R}^3$ is a vector function $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$, $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$ defined on an interval I which satisfies $\mathbf{x}$ is of class C^k and $\mathbf{x}'(s) \neq 0$ for all $s \in I$. A curve $\mathbf{x}$ is continuously differentiable if $\mathbf{x}'(s)$ exists for all $s \in I$ and the derivative $\mathbf{x}'(s)$ is a continuous function; thinking dynamically, the vector $\mathbf{x}'(s)$ is the velocity of the curve at time s. We call $\mathbf{x}(s)$ *naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^k and $\|\mathbf{x}'(s)\| = 1$, for each $s \in I$ [23].

Let $\mathbf{x}(s)$ be *biregular*, that is, $\mathbf{x}'(s) \times \mathbf{x}''(s) \neq 0$, for each $s \in I$. We consider a trihedron $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ along $\mathbf{x}(s)$, so-called *Frenet frame*, where [23]

$$\mathbf{t}(s) = \mathbf{x}'(s), \quad \mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|}, \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

The curvature κ, a non-negative scalar field, is defined by setting $\kappa(s) = \|\mathbf{t}'(s)\|$ and torsion is defined by setting $\tau(s) = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle$. If the naturally parametrized curve $\mathbf{x}$ has unit speed and strictly positive curvature, then the following equations hold [23]

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

where $\kappa \neq 0$ for the Frenet frame to be defined.

Let $\mathbf{x}$ and $\mathbf{y}$ be naturally parametrized curves in $\mathbb{E}^3$ with parameter u and v, respectively. Let $\{t_{\mathbf{x}}(u), n_{\mathbf{x}}(u), b_{\mathbf{x}}(u), \kappa_{\mathbf{x}}(u), \tau_{\mathbf{x}}(u)\}$ and $\{t_{\mathbf{y}}(v), n_{\mathbf{y}}(v), b_{\mathbf{y}}(v), \kappa_{\mathbf{y}}(v), \tau_{\mathbf{y}}(v)\}$ be Frenet elements of $\mathbf{x}$ and $\mathbf{y}$, respectively. Some special curve pairs is studied by S. Yuce and A. Sabuncuoglu and the following results are given [24, 25].

Let's assume that $(\mathbf{x}, \mathbf{y})$ curve pair is Bertrand curve pair. In this sitation, since the normal vectors of the $\mathbf{x}$ and $\mathbf{y}$ have the same direction, they are written as

$$\mathbf{t}_{\mathbf{x}}(u) = \cos\theta \mathbf{t}_{\mathbf{y}}(v) - \sin\theta \mathbf{b}_{\mathbf{y}}(v), \tag{2}$$

$$\mathbf{n}_{\mathbf{x}}(u) = \mathbf{n}_{\mathbf{y}}(v), \tag{3}$$

$$\mathbf{b}_{\mathbf{x}}(u) = \sin\theta \mathbf{t}_{\mathbf{y}}(v) + \cos\theta \mathbf{b}_{\mathbf{y}}(v) \tag{4}$$

and

$$\kappa_{\mathbf{x}}(u) = \kappa_{\mathbf{y}}(v)\cos\theta + \tau_{\mathbf{y}}(v)\sin\theta, \tag{5}$$

$$\tau_{\mathbf{x}}(u) = -\kappa_{\mathbf{y}}(v)\sin\theta + \tau_{\mathbf{y}}(v)\cos\theta, \tag{6}$$

where θ is the constant angle between the mutually tangent vectors.

Let's assume that $(\mathbf{x}, \mathbf{y})$ curve pair is Mannheim curve pair. Since the normal vector of the $\mathbf{x}$ and binormal vector of the curve $\mathbf{y}$ have the same direction, they are written as

$$\mathbf{t}_{\mathbf{x}}(u) = \cos\theta \mathbf{t}_{\mathbf{y}}(v) + \sin\theta \mathbf{n}_{\mathbf{y}}(v), \tag{7}$$

$$\mathbf{n}_{\mathbf{x}}(u) = \mathbf{b}_{\mathbf{y}}(v), \tag{8}$$

$$\mathbf{b}_{\mathbf{x}}(u) = -\sin\theta \mathbf{t}_{\mathbf{y}}(v) + \cos\theta \mathbf{n}_{\mathbf{y}}(v) \tag{9}$$

and

$$\kappa_{\mathbf{x}}(u) = \tau_{\mathbf{y}}(v)\sin\theta \frac{dv}{du}, \tag{10}$$

$$\tau_{\mathbf{x}}(u) = -\tau_{\mathbf{y}}(v)\cos\theta \frac{dv}{du}, \tag{11}$$

where θ is the constant angle between the mutually tangent vectors.

Let's assume that $(\mathbf{x},\mathbf{y})$ curve pair be involute-evolute partner curve. Since the mutual tangent vectors of the $\mathbf{x}$ and $\mathbf{y}$ curves are perpendicular, the following equations are available

$$\mathbf{t_x}(u) = \mathbf{n_y}(v), \tag{12}$$
$$\mathbf{n_x}(u) = \cos\theta\mathbf{t_y}(v)+\sin\theta\mathbf{b_y}(v), \tag{13}$$
$$\mathbf{b_x}(u) = -\sin\theta\mathbf{t_y}(v)+\cos\theta\mathbf{b_y}(v) \tag{14}$$

where θ is the constant angle between $\mathbf{t_x}$ and $\mathbf{n_y}$, and

$$\kappa_{\mathbf{x}}(u)=\frac{\sqrt{\kappa_{\mathbf{y}}^2+\tau_{\mathbf{y}}^2}}{(c-s)\kappa_{\mathbf{y}}}. \tag{15}$$

Let M be a regular surface in $\mathbb{R}^3$ parameterized by $\chi(u,v)$. Some basic concepts of M surface is studied by M.P. Do Cormo and these concepts are given below [3].

The standart unit normal vector field $\mathbf{n}$ on surface M can be defined by

$$\mathbf{n}=\frac{\chi_u\times\chi_v}{\|\chi_u\times\chi_v\|}. \tag{16}$$

Also, the first and second fundamental forms of the surface M are as follows

$$I=Edu^2+2Fdudv+Gdv^2,$$
$$II=edu^2+2fdudv+gdv^2,$$

where the E,F and G components are called the coefficients of the first fundemental form of the surface, and the e,f and g components are called the coefficients of the second fundemental form, respectively. The following equations are given for the first and second fundamental form coefficients of the surface

$$E=\langle\chi_u,\chi_u\rangle\,,\quad F=\langle\chi_u,\chi_v\rangle\,,\quad G=\langle\chi_v,\chi_v\rangle \tag{17}$$

and

$$e=\langle\chi_{uu},\mathbf{n}\rangle\,,\quad f=\langle\chi_{uv},\mathbf{n}\rangle\,,\quad g=\langle\chi_{vv},\mathbf{n}\rangle\,. \tag{18}$$

On the other hand, the Gaussian curvature K and the mean curvature H of the surface M are as follows

$$K=\frac{eg-f^2}{EG-F^2} \tag{19}$$

and

$$H=\frac{Eg+Ge-2Ff}{2(EG-F^2)}. \tag{20}$$

Theorem 1. *Let M be a regular surface in $\mathbb{R}^3$. If the Gaussian curvature of the surface M is zero, the surface is called the developable surface [26] .*

Theorem 2. *Let M be a regular surface in $\mathbb{R}^3$. If the mean curvature of the surface M is zero, the surface is called the minimal surface [26].*

3. Translation Surfaces Created with Curve Pairs

Translation surfaces are formed by the sum of the two curves, from Eq. (1), translation surface is as follows

$$\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v) \tag{21}$$

where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. If the partial derivatives of the translation surface given above are taken according to u and v, we have

$$\chi_u = \mathbf{t_x}, \tag{22}$$
$$\chi_v = \mathbf{t_y}, \tag{23}$$
$$\chi_{uu} = \kappa_{\mathbf{x}}\mathbf{n_x}, \tag{24}$$
$$\chi_{vv} = \kappa_{\mathbf{y}}\mathbf{n}_y, \tag{25}$$
$$\chi_{uv} = \frac{d}{dv}\mathbf{t_x}. \tag{26}$$

The unit normal of the translation surface from Eqs. (16), (22) and (23), we get

$$\mathbf{n} = \frac{\mathbf{t_x} \times \mathbf{t_y}}{\|\mathbf{t_x} \times \mathbf{t_y}\|}. \tag{27}$$

The coefficients of the first and second fundamental forms of the translation surface are obtained from Eqs. (17), (18) and Eqs. (22)-(26), as

$$E = \langle \chi_u, \chi_u \rangle = 1, \tag{28}$$
$$F = \langle \chi_u, \chi_v \rangle = \langle \mathbf{t_x}, \mathbf{t_y} \rangle, \tag{29}$$
$$G = \langle \chi_v, \chi_v \rangle = 1 \tag{30}$$

and

$$e = \langle \chi_{uu}, \mathbf{n} \rangle = \frac{\kappa_{\mathbf{x}}}{\|\mathbf{t_x} \times \mathbf{t_y}\|}\langle \mathbf{n_x}, \mathbf{t_x} \times \mathbf{t_y} \rangle, \tag{31}$$
$$f = \langle \chi_{uv}, \mathbf{n} \rangle = \frac{1}{\|\mathbf{t_x} \times \mathbf{t_y}\|}\langle \frac{d}{dv}\mathbf{t_x}, \mathbf{t_x} \times \mathbf{t_y} \rangle, \tag{32}$$
$$g = \langle \chi_{vv}, \mathbf{n} \rangle = \frac{\kappa_{\mathbf{y}}}{\|\mathbf{t_x} \times \mathbf{t_y}\|}\langle \mathbf{n_y}, \mathbf{t_x} \times \mathbf{t_y} \rangle. \tag{33}$$

3.1. **Let x and y Bertrand partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the Bertrand partner curve. In this case, from Eq. (2) and (27) the unit normal of the translation surface is

$$\mathbf{n} = \frac{(\cos\theta\mathbf{t_y} - \sin\theta\mathbf{b_y}) \times \mathbf{t_y}}{\|(\cos\theta\mathbf{t_y} - \sin\theta\mathbf{b_y}) \times \mathbf{t_y}\|} = -\mathbf{n_y}. \tag{34}$$

Since the principal normal vector fields of Bertrand curve pairs are linearly dependent, at the same time $\mathbf{n} = -\mathbf{n_x}$.

The coefficients of the first fundamental form from Eq. (2) and Eqs. (28)-(30), are obtained as

$$\begin{aligned} E &= \langle \chi_u, \chi_u \rangle = 1, \\ F &= \langle \chi_u, \chi_v \rangle = \langle (\cos\theta \mathbf{t_y} - \sin\theta \mathbf{b_y}), \mathbf{t_y} \rangle = \cos\theta, \\ G &= \langle \chi_v, \chi_v \rangle = 1. \end{aligned}$$

The coefficients of the second fundamental form from Eqs. (2),(5) and Eqs. (31)-(33), are as follows

$$\begin{aligned} e &= \langle \kappa_{\mathbf{x}} \mathbf{n_x}, -\mathbf{n_x} \rangle = -\kappa_{\mathbf{x}}, \\ f &= \langle (\kappa_{\mathbf{y}} \cos\theta + \tau_{\mathbf{y}} \sin\theta) \mathbf{n_y}, -\mathbf{n_y} \rangle = -\kappa_{\mathbf{x}}, \\ g &= \langle \kappa_{\mathbf{y}} \mathbf{n_y}, -\mathbf{n_y} \rangle = -\kappa_{\mathbf{y}}. \end{aligned}$$

The Gaussian and mean curvatures of translation surfaces, whose generating curves are Bertrand partner curves from Eqs. (19) and (20), are calculated as

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\kappa_{\mathbf{x}}(\kappa_{\mathbf{y}} - \kappa_{\mathbf{x}})}{\sin^2\theta} \tag{35}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_{\mathbf{x}} - \kappa_{\mathbf{y}} + 2\cos\theta \kappa_{\mathbf{x}}}{2\sin^2\theta}. \tag{36}$$

Theorem 3. *Let $\chi(u, v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Bertrand partner curves to be developable surfaces the necessary and sufficient condition is that this $\mathbf{y}$ is helix.*

Proof. Considering that $\kappa_{\mathbf{x}} \neq 0$, from Eqs. (5), (35) and Theorem 1, it becomes

$$\kappa_{\mathbf{x}} = \kappa_{\mathbf{y}}$$

and

$$\kappa_{\mathbf{y}} \cos\theta + \tau_{\mathbf{y}} \sin\theta = \kappa_{\mathbf{y}}.$$

So, we get

$$\frac{\tau_{\mathbf{y}}}{\kappa_{\mathbf{y}}} = \frac{1 - \cos\theta}{\sin\theta}.$$

Since θ is a constant angle, $\frac{\tau_{\mathbf{y}}}{\kappa_{\mathbf{y}}}$ =constant. So generating curve $\mathbf{y}$ is helix. □

Theorem 4. *Let $\chi(u, v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are a pair of Bertrand curves. The necessary and sufficient condition for the surface χ to be a minimal surface is that the curve $\mathbf{x}$ is a helix.*

Proof. From Eqs. (5) and (6), we can easily see that

$$\kappa_{\mathbf{y}} = \kappa_{\mathbf{x}}(v)\cos\theta - \tau_{\mathbf{x}}(v)\sin\theta. \tag{37}$$

Using Eqs. (36), (37) and Theorem 2, the following equation can be given

$$\kappa_{\mathbf{x}} - \cos\theta\kappa_{\mathbf{x}} = \tau_{\mathbf{x}}\sin\theta$$

and

$$\frac{\tau_{\mathbf{x}}}{\kappa_{\mathbf{x}}} = \frac{1-\cos\theta}{\sin\theta}.$$

Since θ is a constant angle, $\frac{\tau_{\mathbf{x}}}{\kappa_{\mathbf{x}}}$ =constant. So generating curve $\mathbf{x}$ is helix. □

Example 1. *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be naturally parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\cos\frac{u}{5}, \sin\frac{u}{5}, \frac{\sqrt{24}}{5}u\right).$$

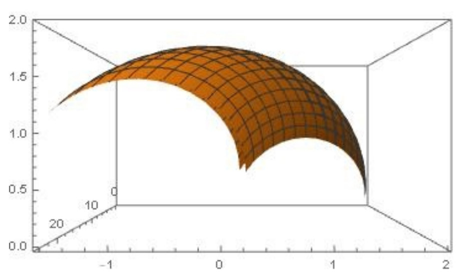
The naturally parametrized curve $\mathbf{y}$ which is the Bertrand partner curve of the $\mathbf{x}$ curve is as follows

$$\mathbf{y}(v) = \left(\frac{24}{25}\cos\frac{v}{5}, \frac{24}{25}\sin\frac{v}{5}, \frac{\sqrt{24}}{5}v\right).$$

The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ Bertrand partner curves is parameterized as follows

$$\chi(u,v) = \left(\cos\frac{u}{5} + \frac{24}{25}\cos\frac{v}{5}, \sin\frac{u}{5} + \frac{24}{25}\sin\frac{v}{5}, \frac{\sqrt{24}}{5}u + \frac{\sqrt{24}}{5}v\right).$$

In Fig. (1), we present the graph of the above translation surface and its generating Bertrand partner curves $\mathbf{x}$ and $\mathbf{y}$.



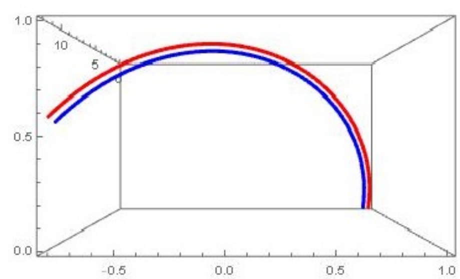


FIGURE 1. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for Bertrand partner curve.

3.2. **Let x and y Mannheim partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the Mannheim partner curves. In this case, from Eq. (7) and (27), the unit normal of the translation surface is

$$\mathbf{n} = \frac{(\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}) \times \mathbf{t_y}}{\|(\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}) \times \mathbf{t_y}\|} = -\mathbf{b_y}. \tag{38}$$

Since the principal normal vector and binormal vector fields of Mannheim curve pairs are linearly dependent, at the same time $\mathbf{n} = -\mathbf{n_x}$. The coefficients of the first fundamental form from Eq. (7) and Eqs. (28)-(30), are as follow

$$\begin{aligned} E &= \langle \chi_u, \chi_u \rangle = 1, \\ F &= \langle \chi_u, \chi_v \rangle = \langle (\cos\theta \mathbf{t_y} + \sin\theta \mathbf{n_y}), \mathbf{t_y} \rangle = \cos\theta, \\ G &= \langle \chi_v, \chi_v \rangle = 1. \end{aligned}$$

The coefficients of the second fundamental form from Eqs. (7),(10) and Eqs. (31)-(33), are obtained as

$$\begin{aligned} e &= \langle \kappa_\mathbf{x} \mathbf{n_x}, -\mathbf{n_x} \rangle = -\kappa_\mathbf{x}, \\ f &= \langle -\kappa_\mathbf{y} \sin\theta \mathbf{t_y} + \kappa_\mathbf{y} \cos\theta \mathbf{n_y} + \tau_\mathbf{y} \sin\theta \mathbf{b_y}, -\mathbf{b_y} \rangle = -\tau_\mathbf{y} \sin\theta, \\ g &= \langle \kappa_\mathbf{y} \mathbf{n_y}, \mathbf{b_y} \rangle = 0. \end{aligned}$$

If we calculate the Gaussian and mean curvatures of translation surfaces, whose generating curves are Mannheim partner curves, from Eqs. (19) and (20), we have

$$K = \frac{eg - f^2}{EG - F^2} = -\tau_\mathbf{y}^2 \tag{39}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_\mathbf{x} + \tau_\mathbf{y} \sin 2\theta}{2\sin^2\theta}. \tag{40}$$

Theorem 5. *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Mannheim partner curves to be developable surfaces, the necessary sufficient condition is that the curve $\mathbf{y}$ is a planar curve.*

Proof. It is easily seen from Eq. (39) and Theorem 1 that $\tau_\mathbf{y} = 0$. This means that the curve $\mathbf{y}$ is a planar curve. □

Theorem 6. *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be a translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curve. For translation surfaces, whose generating curves are Mannheim partner curves to be minimal surfaces, the necessary sufficient condition is that the curve $\mathbf{y}$ is a planar curve or $v = c_1 u + c_2,\ c_1, c_2 \in \mathbb{R}$.*

Proof. From Eqs. (10), (40) and Theorem 2, the following equation can be given

$$\tau_\mathbf{y} \sin 2\theta = \tau_\mathbf{y} \sin\theta \frac{dv}{du}$$

and

$$2\tau_{\mathbf{y}}\cos\theta = \tau_{\mathbf{y}}\frac{dv}{du}.$$

Here $\tau_{\mathbf{y}} = 0$ is an obvious solution. So $\mathbf{y}$ is a planar curve. Let $\tau_{\mathbf{y}} \neq 0$ then, we get

$$2\cos\theta \int du = \int dv.$$

If $2\cos\theta = c_1, c_1 \in \mathbb{R}$ is selected here, we obtain

$$v = c_1 u + c_2,\ \ c_1, c_2 \in \mathbb{R}.$$

□

Example 2. *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be arbitrary parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\frac{8}{5}\cos u, \frac{8}{5}\sin u, \frac{4}{5}u\right).$$

The arbitrary parametrized curve $\mathbf{y}$ which is the Mannheim partner curve of the curve $\mathbf{x}$ is as follows

$$\mathbf{y}(v) = \left(-\frac{8}{5}(\sin v + \cos v), \frac{8}{5}(\sin v + \cos v), \frac{4}{5}v\right).$$

The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ Mannheim partner curves is parameterized as follows

$$\chi(u,v) = \left(\frac{8}{5}\cos u - \frac{8}{5}(\sin v + \cos v), \frac{8}{5}\sin u + \frac{8}{5}(\sin v + \cos v), \frac{4}{5}u + \frac{4}{5}v\right).$$

In Fig. (2), we present the graph of the above translation surface and its generating Mannheim partner curves $\mathbf{x}$ and $\mathbf{y}$.

3.3. **Let x and y involute-evolute partner curves.** Let the curves $\mathbf{x}$ and $\mathbf{y}$, which are the generating curves of the translation surface parameterized by Eq. (1), be the involute-evolute partner curves. So, from Eq. (12) and (27), the unit normal of the translation surface is

$$\mathbf{n} = \frac{\mathbf{n_y} \times \mathbf{t_y}}{\|\mathbf{n_y} \times \mathbf{t_y}\|} = -\mathbf{b_y}. \tag{41}$$

The coefficients of the first fundamental form from Eq. (12) and Eqs. (28)-(30), are as follows

$$\begin{aligned} E &= \langle \chi_u, \chi_u \rangle = 1, \\ F &= \langle \chi_u, \chi_v \rangle = \langle \mathbf{n_y}, \mathbf{t_y} \rangle = 0, \\ G &= \langle \chi_v, \chi_v \rangle = 1. \end{aligned}$$

If we calculate the coefficients of the second fundamental form from Eqs. (12),(13),(15) and Eqs. (31)-(33), we can easily see that

$$e = \langle \kappa_{\mathbf{x}} \mathbf{n_x}, -\mathbf{b_y} \rangle = -\kappa_{\mathbf{x}} \sin\theta,$$

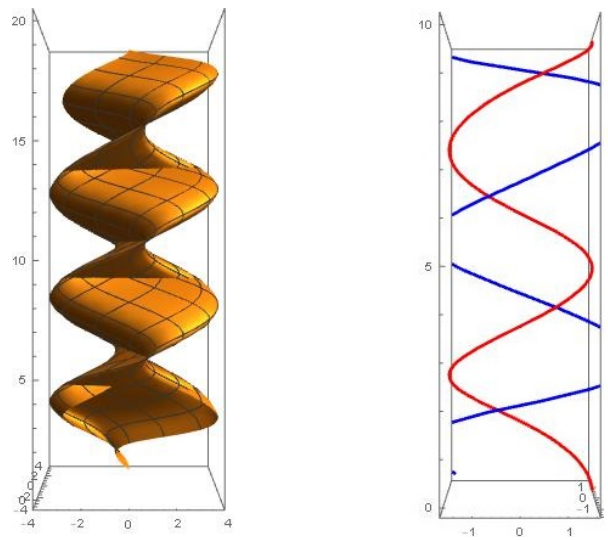


FIGURE 2. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for Mannhiem partner curve.

$$\begin{aligned} f &= \langle -\kappa_{\mathbf{y}}\mathbf{t}_{\mathbf{y}} + \tau_{\mathbf{y}}\mathbf{b}_{\mathbf{y}}, -\mathbf{b}_{\mathbf{y}} \rangle = -\tau_{\mathbf{y}}, \\ g &= \langle \kappa_{\mathbf{y}}\mathbf{n}_{\mathbf{y}}, \mathbf{b}_{\mathbf{y}} \rangle = 0. \end{aligned}$$

The Gaussian and mean curvatures of translation surfaces, whose generating curves are involute-evolute partner curves are obtained from Eqs. (19) and (20), as follows

$$K = \frac{eg - f^2}{EG - F^2} = -\tau_{\mathbf{y}}^2 \tag{42}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{-\kappa_{\mathbf{x}} \sin\theta}{2}. \tag{43}$$

Theorem 7. *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are the involute-evolute partner curves. The necessary and sufficient condition for the surface χ to be developable surface is that the curve $\mathbf{y}$ is a planar curve.*

Proof. It is easily seen from Eq. (42) and Theorem 1 that $\tau_{\mathbf{y}} = 0$. This means that the curve $\mathbf{y}$ is a planar curve. □

Theorem 8. *Let $\chi(u,v) = \mathbf{x}(u) + \mathbf{y}(v)$ be translation surface where $\mathbf{x}$ and $\mathbf{y}$ are generating curves. Suppose that the generating curves are the involute-evolute partner curves. In this case, the translation surface χ cannot be a minimal surface.*

Proof. Since $\kappa_{\mathbf{x}} \neq 0$, considering Eq. (43), it is seen that $H \neq 0$. Therefore, such translation surfaces cannot be minimal. $\square$

Example 3. *Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be arbitrary parametrized curve in $\mathbb{R}^3$ parameterized by*

$$\mathbf{x}(u) = \left(\frac{8}{5}\cos u, \frac{8}{5}\sin u, \frac{4}{5}u\right).$$

The arbitrary parametrized curve $\mathbf{y}$ involute partner curve of the $\mathbf{x}$ curve is as follows

$$\mathbf{y}(v) = \left(\frac{8}{5}\cos v - \frac{2}{5}\sin v + \frac{2}{5}v\sin v, \frac{8}{5}\sin v + \frac{2}{5}\cos v - \frac{2}{5}v\cos v, \frac{3}{5}v\right).$$

The translation surface generating by the $\mathbf{x}$ and $\mathbf{y}$ involute-evolute partner curves is parameterized as follows

$$\chi(u,v) = \left(\tfrac{8}{5}\cos u + \tfrac{8}{5}\cos v - \tfrac{2}{5}\sin v + \tfrac{2}{5}v\sin v, \tfrac{8}{5}\sin u + \tfrac{8}{5}\sin v + \tfrac{2}{5}\cos v2\tfrac{2}{5}v\cos v, \tfrac{4}{5}u + \tfrac{3}{5}v\right).$$

In Fig. (3), we present the graph of the above translation surface and its generating involute-evolute partner curves $\mathbf{x}$ and $\mathbf{y}$.

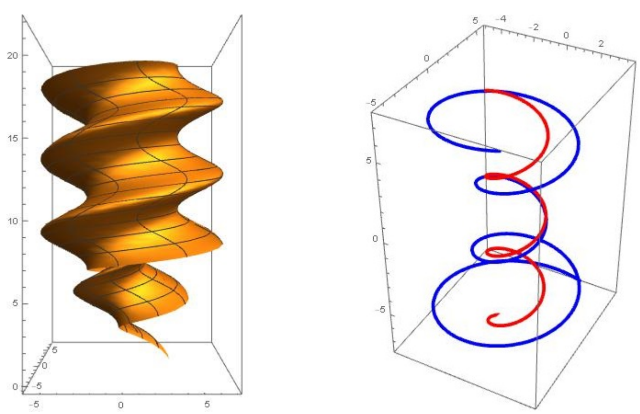


FIGURE 3. Translation surface and its generating curves $\mathbf{x}$(Red) and $\mathbf{y}$(Blue) for involute-evolute partner curves.

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