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Research Article

Generalized product-type operators between Bloch-type spaces

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Abstract

In this paper, we consider generalized product type operators $D^n uC_\phi$ and $T^n_{u_1,u_2,\phi}$. Then we provide several characterizations, as equivalent statements, for the boundedness and compactness of these operators between Bloch type spaces $\mathcal{B}_\alpha(\mathbb{U})$, for all $0 < \alpha < \infty$.

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1. Introduction

Let $\mathbb{U} = \{w \in \mathbb{C} : |w| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$ and the space of all analytic functions on $\mathbb{U}$ denoted by $\mathcal{H}(\mathbb{U})$. Let $0 < \alpha < \infty$, as usual, the weighted type space $H^\infty_\alpha(\mathbb{U})$, is a Banach space of bounded functions $h \in \mathcal{H}(\mathbb{U})$, such that $\sup_{z\in\mathbb{U}}(1 - |z|^2)^\alpha |h(z)| < \infty$. Also $\mathcal{W}^n_\alpha(\mathbb{U})$, denoted the space of all analytic functions $h \in \mathcal{H}(\mathbb{U})$, whose

$$\|h^{(n)}\|_\alpha = \sup_{z\in\mathbb{U}}(1 - |z|^2)^\alpha |h^{(n)}(z)|,$$

be finite. These spaces become Banach spaces, under the following norm.

$$\|h\|_{\mathcal{W}^n_\alpha(\mathbb{U})} = \sum_{j=0}^{n-1} |h^{(j)}(0)| + \|h^{(n)}(z)\|_\alpha < \infty.$$

For more information about $\mathcal{W}^n_\alpha$ spaces, we refere to [17,19]. For $n = 1, 2$, space $\mathcal{W}^n_\alpha(\mathbb{U})$ is called the Bloch type space $\mathcal{B}_\alpha(\mathbb{U})$ and the Zygmund type space $\mathcal{Z}_\alpha(\mathbb{U})$, respectively. More information about the Bloch type and Zygmund type spaces can be found in [18,19,25,26].

Let $\mathcal{S} = \mathcal{S}(\mathbb{U})$ be the class of all holomorphic self-maps of $\mathbb{U}$. Then for a nonnegative integer n, map $\phi \in \mathcal{S}$ and $u \in \mathcal{H}(\mathbb{U})$, the generalized product type operator $D^n uC_\phi$ is given by

$$D^n uC_\phi h(z) = \Big(u(z) \cdot h \circ \phi(z)\Big)^{(n)}, \quad z \in \mathbb{U}.$$

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For $n = 0$, we get the well-known weighted composition operator $u \cdot h \circ \phi$ (see [11–13, 21]). For $n = 1$, we have the derivative of the weighted composition operator which has been studied between Bloch type space, by Hyvarinen and Nieminen in [6]. If $n = 1$ and $u = 1$, then $D^n uC_\phi = DC_\phi$. Li and Stevic' in [9], characterized the boundedness and compactness of operator DC_ϕ between Bloch type spaces in term of the growth of the first and second derivatives of ϕ. Also In [23] Wu and Wulan investigated the compactness of operator DC_ϕ between classical Bloch space $\mathcal{B}$.
Operator $uC_\phi D^n$ which defines as $uC_\phi D^n h(z) = u(z)h^{(n)}\phi(z)$ is another generalization for weighted composition operator. Recently there has been great interest in studing this operator, see for example [10, 19].
In this paper we also consider the following sum of operators, as the generalized Stević-Sharma operator

$$T^n_{u_1,u_2,\phi}h := D^{n-1}u_1C_\phi(h) + u_2C_\phi D^n(h) = \Big(u_1 \cdot h \circ \phi\Big)^{(n-1)} + u_2h^{(n)} \circ \phi. \tag{1.1}$$

when $n = 1$, then we have the well-known stević-Sharma operator $T_{u_1,u_2,\phi}$, We refere to [14, 20, 24], for more information about the stević-Sharma operators on spaces of analytic functions.
The main concern of the present paper is to discuss the boundedness and compactness of the generalized operator $D^n uC_\phi$ between Bloch type space $\mathcal{B}_\alpha(\mathbb{U})$. We break this problem to three different cases $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$ and obtain several characterizations in terms of the powers of $\phi^{(n)}$ and integral operators. We also investigate the boundedness and compactness of the generalized operator $T^n_{u_1,u_2,\phi}$ between Bloch type spaces.

For $n, k \in \mathbb{N}_0$, with condition $k \leq n$, the partial Bell polynomials are triangular

$$B_{n,k}(x_1, ..., x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!...j_{n-k+1}!}(\frac{x_1}{1!})^{j_1}(\frac{x_2}{2!})^{j_2}...(\frac{x_{n-k+1}}{(n-k+1)!})^{j_{n-k+1}},$$

where the sum is taken over all sequences $j_1, j_2, ..., j_{n-k+1}$ of nonnegative integers such that the following two conditions hold: $j_1 + j_2 + ... + j_{n-k+1} = k$, and $j_1 + 2j_2 + ... + (n - k + 1)j_{n-k+1} = n$. Also from the definition, we see that, for example

$$\begin{array}{ll} B_{0,0}(x_1) = 1, & B_{1,1}(x_1) = x_1 \\ B_{1,0}(x_1, x_2) = 0, & B_{n,n}(x_1) = x_1^n \\ B_{n,0}(x_1, x_2, ..., x_{n+1}) = 0, & B_{n,1}(x_1, ..., x_n) = x_n. \end{array} \tag{1.2}$$

For more information about the Bell polynomials, see the standard references [3, 8, 16]. In [17], Stević applied the Bell polynomials for representing the n'th derivative of a composition operator on analytic function spaces, as follows

$$(h \circ \phi)^{(n)}(z) = h'(\phi(z))\phi^{(n)}(z) + \sum_{k=2}^{n-1} h^{(k)}(\phi(z))B_{n,k}(\phi', ..., \phi^{(n-k+1)})(z) + h^{(n)}(\phi(z))(\phi'(z))^n. \tag{1.3}$$

Later, in [19],by assuming that

$$S^u_{n,i}(z) = \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z)B_{l,i}(\phi'(z), \phi''(z), ..., \phi^{(l-i+1)}(z)), \tag{1.4}$$

the n'th derivative of a weighted composition operator, by using (1.3), has been obtained by Stević as follows:

Lemma 1.1 ([19]). *Let $f, u \in \mathcal{H}(\mathbb{U})$ and $\phi \in \mathcal{S}$. Then for any $n \in \mathbb{N}_0$,*

$$D^n(uC_\phi(h(z))) = \sum_{i=0}^{n} h^{(i)}(\phi(z)) \sum_{l=i}^{n} \binom{n}{l} u^{(n-1)}(z) B_{l,i}(\phi'(z), \ldots \phi^{(l-i+1)}(z))$$
$$= \sum_{i=0}^{n} S^u_{n,i}(z) \cdot h^{(i)} \circ \phi(z) = \sum_{i=0}^{n} S^u_{n,i}(z) \cdot (C_\phi D^i h(z)). \tag{1.5}$$

After that, so many authors have used different forms of formula (1.3) and lemma 1.1, in their studies. For example, Zhi-Jie Jiang in [7], by using lemma 1.1, characterized the boundedness and compactness of product type operator $D^n C_\phi M_u$ on Nevanlinna spaces. In [1], lemma 1.1, has been used to characterize the boundedness and compactness of operator uC_ϕ on weighted type spaces of analytic functions. We also use lemma 1.1, to get our results.

Throughout this paper, if there exists a positive constant C such that $A \leq CB$, then we write $A \preceq B$. Also the symbol $A \approx B$ means that $A \preceq B \preceq A$.

2. Boundedness

In this section, we characterize the boundedness of operator $D^n uC_\phi$ from Bloch type space $\mathcal{B}_\alpha(\mathbb{U})$ into $\mathcal{B}_\beta(\mathbb{U})$. To this end, we consider the cases $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$, separately. We also investigate the boundedness of operator $T^n_{u_1,u_2,\phi} : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$.
For our purpose, we need some lemmas as follows.

Lemma 2.1. *Let $\alpha > 0$ and $h \in \mathcal{B}_\alpha(\mathbb{U})$. Then we have the following conditions:*

(a) $|h(z)| \lesssim \|h\|_{\mathcal{B}_\alpha(\mathbb{U})}$, *for* $0 < \alpha < 1$ *and* $z \in \mathbb{U}$.

(b) $|h(z)| \lesssim \frac{\|h\|_{\mathcal{B}_\alpha(\mathbb{U})}}{(1-|z|^2)}$, *for* $\alpha = 1$ *and* $z \in \mathbb{U}$.

(c) $|h(z)| \lesssim \frac{1}{(1-|z|^2)^{\alpha-1}} \|h\|_{\mathcal{B}_\alpha(\mathbb{U})}$, *for* $\alpha > 1$ *and* $z \in \mathbb{U}$.

(d) $|h^{(n)}(z)| \lesssim \frac{\|h\|_{\mathcal{B}_\alpha(\mathbb{U})}}{(1-|z|^2)^{\alpha+n-1}}$, *for* $n \in \mathbb{N}$, $\alpha > 0$ *and* $z \in \mathbb{U}$.

Proof. We refer the proof of parts (a), (c) to Lemma 2.1 [22] and part (d) to [26]. Proof of part (b), can also be found in the details of proof of proposition 2.1[1]. □

Lemma 2.2 ([25]). *Let $\alpha > 1$ then the space $\mathcal{B}_\alpha$ can be identified with the space of analytic functions h with*

$$\sup_{z\in\mathbb{U}} (1 - |z|^2)^{\alpha-1} |h(z)| < \infty.$$

In the sequel we deal with the two following integral operators for $f \in \mathcal{H}(U)$ and $z \in \mathbb{U}$,

$$I_u h(z) := \int_0^z h'(w)u(w)dw, \qquad and \quad J_u h(z) := \int_0^z h(w)u'(w)dw.$$

Next theorem, gives us a characterization for boundedness of operator DuC_ϕ between Bloch type spaces, in term of the integral operators.

Theorem 2.3 ([6]). *Let $u \in \mathcal{H}(U)$, $\phi \in \mathcal{S}$, $\alpha > 1$ and $0 < \beta < \infty$. Then $DuC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded if and only if*

$$H^\beta_{u,\phi} := \sup_{n\geq 1} n^\alpha \|I_{u\phi'}(\phi^n)\|_{\mathcal{B}_\beta} < \infty,$$
$$L^\beta_{u,\phi} := \sup_{n\geq 1} n^{\alpha-1} \|I_{u'}(\phi^n) + nJ_{u\phi'}(\phi^{n-1})\|_{\mathcal{B}_\beta} < \infty,$$
$$I^\beta_{u,\phi} := \sup_{n\geq 1} n^{\alpha-1} \|J_{u'}(\phi^{n-1})\|_{\mathcal{B}_\beta} < \infty. \tag{2.1}$$

Now we consider the case $0 < \alpha < 1$, and provide some characterizations for the boundedness of generalized operator $D^n uc_\phi$ from $\mathcal{B}_\alpha(\mathbb{U})$ into $\mathcal{B}_\beta(\mathbb{U})$.

Theorem 2.4. *Let $n \in \mathbb{N}$, $0 < \alpha < 1$, $\beta > 0$, $u \in \mathcal{H}(\mathbb{U})$ and $\phi \in \mathcal{S}$. Then the following conditions are equivalent:*

(a) *Operator $D^n uC_\phi : \mathcal{B}_\alpha(\mathbb{U}) \mapsto \mathcal{B}_\beta(\mathbb{U})$ is bounded.*

(b) *$u \in \mathcal{W}_\beta^{n+1}$ and*

$$sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} < \infty, \qquad for \quad i \in \{1, ..., n+1\}$$

(c) *$u \in \mathcal{W}_\beta^{n+1}$ and operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{\alpha+i}(\mathbb{U}) \mapsto \mathcal{B}_{\beta+2}(\mathbb{U})$ for any $i \in \{1, ..., n+1\}$, is bounded.*

(d) *$u \in \mathcal{W}_\beta^{n+1}$ and for any $i \in \{1, ..., n+1\}$, the quantities of $H^{\beta+2}_{S^u_{n+1,i},\phi}$, $L^{\beta+2}_{S^u_{n+1,i},\phi}$ and $I^{\beta+2}_{S^u_{n+1,i},\phi}$ are finite, as they have been defined in* (2.1).

Proof. (a)$\Longrightarrow$(b). If $D^n uC_\phi : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ be bounded, then by putting $G_0(z) = 1$ and using (1.2), (1.4) and lemma (1.1), we obtain

$$\begin{aligned}\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |u^{(n+1)}(z)| &= \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,0}(z)| = \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |D^{n+1}uC_\phi G_0| \\ &= \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |(D^n uC_\phi G_0)'| \preceq \|D^n uC_\phi G_0\|_{\mathcal{B}_\beta} < \infty\end{aligned}$$

Therefore, $u \in \mathcal{W}_\beta^{n+1}$. Also by the details of the above relations,

$$\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,0}(z)| < \infty. \tag{2.2}$$

In a similar way, for $G_1(z) = z$, we get that

$$\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |\phi(z)S^u_{n+1,0}(z) + S^u_{n+1,1}(z)| \preceq \|D^n uC_\phi G_1\|_{\mathcal{B}_\beta} < \infty. \tag{2.3}$$

also, by an application the triangle inequality, we obtain

$$|S^u_{n+1,1}(z)| \leq |S^u_{n+1,1}(z) + \phi(z)S^u_{n+1,0}(z)| + |\phi(z)S^u_{n+1,0}(z)|. \tag{2.4}$$

But we assumed that $\phi \in \mathcal{S}$, then $|\phi(z)| < 1$ for any $z \in \mathbb{U}$. Therefore relations (2.2), (2.3) and (2.4) give us

$$\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,1}(z)| \leq \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,1}(z) + \phi(z)S^u_{n+1,0}(z)| \tag{2.5}$$

$$+ \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |\phi(z)||S^u_{n+1,0}(z)| < \infty. \tag{2.6}$$

Now, if we assume that $\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,i}(z)| < \infty$, for $1 \leq i < n+1$. By applying $D^n uC_\phi$ to functions $G_j = z^j$, we have that

$$\begin{aligned}&\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |(\phi^j(z)S^u_{n+1,0}(z) + \sum_{i=1}^{n+1}(j)...(j-i+1)(\phi(z))^{j-i}S^u_{n+1,i}(z))| \\ &\quad \preceq \|D^n uC_\phi G_j|_{\mathcal{B}_\beta} < \infty.\end{aligned}$$

So by the triangle inequality, we get that $\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,n+1}(z)| < \infty$. In this way, we have shown that for $i = 1, 2, ..., n+1$,

$$\sup_{z\in\mathbb{U}}(1-|z|^2)^\beta |S^u_{n+1,i}(z)| < \infty,$$

Therefore, for any arbitrary $a \in \mathbb{U}$ and $i \in \{1, ..., n+1\}$,

$$\sup_{|\phi(a)|\leq\frac{1}{2}} \frac{(1-|a|^2)^\beta |S^u_{n+1,i}(a)|}{(1-|\phi(a)|^2)^{t+\alpha-1}} \preceq \sup_{|\phi(a)|\leq\frac{1}{2}} (1-|a|^2)^\beta |S^u_{n+1,i}(a)| < \infty. \tag{2.7}$$

Now, for a fix $0 \neq a \in \mathbb{U}$ and $m \in \{1, 2, ..., n+2\}$, define the analytic functions

$$H^{\alpha}_{m,a}(z) = \frac{(1-|a|^2)^m}{(1-\overline{a}z)^{m+\alpha-1}}, \tag{2.8}$$

we see that $H'^{\alpha}_{m,\phi(a)}(z) = (m+\alpha-1)\overline{\phi(a)}H^{\alpha+1}_{m,\phi(a)}(z)$. So $H^{\alpha}_{m,\phi(a)} \in \mathcal{B}_\alpha(U)$, for each $m \in \{1, ..., n+2\}$. Then similar to the proof of Theorem1 [27], we can consider analytic functions $K_{m,\phi(a)} \in \mathcal{B}_\alpha(U)$, such that

$$K_{m,\phi(a)}(z) = \sum_{j=1}^{n+1} \frac{C_j H^{\alpha}_{m,\phi(a)}(z)}{\prod_{p=1}^{m-1}(j+\alpha-1+p)}, \tag{2.9}$$

for $m \in \{0, ..., n+1\}$. Therefore,

$$K^{(t)}_{m,\phi(a)}(\phi(a)) = \begin{cases} (\frac{\overline{\phi(a)}^t}{(1-|\phi(a)|^2)^{t+\alpha-1}}, & t = m \\ 0, & t \neq m. \end{cases}$$

Then the boundedness of $D^n uC_\phi$ on $\mathcal{B}_\alpha(U)$, implies that $\sup_{a\in\mathbb{D}} \|D^n uC_\phi H^{\alpha}_{m,\phi(a)}\|_{\mathcal{B}_\beta} < \infty$. Hence, if $\phi(a) \neq 0$, by applying (1.4) and lemma1.1, we get

$$\begin{aligned}\frac{(1-|a|^2)^\beta |\phi(a)|^m |S^u_{n+1,m}(a)|}{(1-|\phi(a)|^2)^{m+\alpha-1}} &\leq \sup_{a\in\mathbb{U}} \|D^n uC_\phi K_{m,\phi(a)}(\phi(a))\|_{\mathcal{B}_\beta} \\ &\leq \sum_{j=1}^{n+1} \frac{|c^i_j|}{\prod_{p=1}^{m-1}(j+\alpha+1+p)} \sup_{a\in\mathbb{U}} \|D^n uC_\phi H^{\alpha}_{m,\phi(a)}\|_{\mathcal{B}^\beta} < \infty,\end{aligned}$$

where c^i_j are independent of the choice of a. Therefore, for $i \in \{1, ..., n+1\}$,

$$\sup_{|\phi(a)|>\frac{1}{2}} \frac{(1-|a|^2)^\beta |S^u_{n+1,i}(a)|}{(1-|\phi(a)|^2)^{i+\alpha-1}} < \infty. \tag{2.10}$$

So (2.7) and (2.10), imply that $\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} < \infty$, for $i \in \{1, ..., n+1\}$.
(b)$\Longrightarrow$ (a). Suppose that $h \in \mathcal{B}_\alpha$, by using lemma 1.1 and relation (1.4), we obtain

$$(1-|z|^2)^\beta |(D^n uC_\phi f)'(z)| = (1-|z|^2)^\beta |\sum_{i=0}^{n+1} f^{(i)}(\phi(z)) S^u_{n+1,i}(z)|$$

But $B_{0,0}(\phi'(z)) = 1$ and for $p \in \mathbb{N}$, $B_{p,0}(\phi'(z), ...\phi^{(p+1)}(z)) = 0$, therefore, $S^u_{n+1,0}(z) = u^{(n+1)}(z)$. Then applying lemma2.1, gives us

$$\begin{aligned}(1-|z|^2)^\beta |(D^n uC_\phi f)'(z)| &\leq (1-|z|^2)^\beta \Big(|f(\phi(z))||u^{(n+1)}(z)| + \sum_{i=1}^{n+1} |f^{(i)}(\phi(z))||S^u_{n+1,i}(z)|\Big) \\ &\leq \|f\|_{\mathcal{B}_\alpha} \|u\|_{\mathcal{W}^{n+1}_\beta} + \|f\|_{\mathcal{B}_\alpha} \sum_{i=1}^{n+1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)}{(1-|\phi(z)|^2)^{i+\alpha-1}}. \end{aligned} \tag{2.11}$$

Also for each $j \in \{1, ..., n+1\}$, we have

$$\begin{aligned}|(D^j uC_\phi h)(0)| &= |f \circ \phi(0)||u^{(j)}(0)| + |\sum_{i=1}^{j} f^{(i)}(\phi(0)) S^u_{j,i}(0)| \\ &\preceq \|f\|_{\mathcal{B}_\alpha} |u^{(j)}(0)| + \|f\|_{\mathcal{B}_\alpha} \sum_{i=1}^{j} \frac{|S^u_{j,i}(0)|}{(1-|\phi(0)|^2)^{i+\alpha-1}}. \end{aligned} \tag{2.12}$$

Therefore (2.11) and (2.12) along with the assumptions in part (b), give us the boundedness of operator $D^n uC_\phi$ from Bloch type space $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$.

(b)$\Longleftrightarrow$ (c). According to lemma 2.2, we get that spaces $\mathcal{B}_{\beta+1}$ and $\mathcal{H}^{\infty}_{\beta}$ are norm-equivalent. Also $Dh \in \mathcal{B}_{\beta+2}$ if and only if $h \in \mathcal{B}_{\beta+1}$. So operator $D(S^u_{n+1,i}C_\phi) : \mathcal{B}_{\alpha+i} \mapsto \mathcal{B}_{\beta+2}$ is bounded if and only if operator $S^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \mapsto \mathcal{H}^{\infty}_{\beta}$ be bounded. On the other hand, the space $\mathcal{B}_{\alpha+i}$ can be identified by $\mathcal{H}^{\infty}_{\alpha+i-1}$. Hence we get the boundedness of operator $S^u_{n+1,i}C_\phi : \mathcal{H}^{\infty}_{\alpha+i-1} \mapsto \mathcal{H}^{\infty}_{\beta}$ and it is straightforward to see that it happens, if and only if $\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{\alpha+i-1}} < \infty$.
(c)$\Longleftrightarrow$(d). It is clear, by applying theorem 2.3. □

In the two next theorems, we consider the cases $\alpha = 1$ and $\alpha > 1$. Proofs are similar to theorem (2.4), so we skip the details.

Theorem 2.5. *Let $n \in \mathbb{N}$, $\alpha = 1$, $\beta > 0$, $u \in \mathcal{H}(\mathbb{U})$ and $\phi \in \mathcal{S}$. Then the following conditions are equivalent:*

(a) *Operator $D^n uC_\phi : \mathcal{B}(\mathbb{U}) \mapsto \mathcal{B}_\beta(\mathbb{U})$ is bounded.*
(b) *For each $i \in \{1, ..., n+1\}$,*

$$\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |u(z)^{(n+1)}|}{(1-|\phi(z)|^2)} < \infty, \qquad \sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} < \infty.$$

(c) *For any $i \in \{1, ..., n+1\}$, operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{1+i}(\mathbb{U}) \mapsto \mathcal{B}_{\beta+2}(\mathbb{U})$ is bounded, also*

$$\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |u(z)^{(n+1)}|}{(1-|\phi(z)|^2)} < \infty$$

(d) *For $i \in \{1, ..., n+1\}$, the quantities of $H^{\beta+2}_{S^u_{n+1,i},\phi}$, $L^{\beta+2}_{S^u_{n+1,i},\phi}$ and $I^{\beta+2}_{S^u_{n+1,i},\phi}$ are finite. Also*

$$\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |u(z)^{(n+1)}|}{(1-|\phi(z)|^2)} < \infty.$$

Theorem 2.6. *Let $\alpha > 1$, $0 < \beta < \infty$, $u \in \mathcal{H}(U)$ and $\phi \in \mathcal{S}$. Then we have the following equivalent conditions:*

(a) *Operator $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded.*
(b) *For each $i \in \{0, 1, ..., n+1\}$, we have that*

$$\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} < \infty.$$

(c) *For any $i \in \{0, ..., n+1\}$, operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{\alpha+i}(\mathbb{U}) \mapsto \mathcal{B}_{\beta+2}(\mathbb{U})$ is bounded.*
(d) *For each $i \in \{0, ..., n+1\}$, the quantities of $H^{\beta+2}_{S^u_{n+1,i},\phi}$, $L^{\beta+2}_{S^u_{n+1,i},\phi}$ and $I^{\beta+2}_{S^u_{n+1,i},\phi}$ are finite.*

Corollary 2.7. *Let $0 < \alpha$, $0 < \beta < \infty$, $\phi \in \mathcal{S}$ and $u \in \mathcal{H}(U)$. Then the following statements are equivalent:*

(a) *Operator $uC_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded.*
(b) *Operator $DuC_\phi : \mathcal{B}_{\alpha+n} \mapsto \mathcal{B}_{\beta+1}$ is bounded.*
(c) *The quantities of $\Lambda^u_1, \Lambda^u_2, \Lambda^u_3$ are finite, when*

$$\Lambda^u_1 := \sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^{\beta+1} u''(z)}{(1-|\phi(z)|^2)^{\alpha+n-1}}, \quad \Lambda^u_2 := \sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^{\beta+1}(2u'(z)\phi'(z) + u\phi''(z))}{(1-|\phi(z)|^2)^{\alpha+n}},$$

$$\Lambda^u_3 = \sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^{\beta+1} u(z)\phi'^2(z)}{(1-|\phi(z)|^2)^{\alpha+n+1}}.$$

Proof. (a) $\iff$ (b). It is clear by applying $f \in \mathcal{B}_\alpha(\mathbb{U})$ if and only if $f^{(n)} \in \mathcal{B}_{\alpha+n}(\mathbb{U})$.
(b) $\iff$(c). It is a result of theorem (2.6) in case $n = 1$. Since by putting $n = 1$ and $i = 0, 1, 2$ in definition of $S^u_{n+1,i}$, we get that

$$\begin{aligned}
S^u_{2,0} &= \sum_{l=0}^{2} \binom{2}{l} u^{(2-l)}(z) B_{l,0}(\phi'(z), ...\phi^{(l+1)}(z)) \\
&= u''(z)B_{0,0}(\phi'(z)) + 2u'(z)B_{1,0}(\phi'(z), \phi''(z)) + u(z)B_{2,0}(\phi'(z), \phi''(z), \phi'''(z)) \\
&= u''(z) \cdot 1 + 2u'(z) \cdot 0 + u(z) \cdot 0 = u''(z), \\
S^u_{2,1} &= \sum_{l=1}^{2} \binom{2}{l} u^{(2-l)}(z) B_{l,1}(\phi'(z), ...\phi^{(l+1)}(z)) \\
&= 2u'(z)B_{1,1}(\phi'(z)) + u(z)B_{2,1}(\phi'(z), \phi''(z)) = 2u'(z)\phi'(z) + u(z)\phi''(z), \\
S^u_{2,2} &= \sum_{l=2}^{2} \binom{2}{l} u^{(2-l)}(z) B_{l,2}(\phi'(z), ...\phi^{(l+1)}(z)) = u(z)\phi'^2(z).
\end{aligned}$$

□

Example 2.8. Let $\alpha = \beta = 1$, $u(z) = z - 2$ and $\phi(z) = \frac{z-1}{2}$, for every $z \in \mathbb{U}$. Then $\phi \in \mathcal{S}$, $u \in \mathcal{H}(\mathbb{U})$, also we have that $u'(z) = 1, \phi'(z) = \frac{1}{2}, u''(z) = 0$ and $\phi''(z) = 0$. Now if we consider the case $n = 1$ in corollary 2.7, simple calculations give us

$$\Lambda^u_1 = 0, \quad \Lambda^u_2 = \sup_{z\in\mathbb{U}} \frac{(1 - |z|^2)}{(1 - |\frac{z-1}{2}|^2)^2} = \infty, \quad \Lambda^u_3 = \sup_{z\in\mathbb{U}} \frac{(1 - |z|^2)(z - 2)(1/4)}{(1 - |\frac{z-1}{2}|^2)^3} = \infty.$$

Therefore, according to corollary 2.7, we can conclude that operators $uC_\phi D : \mathcal{B} \mapsto \mathcal{B}$ and $DuC_\phi : \mathcal{B}_2 \mapsto \mathcal{B}_2$ are not bounded .

By applying the definition of Bloch type and Zygmund type spaces and using theorem 2.5 in the case of $n = 1$, we get the following corollary.

Corollary 2.9. *Let $\phi \in \mathcal{S}$ and $u \in \mathcal{H}(U)$. Then the following statements are equivalent:*

(a) *Operator $DuC_\phi : \mathcal{B} \mapsto \mathcal{B}$ is bounded.*
(b) *Operator $uC_\phi : \mathcal{B} \mapsto \mathcal{Z}$ is bounded.*
(c) *Operator $uC_\phi D : \mathcal{Z} \mapsto \mathcal{Z}$ is bounded.*
(d) *The quantities of L_1, L_2, L_3 are finite, when*

$$L_1 := \sup_{z\in\mathbb{U}} \frac{(1 - |z|^2)|u''(z)|}{(1 - |\phi(z)|^2)}, \quad L_2 := \sup_{z\in\mathbb{U}} \frac{(1 - |z|^2)|u(z)\phi'(z)^2|}{(1 - |\phi(z)|^2)^2},$$
$$L_3 := \sup_{z\in\mathbb{U}} \frac{(1 - |z|^2)|2u'(z)\phi'(z) + u(z)\phi''(z)|}{1 - |\phi(z)|^2}.$$

The equivalency between parts (b) and (d) of the above corollary, shows that in the case $n = 1$, our results reduce to the ones obtained by Colona and Li in [2]. Also by the equivalency between parts (c) and (d) of the above consequence, we have the results obtained by the authors in [5].

Theorem 2.10. *Let $0 < \alpha < 1$, $0 < \beta < \infty$, $\phi \in \mathcal{S}$, $u \in \mathcal{H}(U)$ and $\Lambda^{u_2}_i < \infty$, for $i \in \{1, 2, 3\}$. Then the following statements are equivalent:*

(a) *Operator $T^n_{u_1,u_2,\phi} : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded.*
(b) *$u_1 \in \mathcal{W}^{n+1}_\alpha$ and for any $i \in \{1, ..., n + 1\}$, $\sup_{z\in\mathbb{U}} \frac{(1-|z|^2)^{\beta+1}|S^{u_1}_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} < \infty$.*
(c) *$u_1 \in \mathcal{W}^{n+1}_\alpha$ and for any $i \in \{1, ..., n + 1\}$, the quantities of $H^{\beta+3}_{u_1,\phi}$, $L^{\beta+3}_{u_1,\phi}$ and $I^{\beta+3}_{u_1,\phi}$ are finite.*

Proof. (a)$\Longrightarrow$(b). Let $T^n_{u_1,u_2,\phi} = D^{n-1}u_1C_\phi + u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}^\beta$ be bounded. By assuming that $\Lambda_i^{u_2} < \infty$ for $i \in \{1,2,3\}$, according to theorem (2.7), we have the boundedness of operator $u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$. But

$$\|D^{n-1}u_1C_\phi\|_{\mathcal{B}_\beta} \preceq \|D^{n-1}Tu_1,u_2,\phi D^{n-1}\|_{\mathcal{B}_\beta} + \|u_2C_\phi D^n\|_{\mathcal{B}_\beta},$$

so we get the boundedness of $D^{n-1}u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ which is equivalent to boundedness of oprator $D^n u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_{\beta+1}$. Therefore, theorem (2.4), give us the desired result.
(b)$\Longrightarrow$(a). If we suppose (b), then according to theorem (2.4), we have the boundedness of $D^n u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_{\beta+1}$. Hence operator $D^{n-1}u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded. Since we also assume that $\Lambda_i^{u_2}$ for $i \in \{1,2,3\}$ are finite, then by theorem (2.7), we have the boundedness of $u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$. Therefore by the triangle inequality we get that $T^n_{u_1,u_2,\phi} = D^{n-1}u_1C_\phi + u_2C_\phi D^n$ is bounded form $\mathcal{B}_\alpha$ into $\mathcal{B}^\beta$.
(b)$\Longleftrightarrow$(c). It is clear, by theorem (2.4). □

The above theorem, can be obtained in the cases of $\alpha = 1$ and $\alpha > 1$, with similar proofs.

3. Compactness

In this section, we obtain some estimates for compactness of the generalized product-type operator $D^n uC_\phi$, from the Bloch type space $\mathcal{B}_\alpha$ into the Bloch type space $\mathcal{B}_\beta$. Again we consider the cases $0 < \alpha < 1, \alpha = 1$ and $\alpha > 1$ separately. As an application, we investigate the compactness of generalized operator $T^n_{u_1,u_2,\phi}$ between Bloch-type spaces. Recall that the essential norm $\|P\|_e$ of a bounded operator P between Banach spaces X and Y is defined as the distance from P to the compact operators, that is

$$\|P\|_e = inf\{\|P-k\| : \ k \ is \ compact\}.$$

Notice that $\|P\|_e = 0$ if and only if P is compact. Similar to the proof of proposition 3.11 [4], we get the following characterization for the compactness of a bounded linear operator between Bloch type spaces.

Lemma 3.1. *Suppose that $0 < \alpha, \beta < \infty$ and P be a bounded linear operator from $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. Then P is compact if and only if for any bounded sequence $\{h_n\}_0^\infty$ in $\mathcal{B}_\alpha$, which convergences to 0 uniformly on any compact subset of $\mathbb{U}$, we have that $\|P(h_n)\|_{\mathcal{B}_\beta} \to 0$ as $n \to \infty$.*

Next lemma form [13], gives us a characterization for compactness of a weighted composition operator uC_ϕ between weighted type spaces.

Lemma 3.2. *Let $1 < \alpha,\ \beta > 0,\ u \in \mathcal{H}(\mathbb{U})$ and $\phi \in \mathcal{S}$. Then the bounded weighted composition operator $uC_\phi : \mathcal{H}^\infty_{\alpha+i-1} \mapsto \mathcal{H}^\infty_\beta$ is compact if and only if*

$$\|uC_\phi\|_{e,\mathcal{H}^\infty_{\alpha+i-1}\to\mathcal{H}^\infty_\beta} = \lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta|u(z)|}{(1-|\phi(z)|^2)^{\alpha+i-1}} = 0.$$

Theorem 3.3 ([6]). *Let $1 < \alpha,\ \beta > 0,\ u \in \mathcal{H}(\mathbb{U})$ and $\phi \in \mathcal{S}$. Then $DuC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact if and only if the quantities of $E^{\alpha,\beta}_{u,\phi} = F^{\alpha,\beta}_{u,\phi} = O^{\alpha,\beta}_{u,\phi} = 0$, when*

$$E^{\alpha,\beta}_{u,\phi} := \frac{1}{3(3\alpha+4)2^\alpha}\Big(\frac{e}{2(\alpha+1)}\Big)^{\alpha+1} \limsup_{n\to\infty} n^\alpha \|I_{u\phi'}(\phi^n)\|_{\mathcal{B}_\beta},$$
$$F^{\alpha,\beta}_{u,\phi} := \frac{1}{(9\alpha^2+24\alpha+14)2^{\alpha+1}}\Big(\frac{e}{2\alpha}\Big) \limsup_{n\to\infty} n^{\alpha+1} \|I'_u(\phi^n) + nJ_{u\phi'}(\phi^{n-1})\|_{\mathcal{B}_\beta},$$
$$O^{\alpha,\beta}_{u,\phi} := \frac{1}{9\alpha(\alpha^2+3\alpha+2)2^\alpha}\Big(\frac{e}{2\alpha-2}\Big)^{\alpha-1} \limsup_{n\to\infty} n^{\alpha-1} \|J_{u'}(\phi^{n-1})\|_{\mathcal{B}_\beta}. \quad (3.1)$$

In the following theorem, we give equivalent characterizations for compactness of operator $D^n uC_\phi$ between Bloch type spaces, in the case of $0<\alpha<1$.

Theorem 3.4. *Suppose that $n \in \mathbb{N}$, $0<\alpha<1$, $\beta>0$, $u \in \mathcal{H}(\mathbb{U})$, $\phi \in \mathcal{S}$ and $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ be bounded. Then the following statements are equivalent:*

(a) *Operator $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact.*

(b) *For $i \in \{1, ..., n+1\}$,*

$$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} = 0.$$

(c) *Operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{i+\alpha} \mapsto \mathcal{B}_{\beta+2}$, for any $i \in \{1, ..., n+1\}$ is compact.*

(d) *$E^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = F^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = O^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = 0$, as they have been defined in (3.1).*

Proof. (a)$\Longleftrightarrow$(b). We will show that

$$\|D^n uC_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} = \max_{i\in\{1,...,n+1\}} \{\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}}\}. \tag{3.2}$$

But it is enough to prove(3.2) just in the case $\sup_{z\in\mathbb{U}} |\phi(z)| = 1$. So we assume that $\{a_j\}$ be a sequence in $\mathbb{U}$, such that $\frac{1}{2}|\phi(a_j)| \to 1$ as $j\to\infty$. Since we have defined $K_{m,\phi(a_j)}$ in (2.9), such that they are bounded and converges to zero uniformly on compact subsets of $\overline{\mathbb{U}}$, if $j\to\infty$. Then by applying lemma (3.1), for any compact operator $E : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$, we get $\lim_{j\to\infty} \|EK_{m,\phi(a_j)}\|_{\mathcal{B}_\beta} = 0$. Hence, for each $i \in \{0, 1, ..., n+1\}$, we obtain

$$\begin{aligned} \|D^n uC_\phi - E\|_{\mathcal{B}_\alpha\to\mathcal{B}_\beta} &\succeq \limsup_{j\to\infty} \|(D^n uC_\phi - E)(K_{m,\phi(a_j)})\|_{\mathcal{B}_\beta} \\ &\succeq \limsup_{j\to\infty} (\|D^n uC_\phi(K_{m,\phi(a_j)})\|_{\mathcal{B}_\beta} - \|E(K_{m,\phi(a_j)})\|_{\mathcal{B}_\beta}) \\ &\succeq \limsup_{j\to\infty} \frac{(1-|z|^2)^\beta |\phi(a_j)|^i |S^u_{n+1,i}(a_j)|}{(1-|\phi(a_j)|^2)^{i+\beta-1}} \\ &\succeq \limsup_{j\to\infty} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(a_j)|}{(1-|\phi(a_j)|^2)^{i+\alpha-1}} \end{aligned} \tag{3.3}$$

Which follows that, for each $i \in \{1, ..., n+1\}$

$$\|D^n uC_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} = \inf_E \|D^n uC_\phi - E\|_{\mathcal{B}_\alpha\to\mathcal{B}_\beta} \succeq \max_{i\in\{1,...,n+1\}} \{\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}}\}. \tag{3.4}$$

On the other hand, if we consider operator E_t for $t \in [0,1)$, such that $E_t h(z) = h(tz) = h_t(z)$, for $h \in \mathcal{B}_\alpha$. Then, E_t is a compact operator on $\mathcal{B}^\alpha$, with $\|E_t\|_{\mathcal{B}_\alpha\to\mathcal{B}_\beta} \le 1$(see [11]). Also $h_t \to h$ uniformly on compact subsets of $\mathbb{U}$ as $t\to 1$. Now suppose a sequence $\{t_j\} \subset (0,1)$, such that $t_j \to 1$ as $j\to\infty$. Then operator $D^n uC_\phi E_{t_j} : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact. Hence,

$$\|D^n uC_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} \preceq \limsup_{j\to\infty} \|D^n uC_\phi(I - E_{t_j})\|_{\mathcal{B}_\alpha\to\mathcal{B}_\beta}. \tag{3.5}$$

Now if we show that, for $i \in \{1, ..., n+1\}$,

$$\limsup_{j\to\infty} \|D^n uC_\phi(I - E_{t_j})\|_{\mathcal{B}_\alpha\to\mathcal{B}_\beta} \preceq \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}}. \tag{3.6}$$

Then we get our desired result. But for $h \in \mathcal{B}_\alpha$, when $\|h\|_{\mathcal{B}_\alpha} \le 1$, we have

$$\begin{aligned}
&\|(D^n uC_\phi(I - E_{t_j})h\|_{\mathcal{B}_\beta} \\
&= \sum_{t=0}^{n} \Big| \sum_{m=0}^{t} (h - h_{t_j})^m(\phi(0))S^u_{t,m}(0)\Big| + \sup_{z\in\mathbb{U}}(1-|z|^2)^\beta \Big|\sum_{m=0}^{n+1}(h-h_{t_j})^m(\phi(z))S^u_{n+1,m}(z)\Big| \\
&\le \underbrace{\sum_{t=0}^{n} \Big| \sum_{m=0}^{t} (h - h_{t_j})^{(m)}(\phi(0))S^u_{m,t}(0)\Big|}_{A_1} \quad + \underbrace{\sup_{|\phi(z)|\le r_N}(1-|z|^2)^\beta |(h-h_{t_j})(\phi(z))S^u_{n+1,0}(z)|}_{A_2} \\
&\quad + \underbrace{\sup_{|\phi(z)|> r_N}(1-|z|^2)^\beta |(h-h_{t_j})(\phi(z))S^u_{n+1,0}(z)|}_{A_3} \\
&\quad + \underbrace{\sup_{|\phi(z)|\le r_N}(1-|z|^2)^\beta \Big|\sum_{m=1}^{n+1}(h-h_{t_j})^{(m)}(\phi(z))S^u_{n+1,m}(z)\Big|}_{A_4} \\
&\quad + \underbrace{\sup_{|\phi(z)|> r_N}(1-|z|^2)^\beta \Big|\sum_{m=1}^{n+1}(h-h_{t_j})^{(m)}(\phi(z))S^u_{n+1,m}(z)\Big|}_{A_5},
\end{aligned} \tag{3.7}$$

where $t_j \ge \frac{2}{3}$ for $j \ge K$ and $K \in \mathbb{N}$. But for every $z \in \mathbb{U}$ and each $h \in \mathcal{B}_\alpha(U)$,

$$\begin{aligned}
|h(z) - h(t_j z)| &= \Big|\int_{t_j}^1 zh'(zw)dw\Big| \le \int_{t_j}^1 \frac{|z|}{(1-w^2|z|^2)^\alpha}|h'(zw)|(1-w^2|z|^2)^\alpha dw \\
&\le \|h\|_{\mathcal{B}_\alpha}\int_{t_j}^1 \frac{|z|}{(1-w^2|z|^2)^\alpha}dw.
\end{aligned} \tag{3.8}$$

Then, we can see that $\sup_{|z|\le r_N}|h(z) - h(t_j z)| \le \|h\|_{\mathcal{B}_\alpha}\frac{r_N}{(1-{r_N}^2)^\alpha}(1-t_j)$. Therefore, when $j \to \infty$,

$$A_2 \le \|u\|_{\mathcal{W}^{n+1}_\alpha}\sup_{|z|\le r_N}|f(z) - f(t_j z)| \le \|u\|_{\mathcal{W}^{n+1}_\alpha}\frac{r_N}{(1-{r_N}^2)^\alpha}(1-t_j) \to 0. \tag{3.9}$$

For A_3, in the case that $0 < \alpha < 1$, we have

$$\int_{t_j}^1 \frac{|z|}{(1-w|z|)^\alpha}dw = \frac{(1-t_j|z|)^{1-\alpha} - (1-|z|)^{1-\alpha}}{1-\alpha} \le \frac{(1-t_j)^{1-\alpha}}{1-\alpha},$$

and relation (3.7), gives us

$$A_3 \le \|u\|_{\mathcal{W}^{n+1}_\alpha}\frac{(1-t_j)^{1-\alpha}}{1-\alpha} \to 0, \quad as\ j \to \infty. \tag{3.10}$$

Also for $m \in \mathbb{N}$, the sequence $(h - h(t_j))^{(m)} \to 0$, uniformly on compact subsets of $\mathbb{U}$, as $j \to \infty$. So we get that

$$\limsup_{j\to\infty} A_1 = \limsup_{j\to\infty} A_4 = 0. \tag{3.11}$$

On the other hand,

$$A_5 \leq \underbrace{\sum_{m=1}^{n+1} \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta |h^{(m)}(\phi(z))||S^u_{n+1,m}(z)|}_{M_s}$$
$$+ \underbrace{\sum_{m=1}^{n+1} \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta |t_j^m h^{(m)}(t_j\phi(z))||S^u_{n+1,m}(z)|}_{N_k}.$$

But, for $m \in \{1,...,n+1\}$, applying relation (2.10) and lemma 2.1 yields

$$M_s = \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta \frac{(1-|\phi(z)|^2)^{m+\alpha-1}|h^{(m)}(\phi(z))|}{|\phi(z)|^m} \frac{|\phi(z)|^m |S^u_{n+1,m}(z)|}{(1-|\phi(z)|^2)^{m+\alpha-1}}$$
$$\preceq \|h\|_{\mathcal{B}_\alpha} \sup_{|\phi(z)|>r_N} \frac{(1-|z|^2)^\beta |S^u_{n+1,m}(z)|}{(1-|\phi(z)|^2)^{m+\alpha-1}}.$$

So by taking the limit, when $N \to \infty$ in the last inequality, we obtain

$$\limsup_{j\to\infty} M_s \preceq \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,m}(z)|}{(1-|\phi(z)|^2)^{m+\alpha-1}} \tag{3.12}$$

In a similar way, we get

$$\limsup_{j\to\infty} N_k \preceq \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,m}(z)|}{(1-|\phi(z)|^2)^{m+\alpha-1}}. \tag{3.13}$$

Therefore, relations (3.9), (3.10), (3.11), (3.12) and (3.13), imply that

$$\limsup_{j\to\infty} \|D^n uC_\phi(I-E_{t_j})h\|_{\mathcal{B}_\beta} \preceq \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,m}(z)|}{(1-|\phi(z)|^2)^{m+\alpha-1}}. \tag{3.14}$$

Hence (3.4), (3.5) and (3.14), give us

$$\|D^n uC_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} \approx \max_{i\in\{1,...,n+1\}} \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}}.$$

Now the fact that, $\|D^n uC_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} = 0$ if and only if $D^n uC_\phi : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ be compact, gives us the desired result.

(b)$\Longrightarrow$(c). Suppose that for any $i \in \{1,...,n+1\}$,

$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{\alpha+i-1}} = 0.$$

Then according to lemma 3.2, we get that operator $S^u_{n+1,i}C_\phi : H^\infty_{\alpha+i-1} \mapsto \mathcal{H}^\infty_\beta$ is compact for any $i \in \{1,...,n+1\}$. But $\alpha+i>0$, and the spaces $\mathcal{B}_{\alpha+i}$ and $\mathcal{H}^\infty_{\alpha+i-1}$ are norm-equivalent, hence

$$\|S^u_{n+1,i}C_\phi\|_{e,\mathcal{B}_{\alpha+i}\to\mathcal{B}_{\beta+1}} \approx \|S^u_{n+1,i}C_\phi\|_{e,\mathcal{H}^\infty_{\alpha+i-1}\to\mathcal{H}^\infty_\beta} = 0,$$

then we get the compactness of operator $S^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \to \mathcal{B}_{\beta+1}$. On the other hand, it is easy to see that operator $S^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \mapsto \mathcal{B}_{\beta+1}$ is compact if and only if $DS^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \mapsto \mathcal{B}_{\beta+2}$ is compact. So we obtain the desired result.

(c)$\Longrightarrow$(b) Let $DS^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \mapsto \mathcal{B}_{\beta+2}$ be compact, then we easily get that $S^u_{n+1,i}C_\phi : \mathcal{B}_{\alpha+i} \mapsto \mathcal{B}_{\beta+1}$ is compact and so $\|S^u_{n+1,i}C_\phi\|_{e,\mathcal{B}_{\alpha+i}\to\mathcal{B}_{\beta+1}} = 0$. But $\|S^u_{n+1,i}C_\phi\|_{e,\mathcal{B}_{\alpha+i}\to\mathcal{B}_{\beta+1}} \approx \|S^u_{n+1,i}C_\phi\|_{e,\mathcal{H}^\infty_{\alpha+i-1}\to\mathcal{H}^\infty_\beta}$. Therefore, operator $S^u_{n+1,i}C_\phi : \mathcal{H}^\infty_{\alpha+i-1} \mapsto \mathcal{H}^\infty_\beta$ is compact and lemma 3.2 completes the proof.

(c)$\Longleftrightarrow$(d). It is clear by applying theorem 3.3 □

If we consider the case $\alpha = 1$, by applying lemma 2.1 for any $f \in \mathcal{B}_\alpha$, $|h(\phi(z))| \leq \frac{\|h\|_{\mathcal{B}_\alpha}}{(1-|\phi(z)|^2)}$. Then by defining A_3 as in the proof of theorem 3.4, we get that

$$\begin{aligned} A_3 &= \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta |(h-h_{t_j})(\phi(z)) S^u_{n+1,0}(z)| \\ &\leq \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta |(h(\phi(z)) u^{(n+1)}(z)| + \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta |(h(t_j\phi(z)) u^{(n+1)}(z)| \\ &\leq \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta \frac{\|h\|_{\mathcal{B}_\alpha} |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} + \sup_{|\phi(z)|>r_N} (1-|z|^2)^\beta \frac{\|h\|_{\mathcal{B}_\alpha} |u^{(n+1)}(z)|}{(1-|t_j\phi(z)|^2)} \end{aligned}$$

In the last inequality let $N \to \infty$, then we have

$$\limsup_{j\to\infty} A_3 \preceq \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} \tag{3.15}$$

Therefore, by applying theorem 2.5 and lemma 3.1, in the same way as in the proof of theorem 3.4 we can prove the following theorem, just use (3.15) instead of (3.10).

Theorem 3.5. *Suppose that $n \in \mathbb{N}$, $\alpha = 1$, $\beta > 0$, $u \in \mathcal{H}(\mathbb{U})$, $\phi \in \mathcal{S}$ and $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ be bounded. Then the following statements are equivalent:*

(a) *Operator $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact.*
(b) *For any $i \in \{1, ..., n+1\}$,*

$$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} = 0, \quad \textit{and} \quad \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} = 0.$$

(c) *For any $i \in \{1, ..., n+1\}$, operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{i+\alpha} \mapsto \mathcal{B}_{\beta+2}$ is compact and*

$$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} = 0.$$

(d) $E^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = F^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = O^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = 0$*(as they have been defined in* (3.1)*), and*

$$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} = 0.$$

In the case of $\alpha > 1$, by applying part (c) of lemma 2.1 and using theorem 2.6 and lemma 3.1, similar to the proof of theorems 3.4 and 3.5, we obtain the following result.

Theorem 3.6. *Suppose that $n \in \mathbb{N}$, $\alpha > 1$, $\beta > 0$, $u \in \mathcal{H}(\mathbb{U})$, $\phi \in \mathcal{S}$ and $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ be bounded. Then the following statements are equivalent:*

(a) *Operator $D^n uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact.*
(b) *For any $i \in \{0, 1, ..., n+1\}$, $\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |S^u_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} = 0$.*
(c) *Operator $D(S^u_{n+1,i}.C_\phi) : \mathcal{B}_{i+\alpha} \mapsto \mathcal{B}_{\beta+2}$, for any $i \in \{0, 1, ..., n+1\}$ is compact.*
(d) $E^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = F^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = O^{\alpha+i,\beta+2}_{S^u_{n+1,\phi},\phi} = 0$ *(as they have been defined in* (3.1)*), and*

$$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta |u^{(n+1)}(z)|}{(1-|\phi(z)|^2)} = 0.$$

The next consequence, can be obtained by using corollary 2.7 and theorem 3.6

Corollary 3.7. *Let $0 < \alpha$, $0 < \beta < \infty$, $\phi \in \mathcal{S}$, $u \in \mathcal{H}(U)$ and $uC_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded. Then the following statements are equivalent:*

(a) *Operator $uC_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact.*
(b) *Operator $DuC_\phi : \mathcal{B}_{\alpha+n} \mapsto \mathcal{B}_{\beta+1}$ is compact.*

(c) $\tilde{\Lambda}_1^u = \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta+1} u''(z)}{(1-|\phi(z)|^2)^{\alpha+n-1}} = 0,$
$\tilde{\Lambda}_2^u = \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta+1}(2u'(z)\phi'(z)+u\phi''(z))}{(1-|\phi(z)|^2)^{\alpha+n}} = 0,$
$\tilde{\Lambda}_3^u = \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta+1} u(z)\phi'^2(z)}{(1-|\phi(z)|^2)^{\alpha+n+1}} = 0.$

Example 3.8. Let $\alpha = \beta = 1$, $u(z) = 1 - z$, and $\phi(z) = \frac{z-\frac{1}{2}}{1-\frac{z}{2}}$, for every $z \in \mathbb{U}$. Then $\phi \in \mathcal{S}, u \in \mathcal{H}(\mathbb{U})$ and also $u'(z) = -1, \phi'(z) = \frac{3}{4(1-\frac{z}{2})^2}, u''(z) = 0$ and $\phi''(z) = \frac{3}{4(1-\frac{z}{2})^3}$, for every $z \in \mathbb{U}$. Next, consider the case $n = 1$ in corollary 3.7. Then by simple calculations, we get that

$$\tilde{\Lambda}_1^u = 0, \qquad \tilde{\Lambda}_2^u = \limsup_{|\phi(z)|\to 1} \frac{-3(1-|z|^2)^2}{4(1-|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}|^2)^2(1-\frac{z}{2})^3} = 0,$$
$$\tilde{\Lambda}_3^u = \limsup_{|\phi(z)|\to 1} \frac{9(1-|z|^2)^2(1-z)}{16(1-|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}|^2)^3(1-\frac{z}{2})^4} = 0$$

Hence, according to corollary 2.7 and corollary 3.7, operators $uC_\phi D : \mathcal{B} \mapsto \mathcal{B}$ and $DuC_\phi : \mathcal{B}_2 \mapsto \mathcal{B}_2$ are bounded and also compact.

As a corollary of theorem 3.5 and by applying corollary 2.9, the next result is obtained which is similar to theorem 6 in [2].

Corollary 3.9. *Let $\phi \in \mathcal{S}$, $u \in \mathcal{H}(U)$ and $DuC_\phi : \mathcal{B} \mapsto \mathcal{B}$ is bounded. Then the following statements are equivalent:*

(a) *Operator $DuC_\phi : \mathcal{B} \mapsto \mathcal{B}$ is compact.*
(b) *Operator $uC_\phi : \mathcal{B} \mapsto \mathcal{Z}$ is compact.*
(c) *Operator $uC_\phi D : \mathcal{Z} \mapsto \mathcal{Z}$ is compact.*
(d) $\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)|u''(z)|}{(1-|\phi(z)|^2)} = 0, \quad \limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)|u(z)\phi'(z)^2|}{(1-|\phi(z)|^2)^2} = 0,$
$\limsup_{|\phi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\phi'(z)+u(z)\phi''(z)|}{1-|\phi(z)|^2} = 0.$

The following lemma, which gives a characterization for compactness of the weighted composition operator between Bloch type spaces, will help us to get our next result.

Lemma 3.10 ([15]). *Let $1 < \alpha$, $\beta > 0$, $u \in \mathcal{H}(\mathbb{U})$, $\phi \in \mathcal{S}$ and suppose that $uC_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is bounded. Then uC_ϕ maps $\mathcal{B}_\alpha$ compactly into $\mathcal{B}_\beta$, if and only if the following conditions hold*

$$\lim_{|\phi(w)|\to 1^-} |u'(w)| \frac{(1-|w|^2)^\beta}{(1-|\phi(w)|^2)^{\alpha-1}} = 0, \lim_{|\phi(w)|\to 1^-} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\phi(w)|^2)^\alpha} |\phi'(w)| = 0.$$

Theorem 3.11. *Suppose that $0 < \alpha < 1$, $0 < \beta < \infty$, $\phi \in \mathcal{S}$, $u_1, u_2 \in \mathcal{H}(U)$ and also $\lim_{|\phi(z)|\to 1} |u_2'(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha+n-1}} = \lim_{|\phi(z)|\to 1} |u_2(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha+n}} |\phi'(z)| = 0$. Then the following statements are equivalent:*

(a) *Operator $T^n_{u_1,u_2,\phi} : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact.*
(b) *For any $i \in \{1, ..., n+1\}$,*

$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta+1}|S^{u_1}_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} = 0.$$

(c) $E^{\alpha+i,\beta+3}_{S^{u_1}_{n+1},\phi} = F^{\alpha+i,\beta+3}_{S^{u_1}_{n+1},\phi} = O^{\alpha+i,\beta+3}_{S^{u_1}_{n+1},\phi} = 0$, *as they have been defined in* (3.1).

Proof. (a)$\Longrightarrow$(b). Let $T^n_{u_1,u_2,\phi} : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ be compact, then $\|T^n_{u_1,u_2,\phi}\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} = 0$. Also, operator $u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact if and only if $u_2C_\phi : \mathcal{B}_{\alpha+n} \mapsto \mathcal{B}_\beta$ is compact.

But we have supposed that

$$\lim_{|\phi(z)|\to 1} |u_2'(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha-1}} = \lim_{|\phi(z)|\to 1} |u_2(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha}} |\phi'(z)| = 0,$$

and lemma 3.10, gives us the compactness of operator $u_2C_\phi : \mathcal{B}_{\alpha+n} \mapsto \mathcal{B}_\beta$. So $u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact. Then by the triangle inequality we get that

$$\|D^{n-1}u_1C_\phi\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} \preceq \|T^n_{u_1,u_2,\phi}\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} + \|u_2C_\phi D^n\|_{e,\mathcal{B}_\alpha\to\mathcal{B}_\beta} = 0.$$

Therefore $D^{n-1}u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact. Then we get the compactness of $D^n u_1 C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_{\beta+1}$. Now using theorem 3.4, gives us the statement in part (b).

(b)$\Longrightarrow$(a). Let $\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta+1}|S^{u_1}_{n+1,i}(z)|}{(1-|\phi(z)|^2)^{i+\alpha-1}} = 0$, for $i \in \{1, ..., n+1\}$, then by theorem 3.4, we have the compactness of operator $D^n u_1 C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_{\beta+1}$. Therefore, operator $D^{n-1}u_1C_\phi : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact. We also assumed that

$$\lim_{|\phi(z)|\to 1} |u_2'(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha+n-1}} = 0, \quad \lim_{|\phi(z)|\to 1} |u_2(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha+n}} |\phi'(z)| = 0.$$

So by lemma 3.10, we have the compactness of operator $u_2C_\phi : \mathcal{B}_{\alpha+n} \mapsto \mathcal{B}_\beta$. Hence similar to the proof of part (a) to (b), $u_2C_\phi D^n : \mathcal{B}_\alpha \mapsto \mathcal{B}_\beta$ is compact. Then the triangle inequality together with lemma 3.1, imply the compactness of $T^n_{u_1,u_2,\phi} = D^{n-1}u_1C_\phi + u_2C_\phi D^n$ form $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$.

(b)$\Longleftrightarrow$ (c). It is clear by applying theorem 3.4. □

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